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OPTIMAL INEQUALITIES FOR THE CONVEX COMBINATION OF ERROR FUNCTION

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Abstract. For $\lambda \in (0,1)$ and $x,y>0$ we obtain the best possible constants p and r, such that

$$\operatorname{erf}(M_p(x,y;\lambda)) \leqslant \lambda \operatorname{erf}(x)+(1-\lambda)\operatorname{erf}(y) \leqslant \operatorname{erf}(M_r(x,y;\lambda))$$

where $\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$ and $M_p(x,y;\lambda)=(\lambda x^p+(1-\lambda)y^p)^{1/p}(p\neq 0)$, $M_0(x,y;\lambda)=x^{\lambda}y^{1-\lambda}$ are error function and weighted power mean, respectively. Furthermore, using these results, we generalized and complement an inequality due to Alzer.

1. Introduction

For $x\in R$, the error function $\operatorname{erf}(x)$ is defined as

$$\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt.$$

This function, also known as probability integral, has numbers applications in statistics, probability theory, and partial differential equations. It's well-known that the error function is odd, strictly increasing on $(-\infty,+\infty)$, and strictly concave on $[0,+\infty)$ with $\lim_{x\to+\infty}\operatorname{erf}(x)=1$. For the $n-$th derivation we have the representation

$$\frac{d^n}{dx^n}\operatorname{erf}(x)=(-1)^{n-1}\frac{2}{\sqrt{\pi}}e^{-x^2}H_{n-1}(x),$$

where $H_n(x)=(-1)^n e^{x^2}\frac{d^n}{dx^n}(e^{-x^2})$ is a Hermite polynomial.

The error function can be expanded as a power series in the following two ways [35]:

$$\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!(2n+1)}x^{2n+1}=e^{-x^2}\sum_{n=0}^{+\infty}\frac{1}{\Gamma(n+\frac{3}{2})}x^{2n+1}.$$

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It also can be expressed in terms of incomplete gamma function and a confluent hypergeometric function:

$$\operatorname{erf}(x)=\frac{\operatorname{sgn}(x)}{\sqrt{\pi}}\gamma\left(\frac{1}{2},x^2\right)=\frac{2x}{\sqrt{\pi}}{}_1F_1\left(\frac{1}{2};\frac{3}{2};-x^2\right).$$

In the recently past, the error function have been the subject of intensive research. In particular, many properties and inequalities for error function can be found in the literature [1, 7, 12, 18, 22, 23, 25, 30, 31, 33, 34, 37, 39, 40, 41]. In [4, 16, 19, 21, 27], the authors study the properties of complementary error function. The expressions in series, rational chebyshev approximates and derivation properties of inverse error function are given in [5, 6, 9, 10, 11, 20]. Rational approximates for error function can be found in [8, 15, 17, 26, 42]. In [24, 28, 36, 38] the authors concerned with the computation of complex error function. It might be surprising that the error function has application in heat conduction [13, 29].

In [14], Chu obtained the following sharp inequalities:

$$\sqrt{1-e^{-ax^2}}\leqslant \operatorname{erf}(x)\leqslant\sqrt{1-e^{-bx^2}}$$

hold for all $x\geqslant 0$ with the best possible constants $a=1$ and $b=\frac{4}{\pi}$.

Mitrinović [32] proved the elegant inequality:

$$\operatorname{erf}(x)+\operatorname{erf}(y)\leqslant\operatorname{erf}(x+y)+\operatorname{erf}(x)\operatorname{erf}(y)$$

holds for all $x,y>0$.

The following two best possible inequalities were obtained by Alzer [2]:

$$\operatorname{erf}(1)<\frac{\operatorname{erf}(x+\operatorname{erf}(y))}{\operatorname{erf}(y+\operatorname{erf}(x))}<\frac{2}{\sqrt{\pi}}$$

and

$$0<\frac{\operatorname{erf}(x\operatorname{erf}(y))}{\operatorname{erf}(y\operatorname{erf}(x))}\leqslant 1.$$

For $\lambda\in(0,1)$, we denote $A(x,y;\lambda)=\lambda x+(1-\lambda)y$, $G(x,y;\lambda)=x^{\lambda}y^{1-\lambda}$, $H(x,y;\lambda)=\frac{xy}{\lambda y+(1-\lambda)x}$ and $M_r(x,y;\lambda)=(\lambda x^r+(1-\lambda)y^r)^{1/r}(r\neq 0)$, $M_0(x,y;\lambda)=x^{\lambda}y^{1-\lambda}$ are weighted arithmetic mean, weighted geometric mean, weighted harmonic mean and weighted power mean of two positive numbers x and y with $x\neq y$. It is well-known that

$$H(x,y;\lambda)=M_{-1}(x,y;\lambda)<G(x,y;\lambda)=M_0(x,y;\lambda)<A(x,y;\lambda)=M_1(x,y;\lambda).$$

Very recently, Alzer proved the following Theorem 1.1 in [3].

THEOREM 1.1. *Let* $\lambda\in(0,\frac{1}{2})$ *be a real number, then*

$$c_1(\lambda)\operatorname{erf}(H(x,y;\lambda))\leqslant\lambda\operatorname{erf}(x)+(1-\lambda)\operatorname{erf}(y)\leqslant c_2(\lambda)\operatorname{erf}(H(x,y;\lambda)) \tag{1.1}$$

hold for all $x\geqslant 1$ *and* $y\geqslant 1$ *with the best possible factors*

$$c_1(\lambda)=\frac{\lambda+(1-\lambda)\operatorname{erf}(1)}{\operatorname{erf}(1/(1-\lambda))}\quad\textit{and}\quad c_2(\lambda)=1.$$

It is natural to ask that if (1.1) holds for $0 < x, y < 1$. Moreover we ask: what are the best possible constants p and r such that the inequalities

$$\operatorname{erf}(M_p(x,y;\lambda)) \leqslant \lambda \operatorname{erf}(x) + (1-\lambda)\operatorname{erf}(y) \leqslant \operatorname{erf}(M_r(x,y;\lambda))$$

hold for all $x, y \geqslant 1$ (or $0 < x, y < 1$). In what follows, we answer those questions.

2. Lemmas

In the section we present some lemmas, which will used in the proof of our main results.

LEMMA 2.1. *Let $r \neq 0$ and $w(x) = \operatorname{erf}(x^{\frac{1}{r}})$, one has*
(1) If $r \leqslant -1$, then $w(x)$ is strictly convex on $[1, +\infty)$;
(2) If $-1 < r < 0$, then $w(x)$ is strictly concave on $(0, 1]$;
(3) If $0 < r < 1$, then $w(x)$ is strictly concave on $[1, +\infty)$;
(4) If $r \geqslant 1$, then $w(x)$ is strictly concave on $(0, +\infty)$.

Proof. Elementary computation leads to

$$w'(x) = \frac{2}{\sqrt{\pi}} \frac{1}{r} x^{\frac{1}{r}-1} e^{-x^{\frac{2}{r}}} \tag{2.1}$$

and

$$w''(x) = \frac{2}{\sqrt{\pi}} \frac{1}{r^2} x^{\frac{1}{r}-2} e^{-x^{\frac{2}{r}}} [1 - r - 2x^{\frac{2}{r}}]. \tag{2.2}$$

Therefore, Lemma 2.1 follows from (2.2). □

LEMMA 2.2. *Let $u(x) = \operatorname{erf}(e^x)$, then $u(x)$ is strictly concave on $[0, +\infty)$.*

Proof. Simple computation yields

$$u'(x) = \frac{2}{\sqrt{\pi}} e^{x - e^{2x}} > 0 \tag{2.3}$$

and

$$u''(x) = \frac{2}{\sqrt{\pi}} (1 - 2e^{2x}) e^{x - e^{2x}} < 0 \tag{2.4}$$

for $x \geqslant 0$.

Therefore, (2.4) leads to $u(x)$ is strictly concave on $[0, +\infty)$. □

LEMMA 2.3. *Let $0 < \lambda < 1$, $r \geqslant -1 (r \neq 0)$ and $\psi(x) = x^{r-1}(\lambda x^r + 1 - \lambda)^{\frac{1}{r}-1}$ $\times e^{x^2 - (\lambda x^r + 1 - \lambda)^{\frac{2}{r}}}$, then $\psi(x)$ is strictly increasing in $[1, +\infty)$.*

Proof. By logarithmic differentiation,

$$\frac{\psi'(x)}{\psi(x)} = \frac{1}{x(\lambda x^r + 1 - \lambda)} \psi_1(x) \tag{2.5}$$

where $\psi_1(x) = (r-1)(1-\lambda) + 2x^2(\lambda x^r + 1 - \lambda) - 2\lambda x^r(\lambda x^r + 1 - \lambda)^{\frac{2}{r}}$.

Case 1. If $-1 \leqslant r < 0$, Let

$$\psi_{11}(x) = (r-1)(1-\lambda) + 2(1-\lambda)x^2(\lambda x^r + 1 - \lambda)$$

and

$$\psi_{12}(x) = 2\lambda x^2(\lambda x^r + 1 - \lambda) - 2\lambda x^r(\lambda x^r + 1 - \lambda)^{\frac{2}{r}},$$

then

$$\psi_1(x) = \psi_{11}(x) + \psi_{12}(x). \tag{2.6}$$

Since

$$\psi_{11}(1) = (1-\lambda)(1+r) \geqslant 0, \tag{2.7}$$

$$\psi_{11}'(x) = 2(1-\lambda)x[\lambda(2+r)x^r + 2(1-\lambda)] > 0 \tag{2.8}$$

and

$$\psi_{12}(x) = 2\lambda x^2(\lambda x^r + 1 - \lambda)[1 - (\lambda + (1-\lambda)x^{-r})^{\frac{2-r}{r}}] > 0 \tag{2.9}$$

for $x \geqslant 1$.

From (2.6)–(2.9) we clearly see that $\psi_1(x) > 0$ for $x \in (1, +\infty)$ and $-1 \leqslant r < 0$. Therefore, $\psi(x)$ is strictly increasing in $[1, +\infty)$ for $-1 \leqslant r < 0$.

Case 2. If $0 < r < 2$, then (2.7)–(2.9) hold again, so, $\psi(x)$ is strictly increasing in $[1, +\infty)$ for $0 < r < 2$.

Case 3. If $r \geqslant 2$, we let $\psi_2(x) = \log[2x^2(\lambda x^r + 1 - \lambda)] - \log[2\lambda x^r(\lambda x^r + 1 - \lambda)^{\frac{2}{r}}]$. Then

$$\lim_{x \to +\infty} \psi_2(x) = -\frac{2}{r} \log \lambda > 0 \tag{2.10}$$

and

$$\psi_2'(x) = \frac{(2-r)(1-\lambda)}{x(\lambda x^r + 1 - \lambda)} \leqslant 0. \tag{2.11}$$

It follows from (2.10) and (2.11) that $\psi_2(x) > 0$ for all $x \in [1, +\infty)$ and $r \geqslant 2$. Hence, (2.5) lead to $\psi(x)$ is strictly increasing in $[1, +\infty)$ for $r \geqslant 2$. □

LEMMA 2.4. *For $0 < \lambda < 1$, $r \geqslant -1 (r \neq 0)$ and $x \geqslant 1$, we have*

$$c_1(\lambda, r) \leqslant \frac{\lambda \operatorname{erf}(x) + (1-\lambda)\operatorname{erf}(1)}{\operatorname{erf}((\lambda x^r + 1 - \lambda)^{\frac{1}{r}})} \tag{2.12}$$

and

$$c_1(\lambda, r) \leqslant \frac{\lambda \operatorname{erf}(1) + (1-\lambda)\operatorname{erf}(x)}{\operatorname{erf}((\lambda + (1-\lambda)x^r)^{\frac{1}{r}})} \tag{2.13}$$

where

$$c_1(\lambda,r)=\begin{cases}\frac{\lambda+(1-\lambda)\operatorname{erf}(1)}{\operatorname{erf}((1-\lambda)^{\frac{1}{r}})}, & -1\leqslant r<0,\\ \lambda+(1-\lambda)\operatorname{erf}(1), & r>0.\end{cases}$$

Proof. It is not difficult to verify that $0<c_1(\lambda,r)<1$ for $0<\lambda<1$ and $r\geqslant -1$. Since the proof of (2.13) is similarly with (2.12), so we only prove (2.12).

Firstly, we prove that

$$\frac{\lambda+(1-\lambda)\operatorname{erf}(1)}{\operatorname{erf}((1-\lambda)^{\frac{1}{r}})}\leqslant\frac{\lambda\operatorname{erf}(x)+(1-\lambda)\operatorname{erf}(1)}{\operatorname{erf}((\lambda x^r+1-\lambda)^{\frac{1}{r}})}$$

holds for $-1\leqslant r<0$ and $x\geqslant 1$.

Let $G(x)=\operatorname{erf}((1-\lambda)^{\frac{1}{r}})[\lambda\operatorname{erf}(x)+(1-\lambda)\operatorname{erf}(1)]-[\lambda+(1-\lambda)\operatorname{erf}(1)]\operatorname{erf}((\lambda x^r+1-\lambda)^{\frac{1}{r}})$ and $G_1(x)=\frac{\sqrt{\pi}}{2\lambda}e^{x^2}G'(x)$, then one has

$$G(1)=[\operatorname{erf}((1-\lambda)^{\frac{1}{r}})-(\lambda+(1-\lambda)\operatorname{erf}(1))]\operatorname{erf}(1)>0, \tag{2.14}$$

$$\lim_{x\to+\infty}G(x)=0, \tag{2.15}$$

$$G_1(x)=\operatorname{erf}((1-\lambda)^{\frac{1}{r}})-[\lambda+(1-\lambda)\operatorname{erf}(1)]x^{r-1}(\lambda x^r+1-\lambda)^{\frac{1}{r}-1}e^{x^2-(\lambda x^r+1-\lambda)^{\frac{2}{r}}}, \tag{2.16}$$

$$G_1(1)=\operatorname{erf}((1-\lambda)^{\frac{1}{r}})-[\lambda+(1-\lambda)\operatorname{erf}(1)]>0 \tag{2.17}$$

and

$$\lim_{x\to+\infty}G_1(x)=-\infty. \tag{2.18}$$

Therefore, Lemma 2.3 and (2.16) imply that $G_1(x)$ is strictly decreasing in $[1,+\infty)$, thus from (2.17) and (2.18) we conclude that there exists $x_1\in(1,+\infty)$, such that $G_1(x)>0$ for $x\in(1,x_1)$ and $G_1(x)<0$ for $x\in(x_1,+\infty)$. So, $G(x)$ is strictly increasing in $[1,x_1]$ and strictly decreasing in $[x_1,+\infty)$.

It follows from (2.14) and (2.15) together with the piecewise monotonicity of $G(x)$ that $G(x)>0$ for $x\in[1,+\infty)$ and $-1\leqslant r<0$.

Next, we prove that

$$\lambda+(1-\lambda)\operatorname{erf}(1)\leqslant\frac{\lambda\operatorname{erf}(x)+(1-\lambda)\operatorname{erf}(1)}{\operatorname{erf}((\lambda x^r+1-\lambda)^{\frac{1}{r}})}$$

holds for $x\geqslant 1$ and $r>0$.

Let $H(x)=\lambda\operatorname{erf}(x)+(1-\lambda)\operatorname{erf}(1)-[\lambda+(1-\lambda)\operatorname{erf}(1)]\operatorname{erf}((\lambda x^r+1-\lambda)^{\frac{1}{r}})$ and $H_1(x)=\frac{\sqrt{\pi}}{2\lambda}e^{x^2}H'(x)$, then we have

$$H(1)=(1-\lambda)(1-\operatorname{erf}(1))\operatorname{erf}(1)>0, \tag{2.19}$$

$$\lim_{x\to+\infty}H(x)=0, \tag{2.20}$$

$$H_1(x) = 1 - [\lambda + (1-\lambda)\,\mathrm{erf}(1)]x^{r-1}(\lambda x^r + 1 - \lambda)^{\frac{1}{r}-1}e^{x^2-(\lambda x^r+1-\lambda)^{\frac{2}{r}}}, \tag{2.21}$$

$$H_1(1) = (1-\lambda)(1-\mathrm{erf}(1)) > 0 \tag{2.22}$$

and

$$\lim_{x\to+\infty} H_1(x) = -\infty. \tag{2.23}$$

Hence, Lemma 2.3 and (2.21) imply that $H_1(x)$ is strictly decreasing in $[1,+\infty)$. It follows from the monotonicity of $H_1(x)$ and (2.22) together with (2.23) that there exists $x_2 \in (1,+\infty)$, such that $H_1(x) > 0$ for $x \in (1,x_2)$ and $H_1(x) < 0$ for $x \in (x_2,+\infty)$. Therefore, $H(x)$ is strictly increasing in $[1,x_2]$ and strictly decreasing in $[x_2,+\infty)$.

From the piecewise monotonicity of $H(x)$ and (2.19) together with (2.20) we clearly see that $H(x) > 0$ for $x \in [1,+\infty)$ and $r > 0$. $\square$

LEMMA 2.5. *For $0 < \lambda < 1$ and $x \geqslant 1$, we have*

$$\lambda + (1-\lambda)\,\mathrm{erf}(1) \leqslant \frac{\lambda\,\mathrm{erf}(x) + (1-\lambda)\,\mathrm{erf}(1)}{\mathrm{erf}(x^\lambda)} \tag{2.24}$$

and

$$\lambda + (1-\lambda)\,\mathrm{erf}(1) \leqslant \frac{\lambda\,\mathrm{erf}(1) + (1-\lambda)\,\mathrm{erf}(x)}{\mathrm{erf}(x^{1-\lambda})}. \tag{2.25}$$

Proof. Let $E(x) = \lambda\,\mathrm{erf}(x) + (1-\lambda)\,\mathrm{erf}(1) - [\lambda + (1-\lambda)\,\mathrm{erf}(1)]\,\mathrm{erf}(x^\lambda)$, $E_1(x) = \frac{\sqrt{\pi}}{2\lambda}e^{x^2}E'(x)$ and $E_2(x) = \frac{x^{2-\lambda}e^{x^{2\lambda}-x^2}}{\lambda+(1-\lambda)\,\mathrm{erf}(1)}E_1'(x)$, then simple computation leads to

$$E(1) = (1-\lambda)(1-\mathrm{erf}(1))\,\mathrm{erf}(1) > 0, \tag{2.26}$$

$$\lim_{x\to+\infty} E(x) = 0, \tag{2.27}$$

$$E_1(x) = 1 - [\lambda + (1-\lambda)\,\mathrm{erf}(1)]x^{\lambda-1}e^{-x^{2\lambda}+x^2},$$

$$E_1(1) = (1-\lambda)(1-\mathrm{erf}(1)) > 0, \tag{2.28}$$

$$\lim_{x\to+\infty} E_1(x) = -\infty, \tag{2.29}$$

$$E_2(x) = 1 - \lambda + 2\lambda x^{2\lambda} - 2x^2,$$

$$E_2(1) = \lambda - 1 < 0 \tag{2.30}$$

and

$$E_2'(x) = 4x(\lambda^2 x^{2\lambda-2} - 1) < 0 \tag{2.31}$$

for $x \geqslant 1$.

Therefore, Inequalities (2.31) and (2.30) imply that $E_1(x)$ is strictly decreasing in $[1,+\infty)$.

From the monotonicity of $E_1(x)$ and (2.28) together with (2.29) we clearly see that there exists $x_3 \in (1,+\infty)$, such that $E_1(x) > 0$ for $x \in (1,x_3)$ and $E_1(x) < 0$ for

$x \in (x_3, +\infty)$. Thus, $E(x)$ is strictly increasing in $[1, x_3]$ and is strictly decreasing in $[x_3, +\infty)$.

Hence, $E(x) > 0$ follows from the piecewise monotonicity of $E(x)$ and (2.26) together with (2.27).

The proof of (2.25) is similarly with (2.24), so we omit the detail. □

LEMMA 2.6. *For* $0 < \lambda < 1$, $r \geqslant 1$ *and* $0 < x < 1$, *we have*

$$\frac{\lambda \operatorname{erf}(1)}{\operatorname{erf}(\lambda^{\frac{1}{r}})} \leqslant \frac{\lambda \operatorname{erf}(x)}{\operatorname{erf}(\lambda^{\frac{1}{r}} x)} \tag{2.32}$$

and

$$\frac{\lambda \operatorname{erf}(1)}{\operatorname{erf}(\lambda^{\frac{1}{r}})} \leqslant \frac{(1-\lambda) \operatorname{erf}(x)}{\operatorname{erf}((1-\lambda)^{\frac{1}{r}} x)}. \tag{2.33}$$

Proof. We only prove (2.32). For $0 < x < 1$ and $r \geqslant 1$, let $J(x) = \lambda \operatorname{erf}(\lambda^{\frac{1}{r}}) \operatorname{erf}(x) - \lambda \operatorname{erf}(1) \operatorname{erf}(\lambda^{\frac{1}{r}} x)$, then simple computation leads to

$$J(0) = 0, \quad J(1) = 0 \tag{2.34}$$

and

$$J''(x) = -\frac{4\lambda}{\sqrt{\pi}} x e^{-x^2} \left[\operatorname{erf}(\alpha) - \alpha^3 \operatorname{erf}(1) e^{(1-\alpha^2)x^2}\right] \tag{2.35}$$

where $0 < \alpha = \lambda^{\frac{1}{r}} < 1$.

Since

$$\operatorname{erf}(\alpha) - \alpha^3 \operatorname{erf}(1) e^{(1-\alpha^2)x^2} > \operatorname{erf}(\alpha) - \alpha^3 \operatorname{erf}(1) e^{1-\alpha^2} \tag{2.36}$$

for $x \in (0,1)$.

Next, we prove that $I(\alpha) = \operatorname{erf}(\alpha) - \alpha^3 \operatorname{erf}(1) e^{1-\alpha^2} > 0$ for $\alpha \in (0,1)$.

Elementary computations yield

$$I(0) = 0, \quad I(1) = 0 \tag{2.37}$$

$$I'(\alpha) = \operatorname{erf}'(\alpha) - \operatorname{erf}(1)(3\alpha^2 - 2\alpha^4) e^{1-\alpha^2},$$

$$I'(0) = \frac{2}{\sqrt{\pi}}, \quad I'(1) = \frac{2}{e\sqrt{\pi}} - \operatorname{erf}(1) = -0.4276... < 0 \tag{2.38}$$

and

$$I''(\alpha) = \alpha e^{-\alpha^2} I_1(\alpha),$$

where

$$I_1(\alpha) = -\frac{4}{\sqrt{\pi}} - \operatorname{erf}(1) e (6 - 14\alpha^2 + 4\alpha^4), \tag{2.39}$$

$$I_1(0) = -\frac{4}{\sqrt{\pi}} - 6 \operatorname{erf}(1) e = -16.0009... < 0, \tag{2.40}$$

$$I_1(1) = -\frac{4}{\sqrt{\pi}} + 4\,\text{erf}(1)e = 6.9060... > 0. \tag{2.41}$$

It is easy to see that the function $\phi(\alpha) = 6 - 14\alpha^2 + 4\alpha^4$ is strictly decreasing in $(0,1)$, then (2.39) yields to $I_1(\alpha)$ is strictly increasing in $(0,1)$.

It follows from the monotonicity of $I_1(\alpha)$ and (2.40) together with (2.41) that there exists $\alpha_1 \in (0,1)$, such that $I_1(\alpha) < 0$ for $\alpha \in (0,\alpha_1)$ and $I_1(\alpha) > 0$ for $\alpha \in (\alpha_1,1)$. Therefore, $I'(\alpha)$ is strictly decreasing in $[0,\alpha_1]$ and strictly increasing in $[\alpha_1,1]$.

From the piecewise monotonicity of $I'(\alpha)$ and (2.38) we conclude that there exists $\alpha_2 \in (0,1)$, such that $I'(\alpha) > 0$ for $\alpha \in (0,\alpha_2)$ and $I_1(\alpha) < 0$ for $\alpha \in (\alpha_2,1)$. Hence, $I(\alpha)$ is strictly increasing in $[0,\alpha_2]$ and strictly decreasing in $[\alpha_2,1]$.

It follows from the piecewise monotonicity of $I(\alpha)$ and (2.37) that $I(\alpha) > 0$ for $\alpha \in (0,1)$.

Therefore, (2.36) and (2.35) lead to $J(x)$ is concave on $(0,1)$, from (2.34) we have $J(x) \geqslant \min\{J(0),J(1)\} = 0$. $\square$

3. Main results

THEOREM 3.1. *Let $\lambda \in (0,1)$, the double inequalities*

$$\text{erf}(M_p(x,y;\lambda)) \leqslant \lambda\,\text{erf}(x) + (1-\lambda)\,\text{erf}(y) \leqslant \text{erf}(M_r(x,y;\lambda)) \tag{3.1}$$

hold for all $x \geqslant 1$, $y \geqslant 1$ if and only if $p = -\infty$ and $r \geqslant -1$.

Proof. Firstly, we prove that if $r \geqslant -1$ and $p = -\infty$, then (3.1) hold.

The monotonicity of $\text{erf}(x)$ implies that the left-hand side of (3.1) is true with $p = -\infty$. Since the weighted power mean is increasing on R with respect with it's order, this implies that $t \to \text{erf}(M_t(x,y;\lambda))$ is increasing on R. Therefore, it is enough to prove that the right-hand side of (3.1) is valid for $r = -1$, which is followed from (1.1).

Secondly, we prove that the right-hand side of (3.1) imply that $r \geqslant -1$.

For $x \geqslant 1$ and $y \geqslant 1$, from the right-hand side of (3.1) we can let

$$K(x,y) = \text{erf}(M_r(x,y;\lambda)) - \lambda\,\text{erf}(x) - (1-\lambda)\,\text{erf}(y) \geqslant 0.$$

Then simple computation leads to

$$K(y,y) = \frac{\partial}{\partial x}K(x,y)\,|_{x=y} = 0$$

and

$$\frac{\partial^2}{\partial x^2}K(x,y)\,|_{x=y} = \lambda(1-\lambda)\frac{2}{\sqrt{\pi}}\frac{1}{y}e^{-y^2}[r-1+2y^2] \geqslant 0,$$

this leads to $r \geqslant -1$.

Thirdly, we suppose that there exists a real number p such that the left-hand side of (3.1) hold for all $x \geqslant 1$ and $y \geqslant 1$. We divide the proof into two cases.

Case A. If $p \geqslant 0$, for fixed $y \in R$ we have

$$\lim_{x\to+\infty} \operatorname{erf}(M_p(x,y;\lambda)) = 1$$

and

$$\lim_{x\to+\infty} [\lambda \operatorname{erf}(x) + (1-\lambda)\operatorname{erf}(y)] = \lambda + (1-\lambda)\operatorname{erf}(y) < 1.$$

This contradict with the left-hand side of (3.1).

Case B. If $-\infty < p < 0$, then for $x \geqslant 1$, from the left-hand side of (3.1) we let $\beta = \lambda^{\frac{1}{p}}$, $y \to +\infty$ and

$$Q(x) = \lambda \operatorname{erf}(x) + 1 - \lambda - \operatorname{erf}(\beta x) \geqslant 0. \tag{3.2}$$

Hence we get

$$\lim_{x\to+\infty} Q(x) = 0 \tag{3.3}$$

and

$$Q'(x) = \frac{2}{\sqrt{\pi}} e^{-x^2} [\lambda - \beta e^{(1-\beta)x^2}]. \tag{3.4}$$

Since $\beta > 1$, then (3.4) leads to that there exists $\eta_1 \in (1,+\infty)$, such that $Q'(x) > 0$ for $x \in (\eta_1,+\infty)$, this implies that $Q(x)$ is strictly increasing in $[\eta_1,+\infty)$.

It follows from (3.3) and the monotonicity of $Q(x)$ that there exists $\eta_2 \in (1,+\infty)$, such that $Q(x) < 0$ for $x \in (\eta_2,+\infty)$, this is contradict with (3.2). □

THEOREM 3.2. *Let $\lambda \in (0,1)$, the double inequalities*

$$\operatorname{erf}(M_\mu(x,y;\lambda)) \leqslant \lambda \operatorname{erf}(x) + (1-\lambda)\operatorname{erf}(y) \leqslant \operatorname{erf}(M_\nu(x,y;\lambda)) \tag{3.5}$$

hold for all $0 < x,y < 1$ if and only if $\mu \leqslant -1$ and $\nu \geqslant 1$.

Proof. Firstly we prove that if $\mu \leqslant -1$ and $\nu \geqslant 1$, then (3.5) is valid.

For $\mu \leqslant -1$ and $0 < x,y < 1$, we let $s = x^\mu$ and $t = y^\mu$, then $s,t > 1$. It follows from Lemma 2.1(1) that

$$w(\lambda s + (1-\lambda)t) \leqslant \lambda w(s) + (1-\lambda)w(t).$$

This leads to

$$\operatorname{erf}(M_\mu(x,y;\lambda)) \leqslant \lambda \operatorname{erf}(x) + (1-\lambda)\operatorname{erf}(y)$$

for all $0 < x,y < 1$.

For $\nu \geqslant 1$, $0 < x,y < 1$, we let $s = x^\nu$ and $t = y^\nu$, then $0 < s,t < 1$. From Lemma 2.1(4) we clearly see that

$$w(\lambda s + (1-\lambda)t) \geqslant \lambda w(s) + (1-\lambda)w(t).$$

This leads to

$$\operatorname{erf}(M_\nu(x,y;\lambda)) \geqslant \lambda \operatorname{erf}(x) + (1-\lambda)\operatorname{erf}(y)$$

for all $0 < x, y < 1$.

Secondly, we prove that the right-hand side of (3.5) implies $\nu \geqslant 1$. Let

$$T(x,y) = \operatorname{erf}(M_\nu(x,y;\lambda)) - \lambda\operatorname{erf}(x) - (1-\lambda)\operatorname{erf}(y) \geqslant 0.$$

Then

$$T(y,y) = \frac{\partial}{\partial x}T(x,y)|_{x=y} = 0$$

and

$$\frac{\partial^2}{\partial x^2}T(x,y)|_{x=y} = \lambda(1-\lambda)\frac{2}{\sqrt{\pi}}\frac{1}{y}e^{-y^2}[\nu - 1 + 2y^2] \geqslant 0. \tag{3.6}$$

Therefore, (3.6) leads to $\nu \geqslant 1$ for all $0 < x, y < 1$.

Finally, we prove that the left-hand side of (3.5) implies $\mu \leqslant -1$.

Let $y \to 1$, then from the left-hand side of (3.5) we obtain

$$L(x) = \lambda\operatorname{erf}(x) + (1-\lambda)\operatorname{erf}(1) - \operatorname{erf}(M_\mu(x,1;\lambda)) \geqslant 0 \tag{3.7}$$

for $0 < x < 1$.

By elementary computations, we get

$$L(1) = 0 \tag{3.8}$$

and

$$L'(x) = \frac{2\lambda}{\sqrt{\pi}}e^{-x^2}[1 - x^{\mu-1}(\lambda x^\mu + 1 - \lambda)^{\frac{1}{\mu}-1}e^{x^2-(\lambda x^\mu+1-\lambda)^{\frac{2}{\mu}}}]. \tag{3.9}$$

Let

$$L_1(x) = \log 1 - \log[x^{\mu-1}(\lambda x^\mu + 1 - \lambda)^{\frac{1}{\mu}-1}e^{x^2-(\lambda x^\mu+1-\lambda)^{\frac{2}{\mu}}}]. \tag{3.10}$$

Then

$$\lim_{x\to 1^-} L_1(x) = 0 \tag{3.11}$$

and

$$L_1'(x) = \frac{1}{x(\lambda x^\mu + 1 - \lambda)}L_2(x), \tag{3.12}$$

where

$$L_2(x) = (1-\mu)(1-\lambda) + 2\lambda(\lambda x^\mu + 1 - \lambda)^{\frac{2}{\mu}}x^\mu - 2x^2(\lambda x^\mu + 1 - \lambda)$$

and

$$\lim_{x\to 1^-} L_2(x) = (1-\lambda)(-1-\mu). \tag{3.13}$$

In fact, if $\mu > -1$, then by the continuity of $L_2(x)$ and (3.13) we know that there exists a small $\delta_1 > 0$ such that $L_2(x) < 0$ for $x \in (1-\delta_1, 1)$. Therefore, (3.12) leads to $L_1(x)$ is strictly decreasing in $[1-\delta_1, 1]$.

From (3.11) and the monotonicity of $L_1(x)$ in $[1-\delta_1, 1]$ we conclude that there exists a small $\delta_2 > 0$, such that $L_1(x) > 0$ for $x \in (1-\delta_2, 1)$. Hence, (3.9) and (3.10) imply that $L(x)$ is increasing in $[1-\delta_2, 1]$.

It follows from (3.8) and the monotonicity of $L(x)$ in $(1-\delta_2, 1)$ that there exists a small $\delta_3 > 0$, such that $L(x) < 0$ for $x \in (1-\delta_3, 1)$. This is contradict with (3.7). □

The following Theorem 3.3 generalized Theorem 1.1.

THEOREM 3.3. *Let* $0 < \lambda < 1$ *and* $r \geqslant -1$, *then the double inequalities*

$$c_1(\lambda, r)\,\mathrm{erf}(M_r(x,y;\lambda)) \leqslant \lambda\,\mathrm{erf}(x) + (1-\lambda)\,\mathrm{erf}(y) \leqslant c_2(\lambda, r)\,\mathrm{erf}(M_r(x,y;\lambda)) \quad (3.14)$$

hold for all $x, y \geqslant 1$, *and the factors*

$$c_1(\lambda, r) = \begin{cases} \frac{\lambda + (1-\lambda)\,\mathrm{erf}(1)}{\mathrm{erf}((1-\lambda)^{\frac{1}{r}})}, & -1 \leqslant r < 0, \\ \lambda + (1-\lambda)\,\mathrm{erf}(1), & r \geqslant 0, \end{cases} \quad \text{and} \quad c_2(\lambda, r) = 1$$

are the best possible.

Proof. The right-hand side of (3.14) follows from Theorem 3.1, so we only need to prove the left-hand side of (3.14). We divide the proof into three cases.

Case 1. $-1 \leqslant r < 0$. For $x \geqslant 1$ and $y \geqslant 1$, we let $w(z) = \mathrm{erf}(z^{\frac{1}{r}})$, $s = x^r$ and $t = y^r$, then $0 < s, t \leqslant 1$. From (2.1) and (2.2) we clearly see that $w'' < 0$ and $w' < 0$ on $(0,1)$ for $-1 \leqslant r < 0$. Therefore, $-w'$ is positive and increasing in $[0,1]$. Let

$$A_\lambda(s,t) = \lambda w(s) + (1-\lambda)w(t) - c_1(\lambda, r)w(\lambda s + (1-\lambda)t). \quad (3.15)$$

Subcase 1.1. If $0 < s \leqslant t \leqslant 1$, then $s \leqslant \lambda s + (1-\lambda)t \leqslant t$. Differentiating (3.15) leads to

$$\frac{1}{1-\lambda}\frac{\partial}{\partial t}A_\lambda(s,t) = w'(t) - c_1(\lambda, r)w'(\lambda s + (1-\lambda)t) < 0.$$

Thus

$$A_\lambda(s,t) \geqslant A_\lambda(s,1) = \lambda w(s) + (1-\lambda)w(1) - c_1(\lambda, r)w(\lambda s + 1 - \lambda). \quad (3.16)$$

Therefore, $A_\lambda(s,t) \geqslant 0$ follows from (3.16) and (2.12).

Subcase 1.2. If $0 < t \leqslant s \leqslant 1$, then $t \leqslant \lambda s + (1-\lambda)t \leqslant s$. Differentiating (3.15) yields to

$$\frac{1}{\lambda}\frac{\partial}{\partial s}A_\lambda(s,t) = w'(s) - c_1(\lambda, r)w'(\lambda s + (1-\lambda)t) < 0.$$

So

$$A_\lambda(s,t) \geqslant A_\lambda(1,t) = \lambda w(1) + (1-\lambda)w(t) - c_1(\lambda, r)w(\lambda + (1-\lambda)t). \quad (3.17)$$

Hence, $A_\lambda(s,t) \geqslant 0$ follows from (3.17) and (2.13).

Case 2. $r = 0$. For $x \geqslant 1$ and $y \geqslant 1$, we let $u(z) = \mathrm{erf}(e^z)$, $s = \log x$ and $t = \log y$, then $s, t \geqslant 0$. From (2.3) and (2.4) we know that $u'' < 0$ and $u' > 0$ on $[0, +\infty)$. Therefore, u' is positive and decreasing in $[0, +\infty)$. Let

$$B_\lambda(s,t) = \lambda u(s) + (1-\lambda)u(t) - c_1(\lambda, r)u(\lambda s + (1-\lambda)t). \quad (3.18)$$

Subcase 2.1. If $0 \leqslant s \leqslant t$, then $s \leqslant \lambda s + (1-\lambda)t \leqslant t$, (3.18) leads to

$$\frac{1}{\lambda}\frac{\partial}{\partial x}B_\lambda(s,t) = u'(s) - c_1(\lambda,r)u'(\lambda s + (1-\lambda)t) > 0.$$

This implies that

$$B_\lambda(s,t) \geqslant B_\lambda(0,t) = \lambda u(0) + (1-\lambda)u(t) - c_1(\lambda,r)u((1-\lambda)t). \tag{3.19}$$

Hence, $B_\lambda(s,t) \geqslant 0$ follows from (3.19) and (2.25).

Subcase 2.2. If $0 \leqslant t \leqslant s$, then $t \leqslant \lambda s + (1-\lambda)t \leqslant s$, (3.18) yields

$$\frac{1}{1-\lambda}\frac{\partial}{\partial t}B_\lambda(s,t) = u'(t) - c_1(\lambda,r)u'(\lambda s + (1-\lambda)t) > 0.$$

Thus, we have

$$B_\lambda(s,t) \geqslant B_\lambda(s,0) = \lambda u(s) + (1-\lambda)u(0) - c_1(\lambda,r)u(\lambda s). \tag{3.20}$$

Hence, $B_\lambda(s,t) \geqslant 0$ follows from (3.20) and (2.24).

Case 3. $r > 0$. For $x \geqslant 1$ and $y \geqslant 1$, we let $w(z) = \mathrm{erf}(z^{\frac{1}{r}})$, $s = x^r$ and $t = y^r$, then $s,t \geqslant 1$. It follows from (2.1) and (2.2) that $w'' < 0$ and $w' > 0$ in $(1,+\infty)$ for $r \geqslant 0$, therefore, w' is positive and decreasing in $[1,+\infty)$.

Subcase 3.1. If $1 \leqslant s \leqslant t$, then $s \leqslant \lambda s + (1-\lambda)t \leqslant t$, (3.15) leads to

$$\frac{1}{\lambda}\frac{\partial}{\partial s}A_\lambda(s,t) = w'(s) - c_1(\lambda,r)w'(\lambda s + (1-\lambda)t) > 0.$$

Therefore,

$$A_\lambda(s,t) \geqslant A_\lambda(1,t) = \lambda w(1) + (1-\lambda)w(t) - c_1(\lambda,r)w(\lambda + (1-\lambda)t). \tag{3.21}$$

Hence, $A_\lambda(s,t) \geqslant 0$ follows from (3.21) and (2.13).

Subcase 3.2. If $1 \leqslant t \leqslant s$, then $t \leqslant \lambda s + (1-\lambda)t \leqslant s$, from (3.15) we obtain

$$\frac{1}{1-\lambda}\frac{\partial}{\partial t}A_\lambda(s,t) = w'(t) - c_1(\lambda,r)w'(\lambda s + (1-\lambda)t) > 0.$$

Thus

$$A_\lambda(s,t) \geqslant A_\lambda(s,1) = \lambda w(s) + (1-\lambda)w(1) - c_1(\lambda,r)w(\lambda s + 1 - \lambda). \tag{3.22}$$

Therefore, $A_\lambda(s,t) \geqslant 0$ follows from (3.22) and (2.12).

The following (3.23) and (3.24) imply that $c_1(\lambda,r)$ and $c_2(\lambda,r)$ are the best possible.

$$\lim_{y\to 1}\lim_{x\to+\infty}\frac{\lambda\,\mathrm{erf}(x) + (1-\lambda)\,\mathrm{erf}(y)}{\mathrm{erf}(M_r(x,y;\lambda))} = \begin{cases} \frac{\lambda+(1-\lambda)\,\mathrm{erf}(1)}{\mathrm{erf}((1-\lambda)^{\frac{1}{r}})}, & -1 \leqslant r < 0, \\ \lambda + (1-\lambda)\,\mathrm{erf}(1), & r \geqslant 0. \end{cases} \tag{3.23}$$

and

$$\lim_{y\to x}\frac{\lambda\,\mathrm{erf}(x)+(1-\lambda)\,\mathrm{erf}(y)}{\mathrm{erf}(M_r(x,y;\lambda))}=1 \qquad (r\geqslant -1). \tag{3.24}$$

This complete the proof of Theorem 3.3. □

The following Theorem 3.4 complement of Theorem 1.1.

THEOREM 3.4. *Let* $0<\lambda<1$, $r\geqslant 1$, *then the double inequalities*

$$c_3(\lambda,r)\,\mathrm{erf}(M_r(x,y;\lambda))\leqslant\lambda\,\mathrm{erf}(x)+(1-\lambda)\,\mathrm{erf}(y)\leqslant c_4(\lambda,r)\,\mathrm{erf}(M_r(x,y;\lambda)) \tag{3.25}$$

hold for all $0<x,y<1$, *and the factors*

$$c_3(\lambda,r)=\frac{\lambda\,\mathrm{erf}(1)}{\mathrm{erf}(\lambda^{\frac{1}{r}})} \qquad \textit{and} \quad c_4(\lambda,r)=1$$

are the best possible.

Proof. The right-hand side of (3.25) follows from Theorem 3.2, so we only need to prove the left-hand side of (3.25). We let $w(z)=\mathrm{erf}(z^{\frac{1}{r}})$ and

$$D_\lambda(s,t)=\lambda w(s)+(1-\lambda)w(t)-c_3(\lambda,r)w(\lambda s+(1-\lambda)t). \tag{3.26}$$

For $r\geqslant 1$, $0<x,y<1$, let $s=x^r, t=y^r$, then $0<s,t<1$. From (2.1) and (2.2) we see that $w''<0$ and $w'>0$, thus w' is positive and decreasing in $[0,1]$.

Case 1. If $0<s\leqslant t<1$, then $s\leqslant\lambda s+(1-\lambda)t\leqslant t$. It follows from (3.26) that

$$\frac{1}{\lambda}\frac{\partial}{\partial s}D_\lambda(s,t)=w'(s)-c_3(\lambda,r)w'(\lambda s+(1-\lambda)t)>0.$$

This leads to

$$D_\lambda(s,t)>D_\lambda(0,t)=\lambda w(0)+(1-\lambda)w(t)-c_3(\lambda,r)w((1-\lambda)t). \tag{3.27}$$

Hence, $B_\lambda(s,t)>0$ follows from (3.27) and (2.33).

Case 2. If $0<t\leqslant s<1$, then $t\leqslant\lambda s+(1-\lambda)t\leqslant s$. From (3.26) we get

$$\frac{1}{1-\lambda}\frac{\partial}{\partial t}D_\lambda(s,t)=w'(t)-c_3(\lambda,r)w'(\lambda s+(1-\lambda)t)>0.$$

So

$$D_\lambda(s,t)>D_\lambda(s,0)=\lambda w(s)+(1-\lambda)w(0)-c_3(\lambda,r)w(\lambda s). \tag{3.28}$$

Hence, $D_\lambda(s,t)>0$ follows from (3.28) and (2.32).

The following (3.29) and (3.30) imply that $c_3(\lambda,r)$ and $c_4(\lambda,r)$ are the best possible.

$$\lim_{y\to 0}\lim_{x\to 1}\frac{\lambda\,\mathrm{erf}(x)+(1-\lambda)\,\mathrm{erf}(y)}{\mathrm{erf}(M_r(x,y;\lambda))}=\frac{\lambda\,\mathrm{erf}(1)}{\mathrm{erf}(\lambda^{\frac{1}{r}})} \tag{3.29}$$

and

$$\lim_{y\to x}\frac{\lambda\,\mathrm{erf}(x)+(1-\lambda)\,\mathrm{erf}(y)}{\mathrm{erf}(M_r(x,y;\lambda))}=1 \tag{3.30}$$

for $r\geqslant 1$. □

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