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REFINEMENT OF JENSEN'S INEQUALITY WITH APPLICATIONS TO CYCLIC MIXED SYMMETRIC MEANS AND CAUCHY MEANS

ILKO BRNETIĆ, KHURAM ALI KHAN AND JOSIP PEČARIĆ

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Abstract. Generalized refinement of Jensen's inequality is given. Appplications are done for cyclic mixed symmetric means and Cauchy means.

1. Introduction and preliminary results

The aim of this article is to establish generalized refinement of Jensen's inequality and to apply this result for power means, cyclic mixed symmetric mean, generalized means and quasi-arithmetic means and also for Cauchy means.

At the beginning let's remind on classical Jensen's inequality:

Jensen's inequality. If $f: I \to \mathbb{R}$, $I \subseteq \mathbb{R}$ is a convex function, $(x_1, \ldots, x_n) \in I^n$ $(n \geqslant 2)$ and $(\lambda_1, \ldots, \lambda_n)$ positive $n-$tuple such that $\sum_{i=1}^n \lambda_i = 1$, then the following inequality holds

$$f\Big(\sum_{i=1}^n \lambda_i x_i\Big) \leqslant \sum_{i=1}^n \lambda_i f(x_i).$$

Especially, the following inequality is valid

$$f\left(\frac{\sum_{i=1}^n x_i}{n}\right) \leqslant \frac{\sum_{i=1}^n f(x_i)}{n}.$$

Throught this article we are going to use some of the following hypotheses:

($\mathscr{H}_1$) Let $I \subset \mathbb{R}$ be an interval, $\mathbf{x} := (x_1, \ldots, x_n) \in I^n$ such that and $x_{i+n} = x_i$ and $\lambda := (\lambda_1, \ldots, \lambda_n)$ be a positive n-tuple such that $\sum_{i=1}^k \lambda_i = 1$ for some k, $2 \leqslant k \leqslant n$.

($\mathscr{H}_2$) Let $f: I \to \mathbb{R}$ be a convex function.

($\mathscr{H}_3$) Let h, $g: I \to \mathbb{R}$ be continuous and strictly monotone functions.

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2. Refinement of Jensen's inequality

THEOREM 2.1. *Let ($\mathscr{H}_1$), ($\mathscr{H}_2$) be fulfilled. Then*

$$f\left(\frac{\sum_{i=1}^{n} x_i}{n}\right) \leqslant \frac{1}{n}\sum_{i=1}^{n} f\left(\sum_{j=0}^{k-1} \lambda_{j+1} x_{i+j}\right) \leqslant \frac{\sum_{i=1}^{n} f(x_i)}{n}. \tag{1}$$

Proof. First, since f is convex, by Jensen's inequality we have

$$\begin{aligned}\sum_{i=1}^{n} f\left(\sum_{j=0}^{k-1} \lambda_{j+1} x_{i+j}\right) &\leqslant \sum_{i=1}^{n}\sum_{j=0}^{k-1} \lambda_{j+1} f(x_{i+j}) \\ &= \sum_{i=1}^{n} f(x_i) \sum_{j=1}^{k} \lambda_j = \sum_{i=1}^{n} f(x_i).\end{aligned}$$

On the other hand, since f is convex, by Jensen's inequality, we have

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n} f\left(\sum_{j=0}^{k-1} \lambda_{j+1} x_{i+j}\right) &\geqslant f\left(\frac{\sum_{i=1}^{n}\sum_{j=0}^{k-1} \lambda_{j+1} x_{i+j}}{n}\right) \\ &= f\left(\frac{\sum_{i=1}^{n} x_i \sum_{j=1}^{k} \lambda_j}{n}\right) = f\left(\frac{\sum_{i=1}^{n} x_i}{n}\right). \quad \square\end{aligned}$$

The result given in Theorem 2.1. is a generalization of the result given in [3, Theorem 4].

3. Cyclic mixed symmetric means

Assume ($\mathscr{H}_1$) for the positive n-tuple $\mathbf{x}$. We define the power means of order $r \in \mathbb{R}$ as follows:

$$M_r(x_i, \ldots, x_{i+k-1}; \lambda_1, \ldots, \lambda_k) = \begin{cases} \left(\sum_{j=0}^{k-1} \lambda_{j+1} x_{i+j}^r\right)^{\frac{1}{r}}; & r \neq 0, \\ \prod_{j=0}^{k-1} x_{i+j}^{\lambda_{j+1}}; & r = 0, \end{cases}$$

and cyclic mixed symmetric means corresponding to (1) are

$$M_{r,s}(\mathbf{x}, \lambda) := \begin{cases} \left(\frac{1}{n}\sum_{i=1}^{n} M_r^s(x_i, \ldots, x_{i+k-1}; \lambda_1, \ldots, \lambda_k)\right)^{\frac{1}{s}}; & s \neq 0, \\ \left(\prod_{i=1}^{n} M_r(x_i, \ldots, x_{i+k-1}; \lambda_1, \ldots, \lambda_k)\right)^{\frac{1}{n}}; & s = 0. \end{cases} \tag{2}$$

The standard power means of order $r \in \mathbb{R}$ for the positive n-tuple $\mathbf{x}$, are

$$M_r(x_1,\ldots,x_n) = M_r(\mathbf{x}) := \begin{cases} \left(\frac{1}{n}\sum_{i=1}^{n} x_i^r\right)^{\frac{1}{r}}; & r \neq 0, \\ \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}; & r = 0. \end{cases}$$

The bounds for cyclic mixed symmetric means are power means, as given in the following result.

COROLLARY 3.1. *Assume ($\mathscr{H}_1$) for the positive n-tuple $\mathbf{x}$. Let $r,s \in \mathbb{R}$ such that $r \leqslant s$. Then*

$$M_r(\mathbf{x}) \leqslant M_{s,r}(\mathbf{x},\lambda) \leqslant M_s(\mathbf{x}). \tag{3}$$

Proof. Assume $r,\ s \neq 0$. To obtain (3), we apply Theorem 2.1, either for the function $f(x) = x^{\frac{s}{r}}$ $(x > 0)$ and the n-tuples $(x_1^r,\ldots,x_n^r)$ in (1) and then raising the power $\frac{1}{s}$, or $f(x) = x^{\frac{r}{s}}$ $(x > 0)$ and $(x_1^s,\ldots,x_n^s)$ and raising the power $\frac{1}{r}$.
When $r = 0$ or $s = 0$, we get the required results by taking limit. □

Special cases of the refinement given in Corollary 3.1. can be found in [2] (Theorem 4 with Corollaries 4.1.–4.4. as an application). Namely, the result of this theorem is an inequality (3) for $r = 0$, $s = 1$, $n = 3$ and $k = 3$.

Assume ($\mathscr{H}_1$) and ($\mathscr{H}_3$). Then we define the generalized means with respect to (1) as follows:

$$M_{g,h}(\mathbf{x},\lambda) := g^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}(g\circ h^{-1})\Big(\sum_{j=0}^{k-1}\lambda_{j+1}h(x_{i+j})\Big)\right).$$

Let $q : I \to \mathbb{R}$ be a continuous and strictly monotone function then the cyclic quasi-arithmetic means are given by

$$M_q(\mathbf{x}) := q^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} q(x_i)\right).$$

The relation among the generalized means and cyclic quasi-arithmetic means is given in the next corollary.

COROLLARY 3.2. *Assume ($\mathscr{H}_1$) and ($\mathscr{H}_3$). Then*

$$M_h(\mathbf{x}) \leqslant M_{g,h}(\mathbf{x},\lambda) \leqslant M_g(\mathbf{x}) \tag{4}$$

if either $g\circ h^{-1}$ is convex and g is strictly increasing or $g\circ h^{-1}$ is concave and g is strictly decreasing.

Proof. First, we can apply Theorem 2.1 to the function $g \circ h^{-1}$ and the n-tuples $(h(x_1),\ldots,h(x_n))$, then we can apply g^{-1} to the inequality coming from (1). This gives (4). $\square$

For instance, if we put $g(x) = x$ and $h(x) = \ln x$ in Corollary 3.2. we obtain

$$M_0(x_1,\ldots,x_n) \leqslant \frac{1}{n}\sum_{i=1}^{n} M_0(x_i,\ldots,x_{i+k-1};\lambda_1,\ldots,\lambda_k) \leqslant M_1(x_1,\ldots,x_n).$$

which is a special case of Corollary 3.1. as well.

REMARK 3.3. Under the conditions ($\mathscr{H}_1$), we define

$$\Upsilon_1(f) = \Upsilon_1(\mathbf{x},\lambda,f) := \frac{1}{n}\sum_{i=1}^{n} f(x_i) - \frac{1}{n}\sum_{i=1}^{n} f(\sum_{j=0}^{k-1}\lambda_{j+1}x_{i+j}),$$

$$\Upsilon_2(f) = \Upsilon_2(\mathbf{x},\lambda,f) := \frac{1}{n}\sum_{i=1}^{n} f(\sum_{j=0}^{k-1}\lambda_{j+1}x_{i+j}) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right),$$

where $f : I \to \mathbb{R}$ is a function and $2 \leqslant k \leqslant n$. The functionals $f \to \Upsilon_i(f)$ are linear, $i = 1,2$, and Theorem 2.1 imply that

$$\Upsilon_i(f) \geqslant 0, \quad i = 1,2$$

if $f : I \to \mathbb{R}$ is a convex function.

4. m-exponential convexity

For log-convexity, exponential convexity and m-exponential convexity of the functionals obtained from the interpolations of the discrete Jensen's inequality, we refer [1, 4, 5, 7] and references therein.

We apply the method given in [8], to prove the m-exponential convexity and exponential convexity of the functionals $f \to \Upsilon_i(f)$ for $i = 1,2$, together with the Lagrange type and Cauchy type mean value theorems.

DEFINITION 1. [8] A function $g : I \to \mathbb{R}$ is called *m-exponentially convex* in the Jensen sense if

$$\sum_{i,j=1}^{m} a_i a_j g\left(\frac{x_i + x_j}{2}\right) \geqslant 0$$

holds for every $a_i \in \mathbb{R}$ and every $x_i \in I$, $i = 1,2,\ldots,m$.

A function $g : I \to \mathbb{R}$ is *m-exponentially convex* if it is *m-exponentially convex* in the Jensen sense and continuous on I.

Note that 1-exponentially convex functions in the Jensen sense are in fact the nonnegative functions. Also, m-exponentially convex functions in the Jensen sense are n-exponentially convex in the Jensen sense for every $n \in \mathbb{N}$, $n \leqslant m$.

PROPOSITION 4.1. *If $g : I \to \mathbb{R}$ is an m-exponentially convex function, then for every $x_i \in I$, $i = 1,2,\ldots,m$ and for all $n \in \mathbb{N}$, $n \leqslant m$ the matrix $\left[g\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{n}$ is a positive semi-definite matrix. Particularly,*

$$\det\left[g\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{n} \geqslant 0$$

for all $n \in \mathbb{N}$, $n = 1,2,\ldots,m$.

DEFINITION 2. A function $g : I \to \mathbb{R}$ is exponentially convex in the Jensen sense, if it is m-exponentially convex in the Jensen sense for all $m \in \mathbb{N}$.

A function $g : I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

REMARK 4.2. It is easy to see that a positive function $g : I \to \mathbb{R}$ is log-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense, that is

$$a_1^2 g(x) + 2a_1a_2 g\left(\frac{x+y}{2}\right) + a_2^2 g(y) \geqslant 0$$

holds for every $a_1, a_2 \in \mathbb{R}$ and $x, y \in I$.

Similarly, if g is 2-exponentially convex, then g is log-convex. On the other hand, if g is log-convex and continuous, then g is 2-exponentially convex.

In sequel, we need the well known notion of "Divided difference".

DEFINITION 3. The second order divided difference of a function $g : I \to \mathbb{R}$ at mutually different points $y_0, y_1, y_2 \in I$ is defined recursively by

$$\begin{aligned}
[y_i; g] &= g(y_i), \quad i = 0,1,2 \\
[y_i, y_{i+1}; g] &= \frac{g(y_{i+1}) - g(y_i)}{y_{i+1} - y_i}, \quad i = 0,1 \\
[y_0, y_1, y_2; g] &= \frac{[y_1, y_2; g] - [y_0, y_1; g]}{y_2 - y_0}.
\end{aligned} \tag{5}$$

REMARK 4.3. The value $[y_0, y_1, y_2; g]$ is independent of the order of the points y_0, y_1, and y_2. By taking limits this definition may be extended to include the cases in which any two or all three points coincide as follows: $\forall$ y_0, y_1, y_2 $\in I$ such that $y_2 \neq y_0$

$$\lim_{y_1 \to y_0} [y_0, y_1, y_2; g] = [y_0, y_0, y_2; g] = \frac{g(y_2) - g(y_0) - g^{'}(y_0)(y_2 - y_0)}{(y_2 - y_0)^2}$$

provided that g' exists, and furthermore, taking the limits $y_i \to y_0$, $i = 1,2$ in (5), we get

$$[y_0, y_0, y_0; g] \ = \lim_{y_i \to y_0} [y_0, y_1, y_2; g] = \frac{g^{''}(y_0)}{2} \text{ for } i = 1,2$$

provided that g'' exist on I.

Now, we give the m-exponential convexity for the linear functionals $\Upsilon_i(f)$ $(i = 1,2)$.

THEOREM 4.4. *Assume $I \subset \mathbb{R}$ is an interval, and assume $\Lambda = \{\phi_t \mid t \in J\}$ is a family of functions defined on an interval $I \subset \mathbb{R}$, such that the function $t \to [y_0,y_1,y_2;\phi_t]$ $(t \in J)$ is m-exponentially convex in the Jensen sense on I for every three mutually different points $y_0,y_1,y_2 \in I$. Let $\Upsilon_i(f)$ $(i = 1,2)$ be the linear functionals constructed in Remark 3.3. Then $t \to \Upsilon_i(\phi_t)$ $(t \in J)$ is an m-exponentially convex function in the Jensen sense on I for each $i = 1,2$. If the function $t \to \Upsilon_i(\phi_t)$ $(t \in J)$ is continuous for $i = 1,2$, then it is m-exponentially convex on I for $i = 1,2$.*

Proof. Fix $i = 1,2$.

Let $t_k,t_l \in J$, $t_{kl} := \frac{t_k+t_l}{2}$ and $b_k,b_l \in \mathbb{R}$ for $k,l = 1,2,\dots,n$, and define the function ω on I by

$$\omega := \sum_{k,l=1}^{n} b_k b_l \phi_{t_{kl}}.$$

Since the function $t \to [y_0,y_1,y_2;\phi_t]$ $(t \in J)$ is m-exponentially convex in the Jensen sense, we have

$$[y_0,y_1,y_2;\omega] = \sum_{k,l=1}^{n} b_k b_l [y_0,y_1,y_2;\phi_{t_{kl}}] \geqslant 0.$$

Hence ω is a convex function on I. Therefore we have $\Upsilon_i(\omega) \geqslant 0$, which yields by the linearity of Υ_i, that

$$\sum_{k,l=1}^{n} b_k b_l \Upsilon_i(\phi_{t_{kl}}) \geqslant 0.$$

We conclude that the function $t \to \Upsilon_i(\phi_t)$ $(t \in J)$ is an m-exponentially convex function in the Jensen sense on I.

If the function $t \to \Upsilon_i(\phi_t)$ $(t \in J)$ is continuous on I, then it is m-exponentially convex on I by definition. □

As a consequence of the above theorem we can give the following corollaries.

COROLLARY 4.5. *Assume $I \subset \mathbb{R}$ is an interval, and assume $\Lambda = \{\phi_t \mid t \in J\}$ is a family of functions defined on an interval $I \subset \mathbb{R}$, such that the function $t \to [y_0,y_1,y_2;\phi_t]$ $(t \in J)$ is exponentially convex in the Jensen sense on I for every three mutually different points $y_0,y_1,y_2 \in I$. Let $\Upsilon_i(f)$ $(i = 1,2)$ be the linear functionals constructed in Remark 3.3. Then $t \to \Upsilon_i(\phi_t)$ $(t \in J)$ is an exponentially convex function in the Jensen sense on I for $i = 1,2$. If the function $t \to \Upsilon_i(\phi_t)$ $(t \in J)$ is continuous, then it is exponentially convex on I for $i = 1,2$.*

COROLLARY 4.6. *Assume $I \subset \mathbb{R}$ is an interval, and assume $\Lambda = \{\phi_t : t \in J\}$ is a family of functions defined on an interval $I \subset \mathbb{R}$, such that the function $t \to [y_0,y_1,y_2;\phi_t]$ $(t \in J)$ is 2-exponentially convex in the Jensen sense on I for every three mutually different points $y_0,y_1,y_2 \in I$. Let $\Upsilon_i(f)$ $(i = 1,2)$ be the linear functionals constructed in Remark 3.3. Then the following two statements hold for $i = 1,2$:*

(i) *If the function* $t \to \Upsilon_i(\phi_t)$ $(t \in J)$ *is positive and continuous, then it is* 2*-exponentially convex on* I*, and thus* $\log$*-convex.*

(ii) *If the function* $t \to \Upsilon_i(\phi_t)$ $(t \in J)$ *is positive and differentiable, then for every* $s,t,u,v \in J$*, such that* $s \leqslant u$ *and* $t \leqslant v$*, we have*

$$\mathfrak{u}_{s,t}(\Upsilon_i,\Lambda) \leqslant \mathfrak{u}_{u,v}(\Upsilon_i,\Lambda) \tag{6}$$

where

$$\mathfrak{u}_{s,t}(\Upsilon_i,\Lambda) := \begin{cases} \left(\frac{\Upsilon_i(\phi_s)}{\Upsilon_i(\phi_t)}\right)^{\frac{1}{s-t}}, & s \neq t, \\ \exp\left(\frac{\frac{d}{ds}\Upsilon_i(\phi_s)}{\Upsilon_i(\phi_s)}\right), & s = t \end{cases} \tag{7}$$

for $\phi_s, \phi_t \in \Lambda$.

Proof. Fix $i = 1,2$.

(i) The proof follows by Remark 4.2 and Theorem 4.4.

(ii) From the definition of a convex function ψ on I, we have the following inequality (see [9, page 2])

$$\frac{\psi(s) - \psi(t)}{s-t} \leqslant \frac{\psi(u) - \psi(v)}{u-v}, \tag{8}$$

$\forall s,t,u,v \in J$ such that $s \leqslant u, t \leqslant v, s \neq t, u \neq v$.
By (i), $s \to \Upsilon_i(\phi_s)$, $s \in J$ is $\log$-convex, and hence (8) shows with $\psi(s) = \log \Upsilon_i(\phi_s)$, $s \in J$ that

$$\frac{\log \Upsilon_i(\phi_s) - \log \Upsilon_i(\phi_t)}{s-t} \leqslant \frac{\log \Upsilon_i(\phi_u) - \log \Upsilon_i(\phi_v)}{u-v} \tag{9}$$

for $s \leqslant u, t \leqslant v, s \neq t, u \neq v$, which is equivalent to (6). For $s = t$ or $u = v$ (6) follows from (9) by taking limit. □

REMARK 4.7. Note that the results from Theorem 4.4, Corollary 4.5, Corollary 4.6 are valid when two of the points $y_0, y_1, y_2 \in I$ coincide, say $y_1 = y_0$, for a family of differentiable functions ϕ_t such that the function $t \to [y_0,y_1,y_2;\phi_t]$ is m-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense), and moreover, they are are also valid when all three points coincide for a family of twice differentiable functions with the same property. The proofs can be obtained by recalling Remark 4.3 and suitable characterization of convexity.

The following result given in [5] is related to the first condition of Theorem 4.4.

THEOREM 4.8. *Assume* $I \subset \mathbb{R}$ *is an interval, and assume* $\Lambda = \{\phi_t \mid t \in J\}$ *is a family of twice differentiable functions defined on an interval* $I \subset \mathbb{R}$ *such that the function* $t \mapsto \phi_t''(x)$ $(t \in J)$ *is exponentially convex for every fixed* $x \in I$*. Then the function* $t \mapsto [y_0,y_1,y_2;\phi_t]$ $(t \in J)$ *is exponentially convex in the Jensen sense for any three points* $y_0,\ y_1,\ y_2 \in I$.

REMARK 4.9. It comes from either the conditions of Theorem 4.8 or the proof of this theorem that the functions ϕ_t, $t \in J$ are convex.

5. Mean value theorems

Now we formulate mean value theorems of Lagrange and Cauchy type for the linear functionals $\Upsilon_i(f)$ $(i = 1,2)$ defined in Remark 3.3.

THEOREM 5.1. *Let* $\Upsilon_i(f)$ $(i = 1,2)$ *be the linear functionals constructed in Remark* 3.3 *and* $g \in C^2[a,b]$. *Then there exists* $\xi \in [a,b]$ *such that*

$$\Upsilon_i(g) = \frac{1}{2} g''(\xi) \Upsilon_i\left(x^2\right); \quad i = 1,2.$$

Proof. Fix $i = 1,2$.

Since $g \in C^2[a,b]$, there exist the real numbers $m = \min_{x\in[a,b]} g''(x)$ and $M = \max_{x\in[a,b]} g''(x)$. It is easy to show that the functions ϕ_1 and ϕ_2 defined on $[a,b]$ by

$$\phi_1(x) = \frac{M}{2} x^2 - g(x),$$

and

$$\phi_2(x) = g(x) - \frac{m}{2} x^2,$$

are convex.

By applying the functional Υ_i to the functions ϕ_1 and ϕ_2, we have the properties of Υ_i that

$$\Upsilon_i\left(\frac{M}{2} x^2 - g(x)\right) \geqslant 0,$$

$$\Rightarrow \Upsilon_i(g) \leqslant \frac{M}{2} \Upsilon_i\left(x^2\right), \tag{10}$$

and

$$\Upsilon_i\left(g(x) - \frac{m}{2} x^2\right) \geqslant 0$$

$$\Rightarrow \frac{m}{2} \Upsilon_i\left(x^2\right) \leqslant \Upsilon_i(g). \tag{11}$$

From (10) and from (11), we get

$$\frac{m}{2} \Upsilon_i\left(x^2\right) \leqslant \Upsilon_i(g) \leqslant \frac{M}{2} \Upsilon_i\left(x^2\right).$$

If $\Upsilon_i\left(x^2\right) = 0$, then nothing to prove. If $\Upsilon_i\left(x^2\right) \neq 0$, then

$$m \leqslant \frac{2\Upsilon_i(g)}{\Upsilon_i(x^2)} \leqslant M.$$

Hence we have

$$\Upsilon_i(g) = \frac{1}{2} g''(\xi) \Upsilon_i\left(x^2\right). \quad \square$$

THEOREM 5.2. *Let $\Upsilon_i(f)$ $(i=1,2)$ be the linear functionals constructed in Remark 3.3 and $g,h \in C^2[a,b]$. Then there exists $\xi \in [a,b]$ such that*

$$\frac{\Upsilon_i(g)}{\Upsilon_i(h)} = \frac{g''(\xi)}{h''(\xi)}; \quad i=1,2,$$

provided that $\Upsilon_i(h) \neq 0$ $(i=1,2)$.

Proof. Fix $i=1,2$.
Define $L \in C^2[a,b]$ by

$$L := c_1 g - c_2 h,$$

where

$$c_1 := \Upsilon_i(h)$$

and

$$c_2 := \Upsilon_i(g).$$

Now using Theorem 5.1 for the function L, we have

$$\left(c_1 \frac{g''(\xi)}{2} - c_2 \frac{h''(\xi)}{2}\right) \Upsilon_i\left(x^2\right) = 0. \tag{12}$$

Since $\Upsilon_i(h) \neq 0$, Theorem 5.1 implies that $\Upsilon_i\left(x^2\right) \neq 0$, and therefore (12) gives

$$\frac{\Upsilon_i(g)}{\Upsilon_i(h)} = \frac{g''(\xi)}{h''(\xi)}. \quad \square$$

6. Applications to Cauchy means

In this section we apply the results of previous sections to generate new Cauchy means. We mention that the functionals $\Upsilon_i(f)$, $i=1,2$ defined in Remark 3.3 under the assumption $(\mathscr{H}_1)$, are linear on the vector space of real functions defined on the interval $I \subset \mathbb{R}$, and $\Upsilon_i(f) \geqslant 0$ for every convex function on I.

EXAMPLE 6.1. Let $I=\mathbb{R}$ and consider the class of convex functions

$$\Lambda_1 := \{\phi_t : \mathbb{R} \to [0,\infty) \mid t \in \mathbb{R}\},$$

where

$$\phi_t(x) := \begin{cases} \frac{1}{t^2} e^{tx}; & t \neq 0, \\ \frac{1}{2} x^2; & t = 0. \end{cases}$$

Then $t \mapsto \phi_t''(x)$ $(t \in \mathbb{R})$ is exponentially convex for every fixed $x \in \mathbb{R}$ (see [6]), thus by Theorem 4.8, the function $t \mapsto [y_0, y_1, y_2; \phi_t]$, $t \in \mathbb{R}$ is exponentially convex in the Jensen sense for every three mutually different points $y_0, y_1, y_2 \in \mathbb{R}$.

Now fix $i = 1, 2$. By applying Corollary 4.5 with $\Lambda = \Lambda_1$, we get the exponential convexity of $t \mapsto \Upsilon_i(\phi_t)$ $(t \in \mathbb{R})$ in the Jensen sense. This mapping is also differentiable, therefore exponentially convex, and the expression in (7) has the form

$$\mathfrak{u}_{s,t}(\Upsilon_i, \Lambda_1) = \begin{cases} \left(\frac{\Upsilon_i(\phi_s)}{\Upsilon_i(\phi_t)}\right)^{\frac{1}{s-t}}; & s \neq t, \\ \exp\left(\frac{\Upsilon_i(id\,\phi_s)}{\Upsilon_i(\phi_s)} - \frac{2}{s}\right); & s = t \neq 0, \\ \exp\left(\frac{\Upsilon_i(id\,\phi_0)}{3\Upsilon_i(\phi_0)}\right); & s = t = 0, \end{cases}$$

where "*id*" means the identity function on $\mathbb{R}$.

From (6) we have the monotonicity of the functions $\mathfrak{u}_{s,t}(\Upsilon_i, \Lambda_1)$ in both parameters s and t.

Suppose $\Upsilon_i(\phi_t) > 0$ $(t \in \mathbb{R})$, $a := \min\{x_1, \ldots, x_n\}$, $b := \max\{x_1, \ldots, x_n\}$, and let

$$\mathfrak{M}_{s,t}(\Upsilon_i, \Lambda_1) := \log \mathfrak{u}_{s,t}(\Upsilon_i, \Lambda_1); \quad s, t \in \mathbb{R}.$$

Then from Theorem 5.2 we have

$$a \leqslant \mathfrak{M}_{s,t}(\Upsilon_i, \Lambda_1) \leqslant b,$$

and thus $\mathfrak{M}_{s,t}(\Upsilon_i, \Lambda_1)$ $(s, t \in \mathbb{R})$ are means. The monotonicity of these means is followed by (6).

EXAMPLE 6.2. Let $I = (0, \infty)$ and consider the class of convex functions

$$\Lambda_2 = \{\psi_t : (0, \infty) \to \mathbb{R} \mid t \in \mathbb{R}\},$$

where

$$\psi_t(x) := \begin{cases} \frac{x^t}{t(t-1)}; \ t \neq 0, 1, \\ -\log x; \ t = 0, \\ x \log x; \ t = 1. \end{cases}$$

Then $t \mapsto \psi_t''(x) = x^{t-2} = e^{(t-2)\log x}$ $(t \in \mathbb{R})$ is exponentially convex for every fixed $x \in (0, \infty)$.

Now fix $1 \leqslant i \leqslant 4$. By similar arguments as given in Example 6.1 we get the exponential convexity of $t \mapsto \Upsilon_i(\psi_t)$ $(t \in \mathbb{R})$ in the Jensen sense. This mapping is differentiable too, therefore exponentially convex. It is easy to calculate that (7) can be written as

$$\mathfrak{u}_{s,t}(\mathbf{x}, \mathbf{p}, \Upsilon_i, \Lambda_2) = \begin{cases} \left(\frac{\Upsilon_i(\psi_s)}{\Upsilon_i(\psi_t)}\right)^{\frac{1}{s-t}}; \ s \neq t, \\ \exp\left(\frac{1-2s}{s(s-1)} - \frac{\Upsilon_i(\psi_s \psi_0)}{\Upsilon_i(\psi_s)}\right); \ s = t \neq 0, 1, \\ \exp\left(1 - \frac{\Upsilon_i(\psi_0^2)}{2\Upsilon^i(\psi_0)}\right); \ s = t = 0, \\ \exp\left(-1 - \frac{\Upsilon_i(\psi_0 \psi_1)}{2\Upsilon_i(\psi_1)}\right); \ s = t = 1. \end{cases}$$

Suppose $\Upsilon_i(\psi_t) > 0$ $(t \in \mathbb{R})$, and let $a := \min\{x_1, \ldots, x_n\}$, $b := \max\{x_1, \ldots, x_n\}$. By Theorem 5.2, we can check that

$$a \leqslant \mathfrak{u}_{s,t}(\mathbf{x}, \mathbf{p}, \Upsilon_i, \Lambda_2) \leqslant b; \quad s, t \in \mathbb{R}. \tag{13}$$

The means $\mathfrak{u}_{s,t}(\mathbf{x}, \mathbf{p}, \Upsilon_i, \Lambda_2)$ $(s, t \in \mathbb{R})$ are continuous, symmetric and monotone in both parameters (by use of (6)).

Let $s, t, r \in \mathbb{R}$ such that $r \neq 0$. By the substitutions $s \to \frac{s}{r}$, $t \to \frac{t}{r}$, $(x_1, \ldots, x_n) \to (x_1^r, \ldots, x_n^r)$ in (13), we get

$$\bar{a} \leqslant \mathfrak{u}_{s/r,t/r}(\mathbf{x}^r, \mathbf{p}, \Upsilon^i, \Lambda_2) \leqslant \bar{b},$$

where $\bar{a} := \min\{x_1^r, \ldots, x_n^r\}$ and $\bar{b} := \max\{x_1^r, \ldots, x_n^r\}$. Thus new means can be defined with three parameters:

$$\mathfrak{u}_{s,t,r}(\mathbf{x}, \lambda, \Upsilon_i, \Lambda_2) := \begin{cases} \left(\mathfrak{u}_{s/r,t/r}(\mathbf{x}^r, \lambda, \Upsilon_i, \Lambda_2)\right)^{\frac{1}{r}}; & r \neq 0, \\ \mathfrak{u}_{s,t}(\log \mathbf{x}, \lambda, \Upsilon_i, \Lambda_1); & r = 0, \end{cases}$$

where $\log \mathbf{x} = (\log x_1, \ldots, \log x_n)$.

The monotonicity of these three parameter means is followed by the monotonicity and continuity of the two parameter means.

EXAMPLE 6.3. Let $I = (0, \infty)$, and consider the class of convex functions

$$\Lambda_3 = \{\eta_t : (0, \infty) \to (0, \infty) \mid t \in (0, \infty)\},$$

where

$$\eta_t(x) := \begin{cases} \frac{t^{-x}}{\log^2 t}; & t \neq 1, \\ \frac{x^2}{2}; & t = 1. \end{cases}$$

$t \mapsto \psi_t''(x)$ $(t \in (0, \infty))$ is exponentially convex for every fixed $x \in (0, \infty)$, being the restriction of the Laplace transform of a nonnegative function (see [6] or [10] page 210).

Now fix $1 \leqslant i \leqslant 4$. We can get the exponential convexity of $t \mapsto \Upsilon_i(\psi_t)$ $(t \in \mathbb{R})$ as in Example 6.1. For the class Λ_3, (7) has the form

$$\mathfrak{u}_{s,t}(\Upsilon_i, \Lambda_3) = \begin{cases} \left(\frac{\Upsilon_i(\eta_s)}{\Upsilon_i(\eta_t)}\right)^{\frac{1}{s-t}}; & s \neq t, \\ \exp\left(-\frac{2}{s \log s} - \frac{\Upsilon_i(id\,\eta_s)}{s\Upsilon_i(\eta_s)}\right); & s = t \neq 1, \\ \exp\left(-\frac{\Upsilon_i(id\,\eta_1)}{3\Upsilon_i(\eta_1)}\right); & s = t = 1. \end{cases}$$

The monotonicity of $\mathfrak{u}_{s,t}(\Upsilon_i, \Lambda_3)$ $(s, t \in (0, \infty))$ comes from (6).

Suppose $\Upsilon_i(\eta_t) > 0$ $(t \in (0, \infty))$, and let $a := \min\{x_1, \ldots, x_n\}$, $b := \max\{x_1, \ldots, x_n\}$, and define

$$\mathfrak{M}_{s,t}(\Upsilon_i, \Lambda_3) := -L(s,t) \log \mathfrak{u}_{s,t}(\Upsilon_i, \Lambda_3), \quad s, t \in (0, \infty),$$

where $L(s,t)$ is the well known logarithmic mean

$$L(s,t) := \begin{cases} \frac{s-t}{\log s - \log t}; & s \neq t, \\ t; & s = t. \end{cases}$$

From Theorem 5.2 we have

$$a \leqslant \mathfrak{M}_{s,t}(\Upsilon_i, \Lambda_3) \leqslant b, \quad s,t \in (0,\infty),$$

and therefore we get means.

EXAMPLE 6.4. Let $I = (0,\infty)$ and consider the class of convex functions

$$\Lambda_4 = \{\gamma_t : (0,\infty) \to (0,\infty) \mid t \in (0,\infty)\},$$

where

$$\gamma_t(x) := \frac{e^{-x\sqrt{t}}}{t}.$$

$t \mapsto \psi_t''(x) = e^{-x\sqrt{t}}$, $t \in (0,\infty)$ is exponentially convex for every fixed $x \in (0,\infty)$, being the restriction of the Laplace transform of a non-negative function (see [6] or [10] page 214).

Now fix $1 \leqslant i \leqslant 4$. As before $t \mapsto \Upsilon_i(\psi_t)$ $(t \in \mathbb{R})$ is exponentially convex and differentiable. For the class Λ_4, (7) becomes

$$\mathfrak{u}_{s,t}(\Upsilon_i, \Lambda_4) = \begin{cases} \left(\frac{\Upsilon_i(\gamma_s)}{\Upsilon_i(\gamma_t)}\right)^{\frac{1}{s-t}}; & s \neq t, \\ \exp\left(-\frac{1}{t} - \frac{\Upsilon_i(id\,\gamma_t)}{2\sqrt{t}\,\Upsilon_i(\gamma_t)}\right); & s = t, \end{cases}$$

where id means the identity function on $(0,\infty)$. The monotonicity of $\mathfrak{u}_{s,t}(\Upsilon_i, \Lambda_4)$ $(s,t \in (0,\infty))$ is followed by (6).

Suppose $\Upsilon_i(\eta_t) > 0$ $(t \in (0,\infty))$, let $a := \min\{x_1, \dots, x_n\}$, $b := \max\{x_1, \dots, x_n\}$, and define

$$\mathfrak{M}_{s,t}(\Upsilon_i, \Lambda_4) := -(\sqrt{s} + \sqrt{t}) \log \mathfrak{u}_{s,t}(\Upsilon_i, \Lambda_4), \quad s,t \in (0,\infty).$$

Then Theorem 5.2 yields that

$$a \leqslant \mathfrak{M}_{s,t}(\Upsilon_i, \Lambda_4) \leqslant b,$$

thus we have new means.

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Ilko Brnetić
Faculty of Electrical Engineering and Computing
University of Zagreb
Unska 3, 10000 Zagreb, Croatia
e-mail: ilko.brnetic@fer.hr

Khuram Ali Khan
Department of mathematics
University of Sargodha
Sargodha 40100, Pakistan
e-mail: Khuramsms@gmail.com

Josip Pečarić
Faculty of Textile Technology
University of Zagreb
10000 Zagreb, Croatia
e-mail: pecaric@mahazu.hazu.hr