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NEUMAN–SÁNDOR MEAN, ASYMPTOTIC EXPANSIONS AND RELATED INEQUALITIES

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Abstract. The subject of this paper is a systematic study of inequalities of the form

$$(1-\mu)M_1+\mu M_3 \leqslant M_2 \leqslant (1-\nu)M_1+\nu M_3$$

which cover Neuman-Sándor mean and some classical means. Furthermore, following inequalities with optimal parameters were proved:

$$\mu\frac{1}{H(s,t)}+(1-\mu)\frac{1}{NS(s,t)} \leqslant \frac{1}{A(s,t)} \leqslant \nu\frac{1}{H(s,t)}+(1-\nu)\frac{1}{NS(s,t)}$$

and

$$\mu\frac{1}{H(s,t)}+(1-\mu)\frac{1}{N(s,t)} \leqslant \frac{1}{NS(s,t)} \leqslant \nu\frac{1}{H(s,t)}+(1-\nu)\frac{1}{N(s,t)}.$$

1. Introduction

The Neuman-Sándor mean is a bivariate mean

$$NS(s,t)=\frac{s-t}{2\operatorname{arcsinh}\left(\frac{s-t}{s+t}\right)}.$$

first defined by Neuman and Sándor [21]. Beside this mean, our analysis will include following well known means: harmonic mean (H), geometric mean (G), logarithmic mean (L), arithmetic mean (A), centroidal mean (C), root mean square (Q), and contraharmonic mean (N).

$$H(s,t)=\frac{2st}{s+t}, \qquad G(s,t)=\sqrt{st},$$

$$L(s,t)=\frac{t-s}{\log t-\log s}, \qquad A(s,t)=\frac{s+t}{2},$$

$$C(s,t)=\frac{2}{3}\cdot\frac{s^2+st+t^2}{s+t}, \qquad Q(s,t)=\sqrt{\frac{s^2+t^2}{2}},$$

$$N(s,t)=\frac{s^2+t^2}{s+t}.$$

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Neuman-Sándor mean has been subject of investigation in various papers recently. Inequalities

$$\mu Q + (1-\mu)A < NS < \nu Q + (1-\nu)A \tag{1.1}$$

and

$$\mu N + (1-\mu)A < NS < \nu N + (1-\nu)A. \tag{1.2}$$

were studied in [20].

In [28] authors proved that double inequalities

$$\mu_1 H(s,t) + (1-\mu_1)Q(s,t) < NS(s,t) < \nu_1 H(s,t) + (1-\nu_1)Q(s,t) \tag{1.3}$$

$$\mu_2 G(s,t) + (1-\mu_2)Q(s,t) < NS(s,t) < \nu_2 G(s,t) + (1-\nu_2)Q(s,t) \tag{1.4}$$

$$\mu_3 H(s,t) + (1-\mu_3)N(s,t) < NS(s,t) < \nu_3 H(s,t) + (1-\nu_3)N(s,t) \tag{1.5}$$

hold for all $s,t > 0$ with $s \neq t$ if and only if $\mu_1 \geqslant 2/9$, $\nu_1 \leqslant 1 - 1/[\sqrt{2}\log(1+\sqrt{2})]$, $\mu_2 \geqslant 1/3$, $\nu_2 \leqslant 1 - 1/[\sqrt{2}\log(1+\sqrt{2})]$, $\mu_3 \geqslant 1 - 1/[2\log(1+\sqrt{2})]$ and $\nu_3 \leqslant 5/12$.

Qian, Chu [23] proved that the double inequality

$$\mu C(s,t) + (1-\mu)A(s,t) < NS(s,t) < \nu(s,t)C + (1-\nu)A(s,t) \tag{1.6}$$

holds for all $s,t > 0$ with $s \neq t$ if and only if $\mu \leqslant (3 - 3\log(1+\sqrt{2}))/\log(1+\sqrt{2})$ and $\nu \geqslant \frac{1}{2}$. Other similar results can be found in [16, 22, 24, 25, 26, 27].

The aim of this paper is to give a systematic study of double inequalities of the type

$$(1-\mu)M_1 + \mu M_3 \leqslant M_2 \leqslant (1-\nu)M_1 + \nu M_3 \tag{1.7}$$

which apart from Neuman-Sándor mean also contains two classical means from the list given at the beginnig of this section. The analysis of these inequalities will be made using technique of asymptotic expansions which was subject of research in [10]–[15]. After finding coefficients in asymptotic expansion of Neuman-Sándor mean we will be able to establish asymptotic inequalities between means of our interest. That technique will provide us to easily cover large number of relations between means.

2. Asymptotic expansion of Neuman-Sándor mean

Asymptotic expansion of any of these means has the following form

$$M(x+s,x+t) \sim x + c_1(s,t) + \frac{c_2(s,t)}{x} + \frac{c_3(s,t)}{x^2} + \dots \tag{2.1}$$

where $c_n(s,t)$ is a homogeneous polynomial of order n.

The coefficients c_n will have simpler form if they are presented in terms of variables α and β where

$$\alpha = \frac{t+s}{2}, \quad \beta = \frac{t-s}{2}.$$

Let's assume for a moment that $\alpha = 0$ and that mean M has asymptotic expansion. Because of the homogeneity we have

$$M(x+s,x+t) = M(x-\beta,x+\beta) = \beta M\left(\frac{x}{\beta}-1, \frac{x}{\beta}+1\right) \sim \beta\left(\frac{x}{\beta} + c_1 + c_2\frac{\beta}{x} + c_3\frac{\beta^2}{x^2} + \dots\right)$$

i. e. coefficient by x^{-n} equals β^{n+1} multiplied by constant. Now using Theorem 7.1 from [14] we obtain that for general α, c_n is a homogeneous polynomial of degree n in variables α and β and thus in variables s and t.

We shall need the first few coefficients in asymptotic expansion of the above mentioned means. These expansions were derived in [14]:

$$\begin{aligned}
H(x+s,x+t) &\sim x+\alpha-\beta^2x^{-1}+\alpha\beta^2x^{-2}-\alpha^2\beta^2x^{-3}+\dots\\
G(x+s,x+t) &\sim x+\alpha-\frac{1}{2}\beta^2x^{-1}+\frac{1}{2}\alpha\beta^2x^{-2}-\frac{1}{8}\beta^2(4\alpha^2+\beta^2)x^{-3}+\dots\\
L(x+s,x+t) &\sim x+\alpha-\frac{1}{3}\beta^2x^{-1}+\frac{1}{3}\alpha\beta^2x^{-2}-\frac{1}{45}\beta^2(15\alpha^2+4\beta^2)x^{-3}+\dots\\
A(x+s,x+t) &\sim x+\alpha \qquad\qquad (2.2)\\
C(x+s,x+t) &\sim x+\alpha+\frac{1}{3}\beta^2x^{-1}-\frac{1}{3}\alpha\beta^2x^{-2}+\frac{1}{3}\alpha^2\beta^2x^{-3}+\dots\\
Q(x+s,x+t) &\sim x+\alpha+\frac{1}{2}\beta^2x^{-1}-\frac{1}{2}\alpha\beta^2x^{-2}+\frac{1}{8}\beta^2(4\alpha^2-\beta^2)x^{-3}+\dots\\
N(x+s,x+t) &\sim x+\alpha+\beta^2x^{-1}-\alpha\beta^2x^{-2}+\alpha^2\beta^2x^{-3}+\dots
\end{aligned}$$

The following lemma will be used for obtaining asymptotic expansion of Neuman-Sándor mean. It is a special case of Lemma 1.1 from [15].

LEMMA 2.1. *Let function $f(x)$ have following asymptotic expansion $(a_0\neq 0)$:*

$$f(x)\sim\sum_{n=0}^{\infty}a_nx^{-n},\qquad x\to\infty.$$

Then

$$\frac{1}{f(x)}\sim\sum_{n=0}^{\infty}b_nx^{-n},$$

where coefficients b_n are defined by

$$\begin{aligned}
b_0 &= \frac{1}{a_0},\\
b_n &= -\frac{1}{a_0}\sum_{k=1}^{n}a_kb_{n-k}.
\end{aligned}$$

Now for $\alpha=0$ we have

$$NS(x+s,x+t)=NS(x-\beta,x+\beta)=\frac{\beta}{\operatorname{arcsinh}(\beta/x)}=x\sum_{n=0}^{\infty}b_kx^{-k}.$$

Coefficients (b_k) can be calculated by inverting Maclaurin series of arcsinh function, using Lemma 2.1 with

$$a_k=(-1)^k\binom{k-\frac{1}{2}}{k}\frac{\beta^{2k+1}}{2k+1}.$$

Then the coefficients in the case $\alpha \neq 0$ can be obtained using Theorem 7.1 from the paper [14]. It holds

$$\begin{aligned} c_0 &= 1, \\ c_1 &= \alpha, \\ c_{n+2} &= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} b_{k+2} \alpha^{n-k}. \end{aligned}$$

The first few coefficients of the Neuman-Sándor mean are:

$$\begin{aligned} c_0 &= 1, & c_4 &= \frac{1}{360}\beta^2(60\alpha^2 - 17\beta^2), \\ c_1 &= \alpha, & c_5 &= -\frac{1}{120}\alpha\beta^2(20\alpha^2 - 17\beta^2), \\ c_2 &= \frac{\beta^2}{6}, & c_6 &= \frac{1}{15120}\beta^2(2520\alpha^4 - 4284\alpha^2\beta^2 + 367\beta^4), \\ c_3 &= -\frac{\alpha\beta^2}{6}, & &\vdots \end{aligned}$$

3. Comparison of means. Asymptotic inequalities

It is natural to define a relation between means based on asymptotic expansion, see [11] for details.

DEFINITION 3.1. Let M_1 and M_2 be any two means, and

$$M_1(x+s, x+t) - M_2(x+s, x+t) = c_k(t,s)x^{-k+1} + \mathcal{O}(x^{-k}). \tag{3.1}$$

If $c_k(s,t) > 0$ for all s and t, then we say that mean M_1 is *asymptotically* greater than mean M_2, and write

$$M_1 \succ M_2.$$

Equivalently, we can say mean M_2 is asymptotically smaller than mean M_1 and write

$$M_2 \prec M_1.$$

We can easily see that asymptotic inequality is a necessary relation between two comparable means.

THEOREM 3.2. *If $M_1 \geqslant M_2$, then $M_1 \succ M_2$.*

Proof. For x large enough, the sign of the difference $M_1(x+s, x+t) - M_2(x+s, x+t)$ is the same as the sign of the first term in its asymptotic expansion. □

Consider the asymptotic expansion of linear combination of three means:

$$M_2(x-\beta, x+\beta) - (1-\mu)M_1(x-\beta, x+\beta) - \mu M_3(x-\beta, x+\beta) \sim$$

$$c_m(\mu)x^{-m+1} + c_{m+1}(\mu)x^{-m} + c_{m+2}(\mu)x^{-m-1}\cdots \tag{3.2}$$

where c_j equals the corresponding combination of j-th coefficients of means M_i, that is

$$c_j(\mu) = M_2^{(j)} - (1-\mu)M_1^{(j)} - \mu M_3^{(j)},\ j \in \mathbb{N}_0, \tag{3.3}$$

and c_m denotes the first coefficient that is not zero function of μ. Since we deal here with comparable means, we shall assume $M_1 \leqslant M_2 \leqslant M_3$. From the Theorem 3.2 and formula (3.3) follows that $c_m(\mu)$ is decreasing. Now suppose that mean M_2 is greater than convex combination of means M_1 and M_3:

$$M_2(s,t) \geqslant (1-\mu)M_1(s,t) + \mu M_3(s,t) \tag{3.4}$$

for all values of arguments (s,t), and let μ be such that $c_m(\mu) = 0$. Taking smaller μ will decrease $c_m(\mu)$ to some negative value and the following asymptotic inequality will hold

$$M_2(s,t) \prec (1-\mu)M_1(s,t) + \mu M_3(s,t) \tag{3.5}$$

which along with the Theorem 3.2 contradicts the inequality (3.4). Hence, μ such that $c_m(\mu) = 0$ is optimal. Analogous conclusion is made for the reverse inequality in (3.4). In the following table optimal values of parameter μ are given.

H	G	L	A	NS	C	Q	N	$\times\beta^4/x^3$	$\geqslant 0$
$4/7$	-1			$3/7$				$11/105$	$+$
$3/7$		-1		$4/7$				$13/210$	$+$
$-1/7$			1	$-6/7$				$17/420$	[26]
$1/8$				-1	$7/8$			$17/360$	[25]
$-2/9$				1		$-7/9$		$1/20$	[28]
$5/12$				-1			$7/12$	$17/360$	[28]
	$-3/4$	1		$-1/4$				$1/60$	$-$
	$-1/4$		1	$-3/4$				$1/15$	$+$
	$1/5$			-1	$4/5$			$1/45$	$-$
	$-1/3$			1		$-2/3$		$7/90$	[28]
	$-5/9$			1			$-4/9$	$1/45$	[24]
		$-1/3$	1	$-2/3$				$11/180$	$+$
		$1/4$		-1	$3/4$			$1/40$	$-$
		$-2/5$		1		$-3/5$		$19/300$	$+$
		$-5/8$		1			$-3/8$	$1/120$	$+$
			$1/2$	-1	$1/2$			$17/360$	[23]
			$2/3$	-1		$1/3$		$1/180$	[20]
			$5/6$	-1			$1/6$	$17/360$	[20]
				$-1/2$	1	$-1/2$		$31/360$	$+$
				$-4/5$	1		$-1/5$	$17/450$	$+$
				$3/5$		-1	$2/5$	$29/300$	[22]

For example, the fourth row reads as

$$\frac{1}{8}H + \frac{7}{8}C - NS \sim \frac{17}{360}\beta^4 x^{-3} + \cdots$$

wherefrom it follows

$$\frac{1}{8}H + \frac{7}{8}C \succ NS. \tag{3.6}$$

There are combinations of means in the table for which the real inequality does not hold such as geometric and logarithmic mean and other marked with the $-$ sign. The reference is given for those inequalities which are proved in literature. Other inequalities, marked with $+$ are proved using computer algebra system (CAS).

By this procedure we get one side of the double inequality (1.7).

Optimal parameter for the other side of that inequality can be derived from the values of means at $(0,1)$. Because of homogeneity and symmetry of means it suffices to observe relations between means on the line segment $\{(s,1-s)\colon s \in [0,1/2]\}$. Now the main inequality (1.7) is equivalent to

$$\mu \leqslant \frac{M_2(s,1-s) - M_1(s,1-s)}{M_3(s,1-s) - M_1(s,1-s)} \leqslant \nu \tag{3.7}$$

and the problem reduces to finding infimum and supremum of the function in the middle. For the most of the combinations of classical means mentioned in the introduction, this function appear to be monotone on the $[0,\frac{1}{2}]$ and takes the minimum and maximum value at the edges of that interval. Therefore, we can impose the condition

$$(1-\nu)M_1(0,1) + \nu M_3(0,1) = M_2(0,1)$$

which makes sense if the value of M_1 differs from value of M_3 at point $(0,1)$. In the following table we give such parameters. Inequalities with those parameters were verified through CAS and $+$ sign stands for those which appeared to be true. Some of them were already proved and the references are given in the introduction and in the previous table. Other still wait for analytic proof. To the best of our knowledge the following inequalities, with the best possible constants, have not been proved yet:

$$H \leqslant G \leqslant \tfrac{4}{7}H + \tfrac{3}{7}NS \tag{3.8}$$

$$H \leqslant L \leqslant \tfrac{3}{7}H + \tfrac{4}{7}NS \tag{3.9}$$

$$\tfrac{1}{4}G + \tfrac{3}{4}NS \leqslant A \leqslant (1-\sigma)G + \sigma NS \tag{3.10}$$

$$\tfrac{1}{3}L + \tfrac{2}{3}NS \leqslant A \leqslant (1-\sigma)L + \sigma NS \tag{3.11}$$

$$\tfrac{2}{5}L + \tfrac{3}{5}Q \leqslant NS \leqslant \tfrac{\sqrt{2}\sigma-1}{\sqrt{2}\sigma}L + \tfrac{1}{\sqrt{2}\sigma}Q \tag{3.12}$$

$$\tfrac{5}{8}L + \tfrac{3}{8}N \leqslant NS \leqslant \tfrac{2\sigma-1}{2\sigma}L + \tfrac{1}{2\sigma}N \tag{3.13}$$

$$\tfrac{1}{2}NS + \tfrac{1}{2}Q \leqslant C \leqslant \tfrac{(3\sqrt{2}-4)\sigma}{3\sqrt{2}\sigma-3}NS + \tfrac{3-4\sigma}{3-3\sqrt{2}\sigma}Q \tag{3.14}$$

$$\tfrac{4}{5}NS + \tfrac{1}{5}N \leqslant C \leqslant \tfrac{2\sigma}{6\sigma-3}NS + \tfrac{3-4\sigma}{3-6\sigma}N \tag{3.15}$$

where $\sigma = \operatorname{arcsinh}(1)$.

H	G	L	A	NS	C	Q	N	$\times\beta^2/x$	$\geqslant 0$
-1	1			0				$\frac{1}{2}$	$+$
-1		1		0				$\frac{2}{3}$	$+$
$1-\sigma$			-1	σ				$\frac{7\sigma-6}{6}$	$+$
$\frac{3-4\sigma}{4\sigma}$				1	$-\frac{3}{4\sigma}$			$\frac{7\sigma-6}{6\sigma}$	$+$
$\frac{\sqrt{2}\sigma-1}{\sqrt{2}\sigma}$				-1		$\frac{1}{\sqrt{2}\sigma}$		$\frac{9\sqrt{2}-14\sigma}{12\sigma}$	$+$
$\frac{1-2\sigma}{2\sigma}$				1			$-\frac{1}{2\sigma}$	$\frac{7\sigma-6}{6\sigma}$	$+$
	-1	1		0				$\frac{1}{6}$	$+$
	$1-\sigma$		-1	σ				$\frac{4\sigma-3}{6}$	$+$
	$\frac{4\sigma-3}{4\sigma}$			-1	$\frac{3}{4\sigma}$			$\frac{15-16\sigma}{24\sigma}$	$+$
	$\frac{\sqrt{2}\sigma-1}{\sqrt{2}\sigma}$			-1		$\frac{1}{\sqrt{2}\sigma}$		$\frac{3\sqrt{2}-4\sigma}{6\sigma}$	$+$
	$\frac{2\sigma-1}{2\sigma}$			-1			$\frac{1}{2\sigma}$	$\frac{9-8\sigma}{12\sigma}$	$+$
		$1-\sigma$	-1	σ				$\frac{3\sigma-2}{6}$	$+$
		$\frac{4\sigma-3}{4\sigma}$		-1	$\frac{3}{4\sigma}$			$\frac{1-\sigma}{2\sigma}$	$+$
		$\frac{\sqrt{2}\sigma-1}{\sqrt{2}\sigma}$		-1		$\frac{1}{\sqrt{2}\sigma}$		$\frac{5\sqrt{2}-6\sigma}{12\sigma}$	$+$
		$\frac{2\sigma-1}{2\sigma}$		-1			$\frac{1}{2\sigma}$	$\frac{4-3\sigma}{6\sigma}$	$+$
			$\frac{3-4\sigma}{\sigma}$	1	$\frac{3\sigma-3}{\sigma}$			$\frac{7\sigma-6}{6\sigma}$	$+$
			$\frac{1-\sqrt{2}\sigma}{\left(\sqrt{2}-1\right)\sigma}$	1		$\frac{(\sigma-1)}{\left(\sqrt{2}-1\right)\sigma}$		$\frac{\left(2+\sqrt{2}\right)\sigma-3}{6(\sqrt{2}-1)\sigma}$	$+$
			$\frac{1-2\sigma}{\sigma}$	1			$\frac{\sigma-1}{\sigma}$	$\frac{7\sigma-6}{6\sigma}$	$+$
				$\frac{\left(3\sqrt{2}-4\right)\sigma}{3\sqrt{2}\sigma-3}$	-1	$\frac{3-4\sigma}{3-3\sqrt{2}\sigma}$		$\frac{\left(8-3\sqrt{2}\right)\sigma-3}{18\left(\sqrt{2}\sigma-1\right)}$	$+$
				$\frac{2\sigma}{6\sigma-3}$	-1		$\frac{3-4\sigma}{3-6\sigma}$	$\frac{7\sigma-6}{18\sigma-9}$	$+$
				$\frac{\left(\sqrt{2}-2\right)\sigma}{2\sigma-1}$		1	$\frac{1-\sqrt{2}\sigma}{2\sigma-1}$	$\frac{3-\left(5\sqrt{2}-4\right)\sigma}{12\sigma-6}$	$+$

For the combination of harmonic and centroidal mean with Neuman-Sándor mean we read from the table:

$$\left(1-\frac{3}{4\sigma}\right)H+\frac{3}{4\sigma}C \prec NS. \tag{3.16}$$

Combining (3.6) with (3.16) suggests that the inequality

$$\left(1-\frac{3}{4\sigma}\right)H+\frac{3}{4\sigma}C \leqslant NS \leqslant \frac{1}{8}H+\frac{7}{8}C \tag{3.17}$$

should be true. Expression like this, with sharp inequalities, was proved in [25]. Now some known inequalities are immediate consequence. Inequality (1.7) is equivalent to

$$\mu \leqslant \frac{M_2(1,t)-M_1(1,t)}{M_3(1,t)-M_1(1,t)} \leqslant \nu. \tag{3.18}$$

Since the means are symmetric, it suffices to consider the case $t \geqslant 1$. Because of the arc-hyperbolic sine function in the definition of Neuman-Sándor mean it is convenient to make a substitution

$$t = \frac{1+\sinh\varphi}{1-\sinh\varphi}, \qquad \varphi \in [0, \operatorname{arcsinh}(1)\rangle\,. \tag{3.19}$$

Let

$$M_{HNSC}(\varphi) = \frac{NS(1,t)-H(1,t)}{C(1,t)-H(1,t)} \tag{3.20}$$

with t defined above. Now we have

$$M_{HNSC}(\varphi) = \frac{\frac{\sinh\varphi}{\varphi(1-\sinh\varphi)} - (\sinh\varphi+1)}{\frac{\sinh\varphi+3}{3(1-\sinh\varphi)} - (\sinh\varphi+1)} = \frac{3}{4}\left(1+\frac{1}{\varphi\sinh\varphi} - \frac{1}{\sinh^2\varphi}\right). \tag{3.21}$$

Denote

$$M(\varphi) = 1+\frac{1}{\varphi\sinh\varphi} - \frac{1}{\sinh^2\varphi} = \frac{4}{3}M_{HNSC}(\varphi) \tag{3.22}$$

Then result from [25]:

$$\frac{3}{4\sigma} < M_{HNSC}(\varphi) < \frac{7}{8}, \quad \varphi \in \langle 0, \operatorname{arcsinh}(1)\rangle \tag{3.23}$$

implies

$$\frac{1}{\sigma} < M(\varphi) < \frac{7}{6}, \quad \varphi \in \langle 0, \operatorname{arcsinh}(1)\rangle\,. \tag{3.24}$$

In the same manner we define functions

$$M_{HANS}(\varphi) = \frac{A(1,t)-H(1,t)}{NS(1,t)-H(1,t)} = \frac{1}{M(\varphi)}, \tag{3.25}$$

$$M_{HNSN}(\varphi) = \frac{NS(1,t)-H(1,t)}{N(1,t)-H(1,t)} = \frac{1}{2}M(\varphi), \tag{3.26}$$

$$M_{ANSC}(\varphi) = \frac{NS(1,t)-A(1,t)}{C(1,t)-A(1,t)} = 3(M(\varphi)-1), \tag{3.27}$$

$$M_{ANSN}(\varphi) = \frac{NS(1,t)-A(1,t)}{N(1,t)-A(1,t)} = M(\varphi)-1, \tag{3.28}$$

$$M_{NSCN}(\varphi) = \frac{C(1,t)-NS(1,t)}{N(1,t)-NS(1,t)} = \frac{4-3M(\varphi)}{6-3M(\varphi)}. \tag{3.29}$$

Inequality (1.5), stated as Theorem 1.3. in [28], follow easily since (3.26) and (3.24) yield

$$\frac{1}{2\sigma} < M_{HNSN}(\varphi) < \frac{7}{12}$$

and further

$$\frac{1}{2\sigma}N(1,t) + \left(1 - \frac{1}{2\sigma}\right)H(1,t) < NS(1,t) < \frac{7}{12}N(1,t) + \frac{5}{12}H(1,t).$$

For the arithmetic and centroidal mean we have

$$3\left(\frac{1}{\sigma} - 1\right) < M_{ANSC}(\varphi) < \frac{1}{2}$$

and

$$3\left(\frac{1}{\sigma} - 1\right)C(1,t) + \left(4 - \frac{3}{\sigma}\right)A(1,t) < NS(1,t) < \frac{1}{2}C(1,t) + \frac{1}{2}A(1,t).$$

Hence, the inequality (1.6) holds.

Similarly,

$$\frac{1}{\sigma} - 1 < M_{ANSN}(\varphi) < \frac{1}{6}$$

gives

$$\left(\frac{1}{\sigma} - 1\right)N(1,t) + \left(2 - \frac{1}{\sigma}\right)A(1,t) < NS(1,t) < \frac{1}{6}N(1,t) + \frac{5}{6}A(1,t)$$

which was proved in [20] as Theorem 3.2. Theorem 6 in [26] is a consequence of (3.25) and (3.24).

Finally, by combining (3.24) and (3.29) we obtain new double inequality for centroidal, Neuman-Sándor and contraharmonic mean.

THEOREM 3.3. *Inequality*

$$(1-\mu)NS(s,t) + \mu N(s,t) < C(s,t) < (1-\nu)NS(s,t) + \nu N(s,t)$$

holds for all $s,t > 0$, *with* $s \neq t$, *if and only if* $\mu \leqslant \frac{1}{5}$ *and* $\nu \geqslant \frac{4\sigma-3}{6\sigma-3}$.

4. Reciprocal of means

Besides the classical combinations we can observe convex combinations of reciprocal of means:

$$\mu\frac{1}{M_1} + (1-\mu)\frac{1}{M_3} \leqslant \frac{1}{M_2} \leqslant \nu\frac{1}{M_1} + (1-\nu)\frac{1}{M_3} \tag{4.1}$$

which cover Neuman-Sándor mean. If the mean M has asymptotic expansion of the form (2.1) with $\alpha = 0$, then by Lemma 2.1 reciprocal of the mean M has the following asymptotic expansion

$$M(x-\beta, x+\beta) \sim x - c_2 x^{-1} + (c_2^2 - c_4)x^{-3} + \cdots.$$

As a consequence, we can find optimal parameter for the one side of the inequality (1.7) in the first table. Same as before, observing linear combinations near point $(0,1)$ will should provide optimal parameter for the other side.

In this case some new inequalities can be proved.

THEOREM 4.1. *Inequality*

$$\mu \frac{1}{H(s,t)} + (1-\mu)\frac{1}{NS(s,t)} < \frac{1}{A(s,t)} < \nu \frac{1}{H(s,t)} + (1-\nu)\frac{1}{NS(s,t)} \tag{4.2}$$

holds for all $s,t \in \langle 0,\infty\rangle$, $s \neq t$, *if and only if* $\mu = 0$ *and* $\frac{1}{7} \leqslant \nu \leqslant 1$.

Proof. Similarly as before, define function $R(\varphi)$

$$R(\varphi) = \frac{1/A(1,t) - 1/NS(1,t)}{1/H(1,t) - 1/NS(1,t)}, \quad \varphi \in \langle 0, \sigma\rangle$$

and t is defined by (3.19). $R(\varphi)$ after arranging becomes

$$R(\varphi) = \frac{(\sinh\varphi - \varphi)(1-\sinh^2\varphi)}{-\varphi + \sinh\varphi + \varphi\sinh^2\varphi} = 1 - \frac{1}{\frac{\varphi}{\sinh\varphi}M(\varphi)}.$$

It is easily seen that $\frac{\varphi}{\sinh\varphi}$ takes values between σ and 1 which together with (3.24) gives

$$0 < R(\varphi) < \frac{1}{7}.$$

Optimality of the parameters is assured by discussion before the theorem. $\square$

THEOREM 4.2. *Inequality*

$$\mu \frac{1}{H(s,t)} + (1-\mu)\frac{1}{N(s,t)} < \frac{1}{NS(s,t)} < \nu \frac{1}{H(s,t)} + (1-\nu)\frac{1}{N(s,t)} \tag{4.3}$$

holds for all $s,t \in \langle 0,\infty\rangle$, *with* $s \neq t$, *if and only if* $\mu = 0$ *and* $\frac{5}{12} \leqslant \nu \leqslant 1$.

Proof. Let

$$R(\varphi) = \frac{1/NS(1,t) + 1/N(1,t)}{1/H(1,t) - 1/N(1,t)}, \quad \varphi \in \langle 0, \sigma\rangle \tag{4.4}$$

where t is defined by (3.19). It is easily seen that

$$\lim_{\varphi\to 0} R(\varphi) = \frac{5}{12} \text{ and } \lim_{\varphi\to\sigma} R(\varphi) = 0. \tag{4.5}$$

It remains to prove R is decreasing.

$$\begin{aligned} R'(\varphi) &= \frac{1}{2\sinh^4(\varphi)}(\sinh\varphi - \sinh^5\varphi + \sinh(2\varphi) - \varphi\cosh\varphi(3+\sinh^4\varphi)) \\ &= \frac{1}{2\sinh^4(\varphi)} g(\varphi) \end{aligned}$$

Taylor series for the function g equals

$$\begin{aligned} g(\varphi) &= \frac{3}{8}\sinh\varphi + \sinh(2\varphi) + \frac{5}{16}\sinh(3\varphi) - \frac{1}{16}\sinh(5\varphi) \\ &\qquad - \varphi\left[\frac{25}{8}\cosh\varphi - \frac{3}{16}\cosh(3\varphi) + \frac{1}{16}\cosh(5\varphi)\right] \\ &= \sum_{n=0}^{\infty}\Bigg[\frac{3}{8} + 2^{2n+1} + \frac{5}{16}3^{2n+1} - \frac{1}{16}5^{2n+1} \\ &\qquad - (2n+1)\left(\frac{25}{8} - \frac{3}{16}3^{2n} + \frac{1}{16}5^{2n}\right)\Bigg]\frac{\varphi^{2n+1}}{(2n+1)!} \end{aligned}$$

The first two coefficients of this series is equal to zero, and for $n \geqslant 2$ we have

$$\begin{aligned} &6 + 32\cdot 2^{2n} + 15\cdot 3^{2n} - 5\cdot 5^{2n} \\ &\qquad < 6 + 32\cdot 2^{2n} + 15\cdot 3^{2n} - 5\cdot 2^{2n} - 15n\cdot 2^{2n} - \frac{20n}{3}\cdot 3^{2n} - 5\cdot 3^{2n} < 0 \end{aligned}$$

and also

$$50 - 3\cdot 3^{2n} + 5^{2n} > 0$$

which makes all coefficients negative. Hence, R is strictly decreasing and the statement of theorem is proved. $\square$

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