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IMPROVED ARITHMETIC–GEOMETRIC MEAN INEQUALITY AND ITS APPLICATION

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Abstract. In this short note, we present a refinement of the well-known arithmetic-geometric mean inequality. As application of our result, we obtain an operator inequality.

1. Introduction

For two invertible positive operators A and B, the geometric mean $A\#B$ and the relative operator entropy $S(A|B)$ are defined by

$$A\#B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{1/2}A^{1/2},$$

$$S(A|B) = A^{1/2}\log\left(A^{-1/}BA^{-1/2}\right)A^{1/2}.$$

It is know that

$$A\#B \leqslant \frac{A+B}{2}. \tag{1.1}$$

For more information on geometric mean, the relative operator entropy, and operator inequality the reader is referred to [1, 3, 4, 5, 7, 8].

The well-known arithmetic-geometric mean inequality says that if $a,b \geqslant 0$, then

$$\sqrt{ab} \leqslant \frac{a+b}{2}.$$

In this short note, we prove that if $a,b > 0$, then

$$\left(1+\frac{(\log a-\log b)^2}{8}\right)\sqrt{ab} \leqslant \frac{a+b}{2}, \tag{1.2}$$

which is a refinement of the arithmetic-geometric mean inequality. As an application of inequality (1.2), we present an improvement of inequality (1.1).

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2. Proof

In this section, we give the proof of inequality (1.2) by using Taylor's theorem and an inequality due to Bhatia[2, Lemma 1].

Proof. Let

$$f(x) = \frac{a^x b^{1-x} + a^{1-x} b^x}{2}, \quad 0 \leqslant x \leqslant 1.$$

It is easy to see that the function $f(x)$ is twice differentiable in $(0,1)$. Simple calculations show that

$$f'(x) = (\log a - \log b) \frac{a^x b^{1-x} - a^{1-x} b^x}{2}, \quad x \in (0,1),$$

$$f''(x) = (\log a - \log b)^2 f(x), \quad x \in (0,1).$$

So, for a given $x \in \left(0, \frac{1}{2}\right)$, by Taylor's theorem, there exists $\xi \in \left(x, \frac{1}{2}\right)$ such that

$$f(x) = \sqrt{ab} + (\log a - \log b)^2 \frac{a^\xi b^{1-\xi} + a^{1-\xi} b^\xi}{4} \left(x - \frac{1}{2}\right)^2. \tag{2.1}$$

Bhatia[1, Lemma 1] proved that

$$f(x) \leqslant (1 - \alpha(x)) \sqrt{ab} + \alpha(x) \frac{a+b}{2}, \tag{2.2}$$

where

$$\alpha(x) = (1-2x)^2, \quad x \in (0,1).$$

It follows from (2.1) and (2.2) that for any $x \in \left(0, \frac{1}{2}\right)$, there exists $\xi(x) \in \left(x, \frac{1}{2}\right)$ such that

$$\sqrt{ab} + (\log a - \log b)^2 \frac{a^{\xi(x)} b^{1-\xi(x)} + a^{1-\xi(x)} b^{\xi(x)}}{4} \left(x - \frac{1}{2}\right)^2$$
$$\leqslant (1 - \alpha(x)) \sqrt{ab} + \alpha(x) \frac{a+b}{2},$$

which is equivalent to

$$\alpha(x) \sqrt{ab} + \alpha(x) (\log a - \log b)^2 \frac{a^{\xi(x)} b^{1-\xi(x)} + a^{1-\xi(x)} b^{\xi(x)}}{16} \leqslant \alpha(x) \frac{a+b}{2}.$$

That is,

$$\sqrt{ab} + (\log a - \log b)^2 \frac{a^{\xi(x)} b^{1-\xi(x)} + a^{1-\xi(x)} b^{\xi(x)}}{16} \leqslant \frac{a+b}{2}.$$

So, the arithmetic-geometric mean inequality and the last inequality complete the proof. □

3. Remarks

Recently, Furuichi [6, Theorem 1] proved that if $a,b>0$, then

$$S\left(\sqrt{\frac{b}{a}}\right)\sqrt{ab}\leqslant\frac{a+b}{2}, \tag{3.1}$$

where

$$S(t)=\frac{t^{\frac{1}{t-1}}}{e\log t^{\frac{1}{t-1}}},\quad t>0,\quad S(1)=\lim_{t\to 1}S(t)=1$$

is the Specht's ratio. After seeing the inequalities (1.2) and (3.1), it is hard not to be curious about the relationship between $1+\frac{(\log a-\log b)^2}{8}$ and $S\left(\sqrt{\frac{b}{a}}\right)$. We may ask whether one of the the following inequalities holds:

$$1+\frac{(\log a-\log b)^2}{8}\leqslant S\left(\sqrt{\frac{b}{a}}\right),$$

$$1+\frac{(\log a-\log b)^2}{8}\geqslant S\left(\sqrt{\frac{b}{a}}\right).$$

The answer is no. In fact, if we choose $a=1$ and $b=100$, then we have

$$1+\frac{(\log a-\log b)^2}{8}=3.6509>1.8571=S\left(\sqrt{\frac{b}{a}}\right).$$

On the other hand, if we choose $a=1$ and $b=100000$, then we have

$$1+\frac{(\log a-\log b)^2}{8}=24.8585<53.5719=S\left(\sqrt{\frac{b}{a}}\right).$$

Next, we further discuss the relationship between between $1+\frac{(\log a-\log b)^2}{8}$ and $S\left(\sqrt{\frac{b}{a}}\right)$. Let

$$F(t)=1+\frac{(\log t)^2}{2},\quad t>0.$$

Then

$$F\left(\sqrt{\frac{b}{a}}\right)=1+\frac{(\log a-\log b)^2}{8}.$$

Simple calculations show that the function $F(t)$ has the following properties.
1. $F(t) = F\left(\frac{1}{t}\right)$ for all $t > 0$.
2. $F(t)$ is a monotone increasing function on $(1,\infty)$.
3. $F(t)$ is a monotone decreasing function on $(0,1)$.

The image of $F(t)$ and $S(t)$ as follows. So, if $\frac{1}{x_0} \leqslant \sqrt{\frac{b}{a}} \leqslant x_0$, where $x_0 \approx 227$, then

$$F\left(\sqrt{\frac{b}{a}}\right) \geqslant S\left(\sqrt{\frac{b}{a}}\right),$$

otherwise

$$F\left(\sqrt{\frac{b}{a}}\right) < S\left(\sqrt{\frac{b}{a}}\right).$$

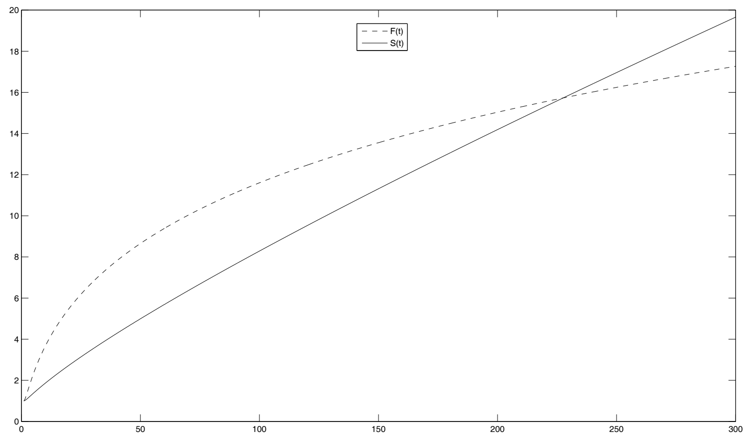

Figure 1: $F(t)$ *and* $S(t)$, $1 < t < 300$.

4. An application

In this section, we present a refinement of inequality (1.1) by using inequality (1.2).

THEOREM 4.1. *Let A and B be invertible positive operators. Then*

$$A\#B + K^*(A\#B)K \leqslant \frac{A+B}{2},$$

where

$$K = \frac{\sqrt{2}}{4}A^{-1}S(A|B).$$

Proof. Let $T = A^{-1/2}BA^{-1/2}$. By inequality (1.2), we have

$$\sqrt{a} + \frac{1}{8}\log a\sqrt{a}\log a \leqslant \frac{a+1}{2}.$$

So,

$$T^{1/2} + \frac{1}{8}\log(T)T^{1/2}\log(T) \leqslant \frac{T+I}{2}.$$

Multiplying $A^{1/2}$ to the above inequality from left hand side and right hand side, we have

$$A\#B + \frac{1}{8}A^{1/2}\log\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}A^{-1}\left(A\#B\right)A^{-1}A^{1/2}\log\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}.$$

This completes the proof. □

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