Journal of
Mathematical
Inequalities
Volume 9, Number 1 (2015), 53–60
doi:10.7153/jmi-09-05

A COMPLETELY MONOTONIC FUNCTION RELATING TO THE q–TRIGAMMA FUNCTION

JIAO-LIAN ZHAO

(*Communicated by N. Elezović*)

Abstract. In the paper, a function relating to the q-trigamma function is proved to be completely monotonic. As by-products, two functions relating to the logarithmic function are also proved to be completely monotonic.

1. Introduction

The classical Euler gamma function $\Gamma(x)$ may be defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, \mathrm{d}t = \frac{1}{x} \prod_{n=1}^\infty \left\{ \left(1 + \frac{1}{n}\right)^x \left(1 + \frac{x}{n}\right)^{-1} \right\} \tag{1}$$

The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, is called the psi or digamma function, and the derivatives $\psi^{(i)}(x)$ for $i \in \mathbb{N}$, the set of all positive integers, are respectively called the polygamma functions. In particular, the functions $\psi'(x)$ and $\psi''(x)$ are called the trigamma and tetragamma functions.

The q-analogue $\Gamma_q(x)$ of the gamma function $\Gamma(x)$ may be defined for $x > 0$ by

$$\Gamma_q(x) = (1-q)^{1-x} \prod_{i=0}^\infty \frac{1-q^{i+1}}{1-q^{i+x}} \tag{2}$$

when $0 < q < 1$, and by

$$\Gamma_q(x) = (q-1)^{1-x} q^{\binom{x}{2}} \prod_{i=0}^\infty \frac{1-q^{-(i+1)}}{1-q^{-(i+x)}} \tag{3}$$

when $q > 1$. The q-psi function $\psi_q(x)$, the q-analogue of the psi function $\psi(x)$, may be defined by

$$\begin{aligned} \psi_q(x) = \frac{\Gamma_q'(x)}{\Gamma_q(x)} &= -\ln(1-q) + \ln q \sum_{k=0}^\infty \frac{q^{k+x}}{1-q^{k+x}} \\ &= -\ln(1-q) + \ln q \sum_{k=1}^\infty \frac{q^{kx}}{1-q^k} \end{aligned} \tag{4}$$

Mathematics subject classification (2010): Primary 33D05; Secondary 26A48.
Keywords and phrases: Completely monotonic function, q-trigamma function, logarithmic function.
The authors were supported in part by the Natural Science Foundation Grant No. 12JK0881 of Ministry of Education, Shaanxi Province, China.

for $0<q<1$ and $x>0$, and by

$$\psi_q(x) = -\ln(q-1) + \ln q\left(x - \frac{1}{2} - \sum_{n\geqslant 0}\frac{q^{-n-x}}{1-q^{-n-x}}\right) \tag{5}$$

for $q>1$ and $x>0$. The functions $\psi_q^{(k)}(x)$, the q-analogues of the polygamma functions $\psi^{(k)}(x)$, for $k\in\mathbb{N}$ are called the q-polygamma functions. The above mentioned functions have the following relations

$$\lim_{q\to 1^{\pm}}\Gamma_q(z) = \Gamma(z), \qquad \Gamma_q(x) = q^{\binom{x-1}{2}}\Gamma_{1/q}(x), \qquad \lim_{q\to 1^{\pm}}\psi_q(x) = \psi(x). \tag{6}$$

For more information, please refer to [2, pp. 493–496] and [6, 7].

We recall from [8, Chapter XIII] and [17, Chapter IV] that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and

$$(-1)^n f^{(n)}(x) \geqslant 0 \tag{7}$$

for $x\in I$ and $n\geqslant 0$. In [17, p. 161, Theorem 12b], it was stated that a necessary and sufficient condition that $f(x)$ should be completely monotonic for $0<x<\infty$ is that

$$f(x) = \int_0^\infty e^{-xt}\,\mathrm{d}\alpha(t), \tag{8}$$

where $\alpha(t)$ is non-decreasing and the integral converges for $0<x<\infty$. In other words, a function is completely monotonic on $(0,\infty)$ if and only if it is a Laplace transform of a positive measure.

For $x>0$, let

$$f(x) = \psi'(x) - \frac{1}{x} - \frac{1}{2x^2}. \tag{9}$$

For $x>0$ and $0<q<1$, let

$$f_q(x) = \psi_q'(x) - \frac{1-q}{1-q^x} - \frac{1}{2}\left(\frac{1-q}{1-q^x}\right)^2 + \frac{1}{2}(1-q)(3-q). \tag{10}$$

It is clear that $\lim_{q\to 1^-} f_q(x) = f(x)$. So, we may regard $f_q(x)$ as the q-analogue of the function $f(x)$.

In recent years, the complete monotonicity of the function (9) was proved, generalized, and applied in [1, 3, 5, 6, 7, 9, 16], [10, Theorem 1.1], [13, Theorem 1.3], [14, pp. 1977–1978], [15, Theorem 2]. For more information on this topic, please refer to related texts in the expository and survey article [12] and closely related references therein.

In [11], it was proved that, for $x>0$ and $0<q<1$, the function

$$f_q(x) = \psi_q'(x) - \frac{(1-q)q^x}{1-q^x} - \frac{1}{2}\left[\frac{(1-q)q^x}{1-q^x}\right]^2, \tag{11}$$

an alternative q-analogue of (9), is completely monotonic on $(0,\infty)$.

The goal of this paper is to prove the complete monotonicity of $f_q(x)$ for $0<q<1$ on $(0,\infty)$. Our main result may be stated as the following theorem.

THEOREM 1. *For* $0 < q < 1$, *the function* $f_q(x)$ *defined by* (10) *is completely monotonic on* $(0,\infty)$.

2. Lemmas

To prove our main result, we need the following lemmas.

LEMMA 1. *For* $i \in \mathbb{N}$ *and* $q \in (0,1)$, *we have*

$$\psi_q^{(i)}(x) = (\ln q)^{i+1} \sum_{k=1}^{\infty} \frac{k^i q^{kx}}{1-q^k}. \tag{12}$$

Proof. This follows from the definition of $\psi_q(x)$ by (4), direct differentiation, and the induction. □

LEMMA 2. *For* $q \in (0,1)$ *and* $x \in (0,\infty)$, *we have*

$$\sum_{k=1}^{\infty} kq^{(k-1)x} = \frac{1}{(1-q^x)^2}. \tag{13}$$

Proof. This can be deduced from the series expansion

$$\frac{1}{(1-x)^2} = \sum_{i=0}^{\infty}(i+1)x^i \tag{14}$$

for $x \in (0,1)$ and replacement of x by q^x in (14). □

LEMMA 3. *For* $0 < q < 1$ *and* $x \in (0,\infty)$, *we have*

$$\psi_q'(x) - \psi_q'(x+1) = (\ln q)^2 \sum_{k=1}^{\infty} kq^{kx}. \tag{15}$$

Proof. By Lemma 1 for $n = 1$, we have

$$\begin{aligned}
\psi_q'(x) - \psi_q'(x+1) &= (\ln q)^2 \sum_{k=1}^{\infty} \frac{kq^{kx}}{1-q^k} - (\ln q)^2 \sum_{k=1}^{\infty} \frac{kq^k q^{kx}}{1-q^k} \\
&= (\ln q)^2 \sum_{k=1}^{\infty} \frac{kq^{kx}(1-q^k)}{1-q^k} \\
&= (\ln q)^2 \sum_{k=1}^{\infty} kq^{kx}.
\end{aligned}$$

Lemma 3 is thus proved. □

REMARK 1. In [4, p. 1245, Theorem 4.4], the identity

$$\psi_q^{(k-1)}(x+1)-\psi_q^{(k-1)}(x)=-\frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}}\left(\frac{q^x}{1-q^x}\right)\ln q \tag{16}$$

for $x\in(0,\infty)$ and $k\in\mathbb{N}$ was deduced. It is not difficult to see that the identity (15) is a special case of (16).

LEMMA 4. *For $0<q<1$ and $i\in\mathbb{N}$, the limit*

$$\lim_{x\to\infty}[f_q(x)]^{(i-1)}=0 \tag{17}$$

is valid, where $f_q(x)$ is defined by (10).

Proof. It is apparent that $\lim_{x\to\infty} f_q(x)=0$.
Differentiating and making use of (12) and (13) result in

$$\begin{aligned}[f_q(x)]^{(i)}&=\psi_q^{(i+1)}(x)-\left(\frac{1-q}{1-q^x}\right)^{(i)}-\left[\frac{(1-q)^2}{2(1-q^x)^2}\right]^{(i)}\\&=\psi_q^{(i+1)}(x)-(1-q)\left(\sum_{\ell=0}^{\infty}q^{x\ell}\right)^{(i)}-\frac{(1-q)^2}{2}\left[\sum_{\ell=0}^{\infty}(\ell+1)q^{x\ell}\right]^{(i)}\\&=\psi_q^{(i+1)}(x)-(1-q)(\ln q)^i\sum_{\ell=1}^{\infty}\ell^i q^{x\ell}-\frac{(1-q)^2}{2}(\ln q)^i\sum_{\ell=1}^{\infty}(\ell+1)\ell^i q^{x\ell}\\&\to 0\end{aligned}$$

as $x\to\infty$ for $0<q<1$. The proof of Lemma 4 is complete. □

LEMMA 5. *The function*

$$h(t)=(\ln t)^2+t(t-1)(t-2)\ln t+\frac{1}{2}(t-1)^3 \tag{18}$$

is completely monotonic on $(0,1]$.

Proof. A straightforward computation yields

$$\begin{aligned}\left[(\ln t)^2\right]^{(i)}&=\frac{(-1)^{i-1}2(i-1)!\ln t}{t^i}+\sum_{k=1}^{i-1}\frac{(-1)^i i!}{k(i-k)}\frac{1}{t^i}\\&=\frac{(-1)^{i-1}2(i-1)!\ln t}{t^i}+\frac{(-1)^i 2(i-1)!}{t^i}\sum_{k=1}^{i-1}\frac{1}{k}\\&=\frac{(-1)^i 2(i-1)!}{t^i}\left[\sum_{k=1}^{i-1}\frac{1}{k}-\ln t\right],\end{aligned}$$

$$
\begin{aligned}
[t(t-1)(t-2)\ln t]' &= t^2-3t+2+\left(3t^2-6t+2\right)\ln t,\\
[t(t-1)(t-2)\ln t]'' &= 6(t-1)\ln t+5t+\frac{2}{t}-9,\\
[t(t-1)(t-2)\ln t]^{(3)} &= 11-\frac{2}{t^2}-\frac{6}{t}+6\ln t,\\
[t(t-1)(t-2)\ln t]^{(i+3)} &= \frac{(-1)^{i+1}2(i+1)!}{t^{i+2}}+\frac{(-1)^{i+1}6i!}{t^{i+1}}+\frac{(-1)^{i-1}6(i-1)!}{t^{i}}\\
&= \frac{(-1)^{i+1}2(i-1)![i(i+1)+3it+3t^2]}{t^{i+2}}.
\end{aligned}
$$

Accordingly,

$$
\begin{aligned}
h'(t) &= \frac{5t^2}{2}-6t+\frac{7}{2}+\left(3t^2-6t+\frac{2}{t}+2\right)\ln t,\\
h''(t) &= \frac{8t^3-12t^2+2t+2+2\left(3t^3-3t^2-1\right)\ln t}{t^2},\\
h^{(3)}(t) &= \frac{14t^3-6t^2-2t-6+\left(6t^3+4\right)\ln t}{t^3},\\
h^{(i+3)}(t) &= \frac{(-1)^{i+1}2(i+2)!}{t^{i+3}}\left[\sum_{k=1}^{i+2}\frac{1}{k}-\ln t\right]+\frac{(-1)^{i+1}2(i-1)![i(i+1)+3it+3t^2]}{t^{i+2}}\\
&= \frac{(-1)^{i+1}2(i+2)!}{t^{i+3}}\left[\sum_{k=1}^{i+2}\frac{1}{k}-\ln t+\frac{i(i+1)t+3it^2+3t^3}{i(i+1)(i+2)}\right]
\end{aligned}
$$

for $i\in\mathbb{N}$. It is clear that

$$
(-1)^{i+3}h^{(i+3)}(t)=\frac{2(i+2)!}{t^{i+3}}\left[\sum_{k=1}^{i+2}\frac{1}{k}-\ln t+\frac{i(i+1)t+3it^2+3t^3}{i(i+1)(i+2)}\right]>0 \tag{19}
$$

on the interval $(0,1]$ for $i\in\mathbb{N}$. This implies that $h^{(3)}(t)$ is strictly increasing on $(0,1]$. From $h^{(3)}(1)=h''(1)=h'(1)=h(1)=0$, we obtain $h^{(3)}(t)\leqslant 0$, $h''(t)\geqslant 0$, $h'(t)\leqslant 0$, and $h(t)\geqslant 0$ on $(0,1]$. In conclusion, the function $h(t)$ is completely monotonic on $(0,1]$. Lemma 5 is proved. □

LEMMA 6. *The function*

$$
p(t)=(\ln t)^2+(t-2)(t-1)^2 \tag{20}
$$

is completely monotonic on $(0,1]$.

Proof. Direct differentiation gives

$$
p'(t)=5-8t+3t^2+\frac{2\ln t}{t},
$$

$$
\begin{aligned}
p''(t) &= \frac{2}{t^2} + 6t - 8 - \frac{2\ln t}{t^2}, \\
p^{(3)}(t) &= 6 - \frac{6}{t^3} + \frac{4\ln t}{t^3}, \\
p^{(i+3)}(t) &= \left[(\ln t)^2\right]^{(i+3)} = \frac{(-1)^{i+1} 2(i+2)!}{t^{i+3}} \left[\sum_{k=1}^{i+2} \frac{1}{k} - \ln t\right]
\end{aligned}
$$

for $i \in \mathbb{N}$. For $t \in (0,1]$, it is obvious that

$$
(-1)^{i+3} p^{(i+3)}(t) = \frac{2(i+2)!}{t^{i+3}} \left[\sum_{k=1}^{i+2} \frac{1}{k} - \ln t\right] > 0, \quad i \in \mathbb{N}.
$$

This implies that $p^{(3)}(t)$ is strictly increasing on $(0,1]$. From $p^{(3)}(1) = p''(1) = p'(1) = p(1) = 0$, it is derived that $p^{(3)}(t) \leqslant 0$, $p''(t) \geqslant 0$, $p'(t) \leqslant 0$, and $p(t) \geqslant 0$ on $(0,1]$. In a word, the function $p(t)$ is completely monotonic on $(0,1]$. The proof of Lemma 6 is complete. □

3. Proof of Theorem 1

Now it is time to supply a proof of Theorem 1.

Direct calculation and utilization of Lemmas 2 and 3 yield

$$
\begin{aligned}
f_q(x) - f_q(x+1) &= \psi_q'(x) - \psi_q'(x+1) - \frac{1-q}{1-q^x} - \frac{(1-q)^2}{2(1-q^x)^2} + \frac{1-q}{1-q^{x+1}} + \frac{(1-q)^2}{2(1-q^{x+1})^2} \\
&= (\ln q)^2 \sum_{k=1}^{\infty} k q^{kx} + (1-q)\left(\frac{1}{1-q^{x+1}} - \frac{1}{1-q^x}\right) \\
&\quad + \frac{1}{2}(1-q)^2 \left[\frac{1}{(1-q^{x+1})^2} - \frac{1}{(1-q^x)^2}\right] \\
&= (\ln q)^2 \sum_{k=1}^{\infty} k q^{kx} + (1-q)\left[\sum_{k=0}^{\infty} q^{k(x+1)} - \sum_{k=0}^{\infty} q^{kx}\right] \\
&\quad + \frac{1}{2}(1-q)^2 \left[\sum_{k=0}^{\infty} (k+1) q^{k(x+1)} - \sum_{k=0}^{\infty} (k+1) q^{kx}\right] \\
&= (\ln q)^2 \sum_{k=1}^{\infty} k q^{kx} + (1-q) \sum_{k=0}^{\infty} \left(q^k - 1\right) q^{kx} \\
&\quad + \frac{1}{2}(1-q)^2 \sum_{k=0}^{\infty} (k+1)\left(q^k - 1\right) q^{kx} \\
&= \sum_{k=1}^{\infty} \left\{\frac{1}{2}(1-q)[(1-q)(k+1)+2]\left(q^k - 1\right) + (\ln q)^2 k\right\} q^{kx}.
\end{aligned}
$$

Let

$$
g_q(t) = \frac{1}{2}(1-q)[(1-q)(t+1)+2]\left(q^t - 1\right) + (\ln q)^2 t
$$

for $0 < q < 1$ and $t \in (0, \infty)$. Then

$$\begin{aligned}
g_q'(t) &= (\ln q)^2 + \frac{1}{2}(\ln q)(q-1)q^t[t(q-1)+q-3] + \frac{1}{2}(q^t-1)(1-q)^2, \\
g_q''(t) &= \frac{1}{2}(q-1)q^t(\ln q)[(q-1)t\ln q + 2q + q\ln q - 3\ln q - 2] \\
&\triangleq \frac{1}{2}(q-1)q^t(\ln q)\varphi(t,q), \\
\varphi(1,q) &= 2[q-1+(q-2)\ln q], \\
\frac{\mathrm{d}\varphi(1,q)}{\mathrm{d}q} &= 2\left[2\left(1-\frac{1}{q}\right)+\ln q\right] \\
&< 0.
\end{aligned}$$

Since $\varphi(1,q)$ is decreasing with respect to $q \in (0,1)$ and $\varphi(1,1) = 0$, so $\varphi(1,q) > 0$ for $q \in (0,1)$. It is obvious that $\varphi(t,q)$ is increasing with respect to t, so $\varphi(t,q) > 0$ for $(t,q) \in [1,\infty) \times (0,1)$. Hence, the second derivative $g_q''(t)$ is positive for $(t,q) \in [1,\infty) \times (0,1)$ and $g_q'(t)$ is increasing with respect to $t \in [1,\infty)$. From Lemma 5, we have

$$g_q'(1) = (\ln q)^2 + q\left(q^2 - 3q + 2\right)\ln q + \frac{1}{2}(q-1)^3 > 0,$$

hence $g_q'(t) > 0$ for $(t,q) \in (1,\infty) \times (0,1)$, equivalently, the function $g_q(t)$ for $0 < q < 1$ is increasing with respect to $t \in [1,\infty)$. By virtue of Lemma 6, we have

$$g_q(1) = (q-2)(q-1)^2 + (\ln q)^2 > 0$$

for $q \in (0,1)$. Thus, the function $g_q(t)$ is positive for $(t,q) \in [1,\infty) \times (0,1)$. As a result,

$$\frac{\mathrm{d}^{i-1}[f_q(x) - f_q(x+1)]}{\mathrm{d}x^{i-1}} = \sum_{k=1}^{\infty} k^{i-1} g_q(k) q^{kx} (\ln q)^{i-1}$$

for $i \in \mathbb{N}$. This means that

$$(-1)^{i-1}[f_q(x) - f_q(x+1)]^{(i-1)} = \sum_{k=1}^{\infty} k^{i-1} g_q(k) q^{kx}\left[(-1)^{i-1}(\ln q)^{i-1}\right] > 0$$

which can be rearranged as

$$(-1)^{i-1}[f_q(x)]^{(i-1)} > (-1)^{i-1}[f_q(x+1)]^{(i-1)}.$$

By induction and Lemma 4, it follows that

$$\begin{aligned}
(-1)^{i-1}[f_q(x)]^{(i-1)} &> (-1)^{i-1}[f_q(x+1)]^{(i-1)} > (-1)^{i-1}[f_q(x+2)]^{(i-1)} \\
&> \cdots > (-1)^{i-1}[f_q(x+k)]^{(i-1)} \geqslant (-1)^{i-1}\lim_{k\to\infty}[f_q(x+k)]^{(i-1)} = 0
\end{aligned}$$

for $(i,k) \in \mathbb{N}^2$. So the function $f_q(x)$ for $0 < q < 1$ is completely monotonic on $(0,\infty)$. The proof of Theorem 1 is complete.

REFERENCES

[1] G. D. ANDERSON, R. W. BARNARD, K. C. RICHARDS, M. K. VAMANAMURTHY, AND M. VUORINEN, *Inequalities for zero-balanced hypergeometric functions*, Trans. Amer. Math. Soc. **347** (1995), no. 5, 1713–1723; Available online at `http://dx.doi.org/10.1090/S0002-9947-1995-1264800-3`.

[2] G. E. ANDREWS, R. A. ASKEY, AND R. ROY, *Special Functions*, Cambridge University Press, Cambridge, 1999.

[3] C.-P. CHEN AND F. QI, *Logarithmically completely monotonic functions relating to the gamma function*, J. Math. Anal. Appl. **321** (2006), no. 1, 405–411; Available online at `http://dx.doi.org/10.1016/j.jmaa.2005.08.056`.

[4] B.-N. GUO AND F. QI, *Properties and applications of a function involving exponential functions*, Commun. Pure Appl. Anal. **8** (2009), no. 4, 1231–1249; Available online at `http://dx.doi.org/10.3934/cpaa.2009.8.1231`.

[5] M. E. H. ISMAIL, L. LORCH, AND M. E. MULDOON, *Completely monotonic functions associated with the gamma function and its q-analogues*, J. Math. Anal. Appl. **116** (1986), 1–9; Available online at `http://dx.doi.org/10.1016/0022-247X(86)90042-9`.

[6] M. E. H. ISMAIL AND M. E. MULDOON, *Inequalities and monotonicity properties for gamma and q-gamma functions*, in: R.V.M. Zahar (Ed.), Approximation and Computation: A Festschrift in Honour of Walter Gautschi, ISNM, Vol. **119**, BirkhRauser, Basel, 1994, 309–323.

[7] M. E. H. ISMAIL AND M. E. MULDOON, *Inequalities and monotonicity properties for gamma and q-gamma functions*, available online at `http://arxiv.org/abs/1301.1749`.

[8] D. S. MITRINOVIĆ, J. E. PEČARIĆ AND A. M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht-Boston-London, 1993.

[9] M. E. MULDOON, *Some monotonicity properties and characterizations of the gamma function*, Aequationes Math. **18** (1978), 54–63; Available online at `http://dx.doi.org/10.1007/BF01844067`.

[10] F. QI, *A completely monotonic function involving the divided difference of the psi function and an equivalent inequality involving sums*, ANZIAM J. **48** (2007), no. 4, 523–532; Available online at `http://dx.doi.org/10.1017/S1446181100003199`.

[11] F. QI, *A completely monotonic function related to the q-trigamma function*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **75** (2013), in press.

[12] F. QI, *Bounds for the ratio of two gamma functions*, J. Inequal. Appl. **2010** (2010), Article ID 493058, 84 pages; Available online at `http://dx.doi.org/10.1155/2010/493058`.

[13] F. QI, *Three classes of logarithmically completely monotonic functions involving gamma and psi functions*, Integral Transforms Spec. Funct. **18** (2007), no. 7, 503–509; Available online at `http://dx.doi.org/10.1080/10652460701358976`.

[14] F. QI AND B.-N. GUO, *Completely monotonic functions involving divided differences of the di- and tri-gamma functions and some applications*, Commun. Pure Appl. Anal. **8** (2009), no. 6, 1975–1989; Available online at `http://dx.doi.org/10.3934/cpaa.2009.8.1975`.

[15] F. QI AND B.-N. GUO, *Necessary and sufficient conditions for functions involving the tri- and tetra-gamma functions to be completely monotonic*, Adv. Appl. Math. **44** (2010), no. 1, 71–83; Available online at `http://dx.doi.org/10.1016/j.aam.2009.03.003`.

[16] F. QI AND B.-N. GUO, *Some completely monotonic functions involving the q-tri- and -tetra-gamma functions and applications*, Available online at `http://arxiv.org/abs/1301.0155`.

[17] D. V. WIDDER, *The Laplace Transform*, Princeton University Press, Princeton, 1946.

(Received November 23, 2013)

Jiao-Lian Zhao
Department of Mathematics, East China Normal University
Shanghai City, 200241, China
and
Department of Mathematics and Informatics
Weinan Teachers University
Weinan City, Shaanxi Province, 714000, China
e-mail: `zhaojl2004@gmail.com, darren2004@126.com`