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PRECISE LOWER BOUND OF $f(A)-f(B)$ FOR $A>B>0$ AND NON–CONSTANT OPERATOR MONOTONE FUNCTION f ON $[0,\infty)$

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Abstract. A and B be strictly positive operators on a Hilbert space H such that $A-B \geqslant m > 0$. Then the following inequalities hold for any non-constant operator monotone function f on $[0,\infty)$:

$$\begin{aligned} f(A)-f(B) &\geqslant f(||B||+m)-f(||B||) \\ &\geqslant f(||A||)-f(||A||-m) > 0. \end{aligned}$$

In particularly, let $A>B$. Then

$$\begin{aligned} f(A)-f(B) &\geqslant f\left(||B||+\frac{1}{||(A-B)^{-1}||}\right)-f(||B||) \\ &\geqslant f(||A||)-f\left(||A||-\frac{1}{||(A-B)^{-1}||}\right) > 0. \end{aligned}$$

We shall state the typical concrete example of these operator inequalities.

Let $A>B$. Then the following inequalities hold as an extension of celebrated Löwner-Heinz inequality

$$\begin{aligned} A^r-B^r &\geqslant \left(||B||+\frac{1}{||(A-B)^{-1}||}\right)^r-||B||^r \\ &\geqslant ||A||^r-\left(||A||-\frac{1}{||(A-B)^{-1}||}\right)^r > 0 \quad \text{for } 0<r\leqslant 1 \end{aligned}$$

and also the following inequalities hold

$$\begin{aligned} \log A-\log B &\geqslant \log\left(||B||+\frac{1}{||(A-B)^{-1}||}\right)-\log||B|| \\ &\geqslant \log||A||-\log\left(||A||-\frac{1}{||(A-B)^{-1}||}\right) > 0. \end{aligned}$$

1. Introduction on operator monotone functions for $A>B>0$

A capital letter means a bounded linear operator on a complex Hilbert space H.

An operator T is said to be *positive* (denoted by $T \geqslant 0$) if $(Tx,x) \geqslant 0$ for all $x \in H$ and an operator T is said to be *strictly positive* (denoted by $T>0$) if T is positive and invertible.

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A continuous real valued function f defined on an interval J is called *operator monotone* if $A \geqslant B$ implies $f(A) \geqslant f(B)$ for all self-adjoint operators A,B with spectra in J.

The well known celebrated Löwner-Heinz inequality asserts that if $A \geqslant B \geqslant 0$, then $A^\alpha \geqslant B^\alpha$ for any $\alpha \in [0,1]$.

This means that $t \mapsto t^\alpha$ is operator monotone. Another well known example of operator monotone is $t \mapsto \log t$ on $(0,\infty)$, that is, $\log A \geqslant \log B$ is weaker than the usual order $A \geqslant B \geqslant 0$.

THEOREM A. [2, Theorem 2.1, 3, Proposition 2.2] *Let A and B be strictly positive operators on a Hilbert space H. If $A > B$, then the following inequality holds:*

$$f(A) > f(B) \tag{1.1}$$

for any non-constant operator monotone function f on $[0,\infty)$.

Theorem A remains valid for A,B self-adjoint operators in [3, Proposition 2.2].

LEMMA B. [3, Lemma 2.1] *Let $A,B \in B(H)$ be invertible positive operators such that $A-B \geqslant m > 0$. Then*

$$B^{-1} - A^{-1} \geqslant \frac{m}{(||A||-m)||A||}. \tag{1.2}$$

THEOREM C. [3, Corollary 2.5] *Let $A > B > 0$. Then*

(i) $A^r - B^r \geqslant ||A||^r - \left(||A|| - \frac{1}{||(A-B)^{-1}||}\right)^r > 0$ *for* $0 < r \leqslant 1$.

(ii) $\log A - \log B \geqslant \log ||A|| - \log\left(||A|| - \frac{1}{||(A-B)^{-1}||}\right)$.

Recently the following interesting Theorem D has been shown by an application of Lemma B and also the method of Theorem A. In fact Theorem D is further extension of Theorem C and also precise estimation of Theorem A.

THEOREM D. [5, Theorem 1] *Let A and B be strictly positive operators on a Hilbert space H. If $A > B$, then the following inequality holds:*

$$f(A) - f(B) \geqslant f(||A||) - f\left(||A|| - \frac{1}{||(A-B)^{-1}||}\right) > 0 \tag{1.3}$$

for any non-constant operator monotone function f on $[0,\infty)$.

We shall show our main Theorem 3.1 and as an application we shall show Corollary 3.2 which is more precise estimation than Theorem D and also Corollary 3.3 which is an improvement of Theorem C by considering simple and useful Lemma 2.2.

2. Basic properties on operator monotone functions for $A>B>0$

LEMMA 2.1. *Let $X-\alpha$ and $X-\beta$ be both strictly positive operators for real numbers α, β and $X>0$. Then the following* (i) *and* (ii) *hold.*

$$\text{(i)} \qquad (X-\alpha)^{-1}-(X-\beta)^{-1}=(\alpha-\beta)(X-\alpha)^{-1}(X-\beta)^{-1} \tag{2.1}$$

and

$$\text{(ii)} \qquad (X-\alpha)^{-1}(X-\beta)^{-1}\geqslant ||X-\alpha||^{-1}||X-\beta||^{-1}. \tag{2.2}$$

Proof. (i)

$$\begin{aligned}(X-\alpha)^{-1}-(X-\beta)^{-1} &= (X-\alpha)^{-1}(X-\beta)^{-1}\{(X-\beta)-(X-\alpha)\}\\ &= (\alpha-\beta)(X-\alpha)^{-1}(X-\beta)^{-1}.\end{aligned}$$

(ii) Since $(X-\beta)(X-\alpha)\leqslant ||(X-\beta)(X-\alpha)||\leqslant ||X-\beta||||X-\alpha||$ and we have only to take inverses of both sides. □

LEMMA 2.2. *Let T be strictly positive operator and let $m>0$. Then the following equality holds for $b\geqslant 0$:*

$$f(T+m)-f(T)=bm+m\int_0^\infty (T+s)^{-1}(T+m+s)^{-1}s^2d\mu(s) \tag{2.3}$$

for an operator monotone function f on $[0,\infty)$.

Proof. It is well known (for examples, [1], [4]) that if f is an operator monotone function on $[0,\infty)$, then there exists a positive measure μ on $[0,\infty)$ such that

$$\begin{aligned}f(t) &= a+bt+\int_0^\infty \frac{ts}{t+s}d\mu(s)\\ &= a+bt+\int_0^\infty \left(s-\frac{s^2}{t+s}\right)d\mu(s)\end{aligned} \tag{2.4}$$

with $a\in\mathbb{R}$ and $b\geqslant 0$.

Then we have the following by (2.4) for two strictly positive operators A and B,

$$f(A)-f(B)=b(A-B)+\int_0^\infty \{(B+s)^{-1}-(A+s)^{-1}\}s^2d\mu(s). \tag{2.5}$$

Replacing A by $T+m$ and B by T in (2.5) for strictly positive operator T, then

$$\begin{aligned}f(T+m)-f(T) &= bm+\int_0^\infty \{(T+s)^{-1}-(T+m+s)^{-1}\}s^2d\mu(s)\\ &= bm+m\int_0^\infty (T+s)^{-1}(T+m+s)^{-1}s^2d\mu(s)\end{aligned} \tag{2.6}$$

since the last equality follows by (2.1) of Lemma 2.1. □

3. Precise lower bound of $f(A)-f(B)$ for $A-B \geqslant m>0$

THEOREM 3.1. *Let A and B be strictly positive operators such that $A-B \geqslant m>0$. Then the following inequalities hold for any non-constant operator monotone function f on $[0,\infty)$:*

$$f(A)-f(B) \geqslant f(||B||+m)-f(||B||) \geqslant f(||A||)-f(||A||-m)>0. \tag{3.1}$$

Proof.

$$\begin{aligned} f(A)-f(B) &\geqslant f(B+m)-f(B) \text{ by monotonicity of } f \text{ since } A \geqslant B+m \\ &= bm+m\int_0^\infty (B+s)^{-1}(B+m+s)^{-1}s^2 d\mu(s) \text{ by Lemma 2.2} \\ &\geqslant bm+m\int_0^\infty ||B+s||^{-1}||B+m+s||^{-1}s^2 d\mu(s) \text{ by (2.2) of Lemma 2.1} \\ &= bm+m\int_0^\infty (||B||+s)^{-1}(||B||+m+s)^{-1}s^2 d\mu(s) \\ &= f(||B||+m)-f(||B||) \text{ by Lemma 2.2} \end{aligned} \tag{3.2}$$

and the second equality follows by $||B+t||=||B||+t$ for $t \geqslant 0$ since $m>0$ and $s \geqslant 0$, so we have the first inequality of (3.1) by (3.2).

On the other hand, the condition $A-B \geqslant m>0$ implies $||A|| \geqslant ||B||+m$ and for $s \geqslant 0$ we have

$$||A||+s \geqslant ||B||+m+s \text{ and } ||A||-m+s \geqslant ||B||+s \tag{3.3}$$

and (3.3) obviously ensures

$$(||A||+s)(||A||-m+s) \geqslant (||B||+m+s)(||B||+s) \tag{3.4}$$

and taking inverses of the both sides of (3.4), we have

$$(||B||+s)^{-1}(||B||+m+s)^{-1} \geqslant (||A||-m+s)^{-1}(||A||+s)^{-1}. \tag{3.5}$$

Then we have

$$\begin{aligned} f(||B||+m)-f(||B||) &= bm+m\int_0^\infty (||B||+s)^{-1}(||B||+m+s)^{-1}s^2 d\mu(s) \text{ by Lemma 2.2} \\ &\geqslant bm+m\int_0^\infty (||A||-m+s)^{-1}(||A||+s)^{-1}s^2 d\mu(s) \text{ by (3.5)} \\ &= f(||A||)-f(||A||-m)>0 \text{ by Lemma 2.2 since } ||A||-m>0 \end{aligned} \tag{3.6}$$

and we have the second inequality of (3.1) by (3.6) and the last one follows by Theorem A. □

COROLLARY 3.2. *Let A and B be strictly positive operators such that $A>B$. Then the following inequalities hold for any non-constant operator monotone function f on $[0,\infty)$:*

$$\begin{aligned} f(A)-f(B) &\geqslant f\left(||B||+\frac{1}{||(A-B)^{-1}||}\right)-f(||B||) \\ &\geqslant f(||A||)-f\left(||A||-\frac{1}{||(A-B)^{-1}||}\right)>0. \end{aligned} \tag{3.7}$$

Proof. Let $A-B\geqslant m>0$. Then $(A-B)^{-1}\leqslant m^{-1}$ and $\frac{1}{||(A-B)^{-1}||}\geqslant m>0$. Since $A-B\geqslant \frac{1}{||(A-B)^{-1}||}$ always holds and we have

$$A-B\geqslant \frac{1}{||(A-B)^{-1}||}\geqslant m>0,$$

and we have only to put $m=\frac{1}{||(A-B)^{-1}||}>0$ in (3.1) of Theorem 3.1. □

We remark that (3.7) of Corollary 3.2 is more precise estimation than (1.3) of Theorem D.

COROLLARY 3.3. *Let A and B be strictly positive operators such that $A>B$. Then the following inequalities hold:*

(i)
$$\begin{aligned} A^r-B^r &\geqslant \left(||B||+\frac{1}{||(A-B)^{-1}||}\right)^r-||B||^r \\ &\geqslant ||A||^r-\left(||A||-\frac{1}{||(A-B)^{-1}||}\right)^r>0 \quad \text{for } 0<r\leqslant 1. \end{aligned} \tag{3.8}$$

(ii)
$$\begin{aligned} \log A-\log B &\geqslant \log\left(||B||+\frac{1}{||(A-B)^{-1}||}\right)-\log||B|| \\ &\geqslant \log||A||-\log\left(||A||-\frac{1}{||(A-B)^{-1}||}\right)>0. \end{aligned} \tag{3.9}$$

Proof. Since t^r for $0<r\leqslant 1$ and $\log t$ are both operator monotone, (i) and (ii) follow by Corollary 3.2. □

Corollary 3.3 is an improvement of Theorem C.

Finally we shall state a simple proof of slightly improvement of Lemma B.

PROPOSITION 3.4. *Let A and B be strictly positive operators on a Hilbert space H such that $A-B\geqslant m>0$. Then the following inequality holds:*

$$\begin{aligned} B^{-1}-A^{-1} &\geqslant m||B||^{-1}(||B||+m)^{-1} \\ &\geqslant m(||A||-m)^{-1}||A||^{-1}. \end{aligned} \tag{3.10}$$

Proof. Since $A \geqslant B+m > 0$ and $(B+m)^{-1} \geqslant A^{-1}$, we have

$$\begin{aligned} B^{-1}-A^{-1} &\geqslant B^{-1}-(B+m)^{-1} \\ &= mB^{-1}(B+m)^{-1} \quad \text{by (2.1) of Lemma 2.1} \\ &\geqslant m||B||^{-1}||B+m||^{-1} \quad \text{by (2.2) of Lemma 2.1} \\ &= m||B||^{-1}(||B||+m)^{-1} \quad \text{by } ||B+m|| = ||B||+m \text{ since } B>0,\ m>0 \\ &\geqslant m(||A||-m)^{-1}||A||^{-1} \quad \text{by (3.5) for } s=0 \end{aligned}$$

and we have (3.10). □

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