

## CERTAIN INEQUALITIES FOR CONVEX FUNCTIONS

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*Abstract.* This is a review paper on some new inequalities for convex functions of one and several variables. The most important result presented for convex functions of one variable is the extension of Jensen's inequality to affine combinations. The most interesting results presented for convex functions of several variables refer to inequalities concerning simplexes and its cones.

### 1. Introduction

We give a short description of the concept of affinity and convexity in a real linear space  $\mathcal{X}$ .

A set  $\mathcal{A} \subseteq \mathcal{X}$  is affine if it contains all binomial affine combinations in  $\mathcal{A}$ , that is, the combinations  $\alpha a + \beta b$  of points  $a, b \in \mathcal{A}$  and coefficients  $\alpha, \beta \in \mathbb{R}$  of the sum  $\alpha + \beta = 1$ . The affine hull of a set  $\mathcal{S} \subseteq \mathcal{X}$  as the smallest affine set that contains  $\mathcal{S}$  is denoted with  $\text{aff}\mathcal{S}$ . A function  $f: \mathcal{A} \rightarrow \mathbb{R}$  is affine if the equality

$$f(\alpha a + \beta b) = \alpha f(a) + \beta f(b) \quad (1)$$

holds for all binomial affine combinations  $\alpha a + \beta b$  of points  $a, b \in \mathcal{A}$ .

A set  $\mathcal{C} \subseteq \mathcal{X}$  is convex if it contains all binomial convex combinations in  $\mathcal{C}$ , that is, the combinations  $\alpha a + \beta b$  of points  $a, b \in \mathcal{C}$  and non-negative coefficients  $\alpha, \beta \in \mathbb{R}$  of the sum  $\alpha + \beta = 1$ . The convex hull of a set  $\mathcal{S} \subseteq \mathcal{X}$  as the smallest convex set that contains  $\mathcal{S}$  is denoted with  $\text{conv}\mathcal{S}$ . A function  $f: \mathcal{C} \rightarrow \mathbb{R}$  is convex if the inequality

$$f(\alpha a + \beta b) \leq \alpha f(a) + \beta f(b) \quad (2)$$

holds for all binomial convex combinations  $\alpha a + \beta b$  of points  $a, b \in \mathcal{C}$ .

Using mathematical induction, it can be proved that the above concept applies to all  $n$ -membered affine or convex combinations. Relying on induction, Jensen (see [2]) has extended the inequality in equation (2).

### 2. Generalization and reversal of the Jensen-Mercer inequality

The section contains the parts of paper [9].

We consider inequalities that are close to the Jensen-Mercer inequality obtained in [4]. The generalization of this inequality is given in Corollary 1, and the reverse inequality is specified in Corollary 2.

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If  $a, b \in \mathbb{R}$  are different points, then every point  $x \in \mathbb{R}$  can be presented by the unique affine combination

$$x = \frac{b-x}{b-a}a + \frac{x-a}{b-a}b. \quad (3)$$

The above combination is convex if, and only if, the point  $x$  belongs to the closed interval  $\text{conv}\{a, b\}$ . Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a convex function, and let  $f_{\{a,b\}}^{\text{line}}: \mathbb{R} \rightarrow \mathbb{R}$  be the function of the secant line passing through the graph points  $A(a, f(a))$  and  $B(b, f(b))$ . Applying the affinity of  $f_{\{a,b\}}^{\text{line}}$  to the combination in equation (3), we get the secant equation

$$f_{\{a,b\}}^{\text{line}}(x) = \frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b). \quad (4)$$

Combining the convexity of  $f$  and the affinity of  $f_{\{a,b\}}^{\text{line}}$ , we obtain the inequality

$$f(x) \leq f_{\{a,b\}}^{\text{line}}(x) \text{ if } x \in \text{conv}\{a, b\}, \quad (5)$$

and the reverse inequality

$$f(x) \geq f_{\{a,b\}}^{\text{line}}(x) \text{ if } x \notin \text{conv}\{a, b\} \setminus \{a, b\}. \quad (6)$$

Assume that  $\text{conv}\{a, b\} = [a, b]$ . A point  $x$  of the ray  $(-\infty, a]$  can be presented by the unique affine combination  $x = (1+p)a - pb$  with the nonnegative coefficient  $p = (a-x)/(b-a)$ . According to equation (6), it follows that

$$f((1+p)a - pb) \geq (1+p)f(a) - pf(b). \quad (7)$$

Replacing  $a$  and  $b$ , we have the presentation and the inequality for a point  $x$  of the ray  $[b, +\infty)$ .

**LEMMA 1.** *Let  $\alpha, \beta, \gamma \in [0, 1]$  be coefficients such that  $\alpha + \beta - \gamma = 1$ . Let  $a, b, c \in \mathbb{R}$  be points such that  $c \in \text{conv}\{a, b\}$ .*

*Then the affine combination  $\alpha a + \beta b - \gamma c$  is in  $\text{conv}\{a, b\}$ , and every convex function  $f: \text{conv}\{a, b\} \rightarrow \mathbb{R}$  satisfies the inequality*

$$f(\alpha a + \beta b - \gamma c) \leq \alpha f(a) + \beta f(b) - \gamma f(c). \quad (8)$$

*Proof.* Involving the convex combination  $c = \kappa a + \lambda b$ , it follows that

$$\alpha a + \beta b - \gamma c = [\alpha(1-\kappa) + \kappa(1-\beta)]a + [\beta(1-\lambda) + \lambda(1-\alpha)]b. \quad (9)$$

The coefficients in square parentheses are nonnegative with the sum equal to 1. So, the right-hand side of the above equality is the convex combination of points  $a$  and  $b$ , and therefore the combination  $\alpha a + \beta b - \gamma c$  belongs to  $\text{conv}\{a, b\}$ .

If  $a = b$ , the inequality in equation (8) takes the trivial form  $f(a) \leq f(a)$ . If  $a \neq b$ , then applying the convexity of  $f$  and affinity of  $f_{\{a,b\}}^{\text{line}}$ , we get

$$f(\alpha a + \beta b - \gamma c) \leq f_{\{a,b\}}^{\text{line}}(\alpha a + \beta b - \gamma c) \quad (10)$$

$$= \alpha f(a) + \beta f(b) - \gamma f_{\{a,b\}}^{\text{line}}(c) \quad (11)$$

$$\leq \alpha f(a) + \beta f(b) - \gamma f(c) \quad (12)$$

finishing the proof.

**COROLLARY 1.** *Let  $\alpha, \beta, \gamma \in [0, 1]$  and  $\gamma_i \in [0, 1]$  be coefficients such that  $\alpha + \beta - \gamma = \sum_{i=1}^n \gamma_i = 1$ . Let  $a, b, c_i \in \mathbb{R}$  be points such that all  $c_i \in \text{conv}\{a, b\}$ .*

*Then the affine combination  $\alpha a + \beta b - \gamma \sum_{i=1}^n \gamma_i c_i$  is in  $\text{conv}\{a, b\}$ , and every convex function  $f : \text{conv}\{a, b\} \rightarrow \mathbb{R}$  satisfies the inequality*

$$f\left(\alpha a + \beta b - \gamma \sum_{i=1}^n \gamma_i c_i\right) \leq \alpha f(a) + \beta f(b) - \gamma \sum_{i=1}^n \gamma_i f(c_i). \tag{13}$$

If  $\alpha = \beta = \gamma = 1$ , then the inequality in equation (13) is reduced to Mercer’s variant of Jensen’s inequality obtained in [4]. Another generalization of Mercer’s result was achieved in [6] using the majorization assumptions.

**LEMMA 2.** *Let  $\alpha, \beta, \gamma \in [1, \infty)$  be coefficients such that  $\alpha + \beta - \gamma = 1$ . Let  $a, b, c \in \mathbb{R}$  be points such that  $c \notin \text{conv}\{a, b\} \setminus \{a, b\}$ .*

*Then the affine combination  $\alpha a + \beta b - \gamma c$  is not in  $\text{conv}\{a, b\} \setminus \{a, b\}$ , and every convex function  $f : \text{conv}\{a, b, c\} \rightarrow \mathbb{R}$  satisfies the inequality*

$$f(\alpha a + \beta b - \gamma c) \geq \alpha f(a) + \beta f(b) - \gamma f(c). \tag{14}$$

*Proof.* Involving the affine combination  $c = \kappa a + \lambda b$ , we have the presentation

$$\alpha a + \beta b - \gamma c = (\alpha - \gamma \kappa)a + (\beta - \gamma \lambda)b, \tag{15}$$

where the right-hand side is the affine combination of points  $a$  and  $b$ . The condition  $\kappa a + \lambda b \notin \text{conv}\{a, b\} \setminus \{a, b\}$  implies that  $\kappa \leq 0$  or  $\kappa \geq 1$ . If  $\kappa \leq 0$ , then  $\alpha - \gamma \kappa \geq \alpha \geq 1$ . If  $\kappa \geq 1$ , then  $\alpha - \gamma \kappa \leq \alpha - \gamma = 1 - \beta \leq 0$ . So, the combination  $\alpha a + \beta b - \gamma c$  does not belong to  $\text{conv}\{a, b\} \setminus \{a, b\}$  by equation (15). Applying the inequality in equation (6), we get the series of inequalities containing the inequality in equation (14).

**COROLLARY 2.** *Let  $\alpha, \beta, \gamma \in [1, \infty)$  and  $\gamma_i \in [0, 1]$  be coefficients such that  $\alpha + \beta - \gamma = \sum_{i=1}^n \gamma_i = 1$ . Let  $a, b, c_i \in \mathbb{R}$  be points such that any  $c_i \notin \text{conv}\{a, b\} \setminus \{a, b\}$  and the convex combination  $\sum_{i=1}^n \gamma_i c_i \notin \text{conv}\{a, b\} \setminus \{a, b\}$ .*

*Then the affine combination  $\alpha a + \beta b - \gamma \sum_{i=1}^n \gamma_i c_i$  is not in  $\text{conv}\{a, b\} \setminus \{a, b\}$ , and every convex function  $f : \text{conv}\{a, b, c_i\} \rightarrow \mathbb{R}$  satisfies the inequality*

$$f\left(\alpha a + \beta b - \gamma \sum_{i=1}^n \gamma_i c_i\right) \geq \alpha f(a) + \beta f(b) - \gamma \sum_{i=1}^n \gamma_i f(c_i). \tag{16}$$

### 3. Inequalities with affine combinations

This section is prepared according to paper [8].

### 3.1. Extension of Jensen’s inequality to affine combinations

Relying on findings of the previous section, we can expose the extension of Jensen’s inequality to affine combinations.

**THEOREM 1.** *Let  $\alpha_i, \beta_j, \gamma_k \geq 0$  be coefficients such that their sums  $\alpha = \sum_{i=1}^n \alpha_i$ ,  $\beta = \sum_{j=1}^m \beta_j$ ,  $\gamma = \sum_{k=1}^l \gamma_k$  satisfy  $\alpha + \beta - \gamma = 1$  and  $\alpha, \beta \in (0, 1]$ . Let  $a_i, b_j, c_k \in \mathbb{R}$  be points such that  $c_k \in \text{conv}\{a, b\}$ , where*

$$a = \frac{1}{\alpha} \sum_{i=1}^n \alpha_i a_i, \quad b = \frac{1}{\beta} \sum_{j=1}^m \beta_j b_j. \tag{17}$$

Then the affine combination

$$\sum_{i=1}^n \alpha_i a_i + \sum_{j=1}^m \beta_j b_j - \sum_{k=1}^l \gamma_k c_k \tag{18}$$

belongs to  $\text{conv}\{a, b\}$ , and every convex function  $f : \text{conv}\{a_i, b_j\} \rightarrow \mathbb{R}$  satisfies the inequality

$$f\left(\sum_{i=1}^n \alpha_i a_i + \sum_{j=1}^m \beta_j b_j - \sum_{k=1}^l \gamma_k c_k\right) \leq \sum_{i=1}^n \alpha_i f(a_i) + \sum_{j=1}^m \beta_j f(b_j) - \sum_{k=1}^l \gamma_k f(c_k). \tag{19}$$

*Proof.* Since  $\alpha = 1 - \beta + \gamma$ , we have that  $\alpha \geq \gamma$ , and similarly  $\beta \geq \gamma$ .

If  $\gamma = 0$ , the combination in equation (18) takes the convex form  $\alpha a + \beta b$  belonging to  $\text{conv}\{a, b\}$ , and the inequality in equation (19) is reduced to Jensen’s inequality.

If  $\gamma > 0$ , then including points  $a, b$  and

$$c = \frac{1}{\gamma} \sum_{k=1}^l \gamma_k c_k \tag{20}$$

in equation (18), we get the combination  $\alpha a + \beta b - \gamma c$  which belongs to  $\text{conv}\{a, b\}$  by Lemma 1. The inequality in equation (19) is trivially true for  $a = b$ . So, we assume that  $a \neq b$  and use the function  $f_{\{a,b\}}^{\text{line}}$ . Applying the affinity of  $f_{\{a,b\}}^{\text{line}}$  to the convex combination in equation (20), and respecting the inequalities  $f_{\{a,b\}}^{\text{line}}(c_k) \geq f(c_k)$ , we have

$$f_{\{a,b\}}^{\text{line}}(c) = \frac{1}{\gamma} \sum_{k=1}^l \gamma_k f_{\{a,b\}}^{\text{line}}(c_k) \geq \frac{1}{\gamma} \sum_{k=1}^l \gamma_k f(c_k). \tag{21}$$

Using the inequality in equation (11), applying Jensen’s inequality to  $f(a)$  and  $f(b)$ , and finally using the inequality in equation (21) respecting minus, we get

$$\begin{aligned} f\left(\sum_{i=1}^n \alpha_i a_i + \sum_{j=1}^m \beta_j b_j - \sum_{k=1}^l \gamma_k c_k\right) &= f(\alpha a + \beta b - \gamma c) \\ &\leq \alpha f(a) + \beta f(b) - \gamma f_{\{a,b\}}^{\text{line}}(c) \\ &\leq \sum_{i=1}^n \alpha_i f(a_i) + \sum_{j=1}^m \beta_j f(b_j) - \sum_{k=1}^l \gamma_k f(c_k) \end{aligned}$$

completing the proof.

A brief scientific-historic background on Jensen’s inequality follows at the end of this subsection. Because of its attractiveness, Jensen’s and related inequalities were studied during the whole last century. So, there are Steffensen’s, Brunk’s and Olkin’s inequality. In this century the research goes on, and we got the Jensen-Mercer and the Mercer-Steffensen inequality. For information as regards these inequalities, one may refer to papers [11], [1], [4], [12] and [9].

A wide area of convex analysis including convex functions and their inequalities is covered in [13]. The practical applications of convex analysis are presented in [16].

### 3.2. Application to quasi-arithmetic means

We assume that

$$\bar{x} = \sum_{i=1}^n \alpha_i a_i + \sum_{j=1}^m \beta_j b_j - \sum_{k=1}^l \gamma_k c_k \tag{22}$$

is an affine combination as in Theorem 1, and  $\varphi : \mathcal{S} \rightarrow \mathbb{R}$  is a strictly monotone continuous function where  $\mathcal{S} = \text{conv}\{a_i, b_j\}$ . The discrete  $\varphi$ -quasi-arithmetic mean of the combination  $\bar{x}$  can be defined as the point

$$M_\varphi(\bar{x}) = \varphi^{-1} \left( \sum_{i=1}^n \alpha_i \varphi(a_i) + \sum_{j=1}^m \beta_j \varphi(b_j) - \sum_{k=1}^l \gamma_k \varphi(c_k) \right) \tag{23}$$

belonging to  $\text{conv}\{a, b\}$ , because the affine combination enclosed in parentheses is located in  $\varphi(\text{conv}\{a, b\})$ .

The order of pair of quasi-arithmetic means  $M_\varphi$  and  $M_\psi$  depends on convexity of the function  $\psi \circ \varphi^{-1}$  and monotonicity of the function  $\psi$ , as follows.

**COROLLARY 3.** *Let  $\bar{x}$  be an affine combination as in equation (22) satisfying all the assumptions of Theorem 1. Let  $\varphi, \psi : \mathcal{S} \rightarrow \mathbb{R}$  be strictly monotone continuous functions where  $\mathcal{S} = \text{conv}\{a_i, b_j\}$ .*

*If  $\psi$  is either  $\varphi$ -convex and increasing or  $\varphi$ -concave and decreasing, then we have the inequality*

$$M_\varphi(\bar{x}) \leq M_\psi(\bar{x}). \tag{24}$$

*If  $\psi$  is either  $\varphi$ -convex and decreasing or  $\varphi$ -concave and increasing, then we have the reverse inequality in equation (24).*

*If  $\psi$  is  $\varphi$ -affine, then the equality is valid in equation (24).*

*Proof.* Prove the case that the function  $\psi$  is  $\varphi$ -convex and increasing. Put the set  $\mathcal{S} = \varphi(\mathcal{S}) = \text{conv}\{\varphi(a_i), \varphi(b_j)\}$ . Applying the inequality in equation (19) to the affine combination

$$\bar{x}_\varphi = \sum_{i=1}^n \alpha_i \varphi(a_i) + \sum_{j=1}^m \beta_j \varphi(b_j) - \sum_{k=1}^l \gamma_k \varphi(c_k) \tag{25}$$

which is in the set  $\varphi(\text{conv}\{a, b\})$ , and the convex function  $f = \psi \circ \varphi^{-1} : \mathcal{J} \rightarrow \mathbb{R}$ , we get

$$\psi \circ \varphi^{-1}(\bar{x}_\varphi) \leq \bar{x}_\psi.$$

Assigning the increasing function  $\psi^{-1}$  to the above inequality, we attain

$$M_\varphi(\bar{x}) = \varphi^{-1}(\bar{x}_\varphi) \leq \psi^{-1}(\bar{x}_\psi) = M_\psi(\bar{x})$$

which finishes the proof.

The inequality in equation (24) may further be applied to the power means. The monotonicity of these power means is also valid. The harmonic-geometric-arithmetic mean inequality for these means is as follows.

**COROLLARY 4.** *Let  $\bar{x}$  be an affine combination as in equation (22) satisfying all assumptions of Theorem 1 with the addition that all  $a_i, b_j > 0$ .*

*Then we have the harmonic-geometric-arithmetic mean inequality*

$$\begin{aligned} \left( \sum_{i=1}^n \frac{\alpha_i}{a_i} + \sum_{j=1}^m \frac{\beta_j}{b_j} - \sum_{k=1}^l \frac{\gamma_k}{c_k} \right)^{-1} &\leq \prod_{i=1}^n a_i^{\alpha_i} \prod_{j=1}^m b_j^{\beta_j} \prod_{k=1}^l c_k^{-\gamma_k} \\ &\leq \sum_{i=1}^n \alpha_i a_i + \sum_{j=1}^m \beta_j b_j - \sum_{k=1}^l \gamma_k c_k. \end{aligned} \tag{26}$$

### 3.2.1. Application to other inequalities

Applying the mean inequality in equation (26), we get that the inequality

$$\left( \frac{\alpha}{a} + \frac{\beta}{b} - \frac{\gamma}{c} \right)^{-1} \leq \frac{a^\alpha b^\beta}{c^\gamma} \leq \alpha a + \beta b - \gamma c \tag{27}$$

holds for coefficients  $\alpha, \beta, \gamma \in [0, 1]$  such that  $\alpha + \beta - \gamma = 1$ , and points  $a, b, c > 0$  such that  $c \in \text{conv}\{a, b\}$ .

As a consequence of equation (27), Bernoulli’s inequalities and Young’s inequality can be extended on the left.

**EXAMPLE 1.** The inequality in equation (27) can be arranged to the inequality

$$\frac{1+x}{1+x-px} \leq (1+x)^p \leq 1+px, \tag{28}$$

which holds for coefficients  $p \in [0, 1]$  and points  $x > -1$ . The reverse inequality is valid for  $p \in [1, \infty)$  and  $x > -1$ .

**EXAMPLE 2.** The inequality in equation (27) can be arranged to the inequality

$$\left( \frac{x^{-p}}{p} + \frac{y^{-q}}{q} \right)^{-1} \leq xy \leq \frac{x^p}{p} + \frac{y^q}{q}, \tag{29}$$

which holds for coefficients  $p, q \in (1, \infty)$  such that  $1/p + 1/q = 1$ , and points  $x, y > 0$ .

### 4. Inequalities on simplexes and their cones

This section is conceived according to paper [10].

#### 4.1. Inequalities on the plane

Let  $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$  be points. We use the standard coordinate addition  $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ , and the scalar multiplication  $\alpha(x, y) = (\alpha x, \alpha y)$ . If  $A(x_A, y_A), B(x_B, y_B)$  and  $C(x_C, y_C)$  are planar points that do not belong to one line, then every point  $P(x, y) \in \mathbb{R}^2$  can be presented by the unique affine combination

$$P = \alpha A + \beta B + \gamma C, \tag{30}$$

where

$$\alpha = \frac{\begin{vmatrix} x & y & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}}{\begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}}, \beta = -\frac{\begin{vmatrix} x & y & 1 \\ x_A & y_A & 1 \\ x_C & y_C & 1 \end{vmatrix}}{\begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}}, \gamma = \frac{\begin{vmatrix} x & y & 1 \\ x_A & y_A & 1 \\ x_B & y_B & 1 \end{vmatrix}}{\begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}}. \tag{31}$$

The above trinomial combination is convex if, and only if, the point  $P$  belongs to the triangle  $\text{conv}\{A, B, C\}$ .

Let  $\mathcal{C}_A$  be the convex cone with the vertex at  $A$  spanned by the vectors  $A - B$  and  $A - C$  containing trinomial affine combinations  $P = A + p(A - B) + q(A - C) = (1 + p + q)A - pB - qC$  where  $p, q \geq 0$ , that is,

$$\mathcal{C}_A = \{(1 + p + q)A - pB - qC : p, q \geq 0\}.$$

Cones  $\mathcal{C}_B$  and  $\mathcal{C}_C$  are defined in the same way.

Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a convex function, and let  $f_{\{A,B,C\}}^{\text{plane}} : \mathbb{R}^2 \rightarrow \mathbb{R}$  be the function of the plane passing through the graph points  $(A, f(A)), (B, f(B))$  and  $(C, f(C))$  of the graph of  $f$ . Because of the affinity of  $f_{\{A,B,C\}}^{\text{plane}}$ , it follows that

$$f_{\{A,B,C\}}^{\text{plane}}(P) = \alpha f(A) + \beta f(B) + \gamma f(C). \tag{32}$$

The following is the basic lemma.

LEMMA 3. *Let  $A, B, C \in \mathbb{R}^2$  be the triangle vertices.*

*Then every convex function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  satisfies the inequality*

$$f(P) \leq f_{\{A,B,C\}}^{\text{plane}}(P) \text{ if } P \in \text{conv}\{A, B, C\}, \tag{33}$$

and the reverse inequality

$$f(P) \geq f_{\{A,B,C\}}^{\text{plane}}(P) \text{ if } P \in \mathcal{C}_A \cup \mathcal{C}_B \cup \mathcal{C}_C. \tag{34}$$

*Proof.* If  $P \in \text{conv}\{A, B, C\}$ , the combination in (30) is convex. Applying Jensen's inequality, and using plane's equation in (32), we obtain

$$f(P) \leq \alpha f(A) + \beta f(B) + \gamma f(C) = f_{\{A,B,C\}}^{\text{plane}}(P). \tag{35}$$

If  $P \in \mathcal{C}_A \cup \mathcal{C}_B \cup \mathcal{C}_C$ , say  $P \in \mathcal{C}_A$  and  $P \neq A$ , we can represent the point  $P$  as the binomial affine combination

$$P = (1 + p + q)A - pB - qC \tag{36}$$

$$= (1 + p + q)A - (p + q) \left( \frac{p}{p + q}B + \frac{q}{p + q}C \right). \tag{37}$$

The sum  $p + q$  is positive because  $P \neq A$ . The term in (37) can be thought of as the ray with the origin at  $A$ . Applying the line inequality in equation (7) to the ray in (37), then using the convexity of  $f$  and the affinity of  $f_{\{A,B,C\}}^{\text{plane}}$ , we get the series of inequalities

$$\begin{aligned} f(P) &\geq (1 + p + q)f(A) - (p + q)f\left(\frac{p}{p + q}B + \frac{q}{p + q}C\right) \\ &\geq (1 + p + q)f(A) - pf(B) - qf(C) \\ &= f_{\{A,B,C\}}^{\text{plane}}((1 + p + q)A - pB - qC) \\ &= f_{\{A,B,C\}}^{\text{plane}}(P) \end{aligned}$$

which includes the inequality in equation (34).

The following are inequalities for convex functions and planar convex combinations with the common center.

**COROLLARY 5.** *Let  $A, B, C \in \mathbb{R}^2$  be the triangle vertices. Let  $\sum_{i=1}^n \alpha_i A_i$  be a convex combination of points  $A_i \in \text{conv}\{A, B, C\}$ , and let  $\sum_{j=1}^m \beta_j B_j$  be a convex combination of points  $B_j \in \mathcal{C}_A \cup \mathcal{C}_B \cup \mathcal{C}_C$  such that*

$$P = \sum_{i=1}^n \alpha_i A_i = \sum_{j=1}^m \beta_j B_j. \tag{38}$$

*Then every convex function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  satisfies the inequality*

$$f(P) \leq \sum_{i=1}^n \alpha_i f(A_i) \leq \sum_{j=1}^m \beta_j f(B_j). \tag{39}$$

*Proof.* The left inequality in equation (39) is the Jensen inequality. The right inequality follows from Lemma 3.1, and the affinity of the function  $f_{\{A,B,C\}}^{\text{plane}}$ .



COROLLARY 6. Let  $A, B, C \in \mathbb{R}^2$  be the triangle vertices. Let  $\sum_{i=1}^n \alpha_i A_i$  be a convex combination of points  $A_i \in \text{conv}\{A, B, C\}$ , and let  $\alpha A + \beta B + \gamma C$  be the unique convex combination such that

$$P = \sum_{i=1}^n \alpha_i A_i = \alpha A + \beta B + \gamma C. \tag{40}$$

Then every convex function  $f : \text{conv}\{A, B, C\} \rightarrow \mathbb{R}$  satisfies the inequality

$$f(P) \leq \sum_{i=1}^n \alpha_i f(A_i) \leq \alpha f(A) + \beta f(B) + \gamma f(C). \tag{41}$$

**4.2. Generalization to higher dimensions**

The results of the previous section can be generalized to higher dimensions by using simplexes. Let  $S_1, \dots, S_{r+1} \in \mathbb{R}^r$  be points. Their convex hull

$$\mathcal{S} = \text{conv}\{S_1, \dots, S_{r+1}\}. \tag{42}$$

is the  $r$ -simplex in the space  $\mathbb{R}^r$  if the points  $S_1 - S_{r+1}, \dots, S_r - S_{r+1}$  are linearly independent. Every point  $P \in \mathbb{R}^r$  can be presented by the unique affine combination

$$P = \sum_{k=1}^{r+1} \alpha_k S_k, \tag{43}$$

where the coefficients  $\alpha_k$  can be determined by generalizing the coefficients in (31). The combination in (43) is convex if and only if the point  $P$  belongs to the  $r$ -simplex  $\text{conv}\{S_1, \dots, S_{r+1}\}$ .

Given the function  $f : \mathbb{R}^r \rightarrow \mathbb{R}$ , let  $f_{\{S_1, \dots, S_{r+1}\}}^{\text{hyperplane}} : \mathbb{R}^r \rightarrow \mathbb{R}$  be the function of the hyperplane (in  $\mathbb{R}^{r+1}$ ) passing through the graph points. Then we have

$$f_{\{S_1, \dots, S_{r+1}\}}^{\text{hyperplane}}(P) = \sum_{k=1}^{r+1} \alpha_k f(S_k). \tag{44}$$

Let  $\mathcal{C}_k$  ( $k = 1, \dots, r + 1$ ) be the convex cone with the vertex at  $S_k$  spanned by the vectors  $S_k - S_j$  for  $k \neq j = 1, \dots, r + 1$  containing  $(r + 1)$ -membered affine combinations  $P = S_k + \sum_{k \neq j=1}^{r+1} p_j (S_k - S_j) = \left(1 + \sum_{k \neq j=1}^{r+1} p_j\right) S_k - \sum_{k \neq j=1}^{r+1} p_j S_j$  where all  $p_j \geq 0$ , that is,

$$\mathcal{C}_k = \left\{ \left(1 + \sum_{k \neq j=1}^{r+1} p_j\right) S_k - \sum_{k \neq j=1}^{r+1} p_j S_j : p_j \geq 0 \right\}.$$

LEMMA 4. Let  $\mathcal{S} = \text{conv}\{S_1, \dots, S_{r+1}\}$  be an  $r$ -simplex in the space  $\mathbb{R}^r$ . Then every convex function  $f : \mathbb{R}^r \rightarrow \mathbb{R}$  satisfies the inequality

$$f(P) \leq f_{\{S_1, \dots, S_{r+1}\}}^{\text{hyperplane}}(P) \text{ if } P \in \mathcal{S}, \tag{45}$$

and the reverse inequality

$$f(P) \geq f_{\{S_1, \dots, S_{r+1}\}}^{\text{hyperplane}}(P) \text{ if } P \in \mathcal{C}_1 \cup \dots \cup \mathcal{C}_{r+1}. \tag{46}$$

*Proof.* To prove equations (45) and (46), we adapt the proof of Lemma 3 as follows.

To prove equation (45), we firstly apply Jensen’s inequality to the convex combination  $P = \sum_{k=1}^{r+1} \alpha_k P_k \in \text{conv}\{S_1, \dots, S_{r+1}\}$ , and then we use the hyperplane equation in (44).

To prove equation (46) for  $P \in \mathcal{C}_k$  other than  $S_k$ , we firstly implement the ray inequality in equation (7) to the binomial affine combination

$$\begin{aligned} P &= \left(1 + \sum_{k \neq j=1}^{r+1} p_j\right) S_k - \sum_{k \neq j=1}^{r+1} p_j S_j \\ &= (1 + p)S_k - pP_k \end{aligned}$$

where  $p = \sum_{k \neq j=1}^{r+1} p_j$  and  $P_k = \sum_{k \neq j=1}^{r+1} (p_j/p)S_j$ , then we apply Jensen’s inequality to the convex combination of  $P_k$ , and thus obtain

$$f(P) \geq \left(1 + \sum_{k \neq j=1}^{r+1} p_j\right) f(S_k) - \sum_{k \neq j=1}^{r+1} p_j f(S_j) = f_{\{S_1, \dots, S_{r+1}\}}^{\text{hyperplane}}(P), \tag{47}$$

which is the desired inequality.

Relying on Lemma 4, we get the generalization of Corollary 5 to higher dimensions.

**COROLLARY 7.** *Let  $\mathcal{S} = \text{conv}\{S_1, \dots, S_{r+1}\}$  be an  $r$ -simplex in the space  $\mathbb{R}^r$ . Let  $\sum_{i=1}^n \alpha_i A_i$  be a convex combination of points  $A_i \in \mathcal{S}$ , and let  $\sum_{j=1}^m \beta_j B_j$  be a convex combination of points  $B_j \in \mathcal{C}_1 \cup \dots \cup \mathcal{C}_{r+1}$  such that*

$$P = \sum_{i=1}^n \alpha_i A_i = \sum_{j=1}^m \beta_j B_j. \tag{48}$$

*Then every convex function  $f : \mathbb{R}^r \rightarrow \mathbb{R}$  satisfies the inequality*

$$f(P) \leq \sum_{i=1}^n \alpha_i f(A_i) \leq \sum_{j=1}^m \beta_j f(B_j) \tag{49}$$

**COROLLARY 8.** *Let  $\mathcal{S} = \text{conv}\{S_1, \dots, S_{r+1}\}$  be an  $r$ -simplex in the space  $\mathbb{R}^r$ . Let  $\sum_{i=1}^n \alpha_i A_i$  be a convex combination of points  $A_i \in \mathcal{S}$ , and let  $\sum_{j=1}^{r+1} \beta_j S_j$  be the unique convex combination such that*

$$P = \sum_{i=1}^n \alpha_i A_i = \sum_{j=1}^{r+1} \beta_j S_j. \tag{50}$$

*Then every convex function  $f : \mathcal{S} \rightarrow \mathbb{R}$  satisfies the inequality*

$$f(P) \leq \sum_{i=1}^n \alpha_i f(A_i) \leq \sum_{j=1}^{r+1} \beta_j f(S_j). \tag{51}$$

Implementing the integral method with convex combinations to the symmetric form of the discrete inequality in equation (51),

$$f\left(\sum_{j=1}^{r+1} \beta_j S_j\right) \leq \sum_{i=1}^n \alpha_i f(A_i) \leq \sum_{j=1}^{r+1} \beta_j f(S_j), \tag{52}$$

we obtain the Hermite-Hadamard inequality

$$f\left(\frac{1}{r+1} \sum_{j=1}^{r+1} S_j\right) \leq \frac{1}{\text{vol}(\mathcal{S})} \int_{\mathcal{S}} f(x_1, \dots, x_r) dx_1 \dots dx_r \leq \frac{1}{r+1} \sum_{j=1}^{r+1} f(S_j). \tag{53}$$

For information as regards the Hermite-Hadamard inequality, one may refer to paper [5].

### 5. Inequalities with positive linear functionals

This section is prepared respecting paper [7].

Let  $X$  be a non-empty set, and let  $\mathbb{X}$  be a subspace of the linear space of all real functions on the domain  $X$ . We assume that  $\mathbb{X}$  contains the unit function  $I$  defined by  $I(x) = 1$  for every  $x \in X$ .

Let  $\mathcal{S} \subseteq \mathbb{R}$  be an interval, and let  $\mathbb{X}_{\mathcal{S}}$  be the set containing all functions  $g \in \mathbb{X}$  with the image in  $\mathcal{S}$ . Then  $\mathbb{X}_{\mathcal{S}}$  is convex set in the space  $\mathbb{X}$ . The same is true for convex sets of Euclidean spaces. Let  $\mathcal{C} \subseteq \mathbb{R}^r$  be a convex set, and let  $(\mathbb{X}^r)_{\mathcal{C}}$  be the set containing all function  $r$ -tuples  $g = (g_1, \dots, g_r) \in \mathbb{X}^r$  with the image in  $\mathcal{C}$ . Then  $(\mathbb{X}^r)_{\mathcal{C}}$  is convex set in the space  $\mathbb{X}^r$ .

A linear functional  $L : \mathbb{X} \rightarrow \mathbb{R}$  is positive (nonnegative) if  $L(g) \geq 0$  for every non-negative function  $g \in \mathbb{X}$ , and  $L$  is unital (normalized) if  $L(I) = 1$ . If  $g \in \mathbb{X}$ , then for every unital positive functional  $L$  the number  $L(g)$  is in the closed interval of real numbers containing the image of  $g$ . Through the paper, the space of all linear functionals on the space  $\mathbb{X}$  will be denoted with  $\mathbb{L}(\mathbb{X})$ .

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be an affine function,  $f(x) = \kappa x + \lambda$  where  $\kappa$  and  $\lambda$  are real constants. If  $g_1, \dots, g_n \in \mathbb{X}$  are functions, and if  $L_1, \dots, L_n \in \mathbb{L}(\mathbb{X})$  are positive functionals providing the unit equality

$$\sum_{i=1}^n L_i(I) = 1, \tag{54}$$

then

$$\begin{aligned} f\left(\sum_{i=1}^n L_i(g_i)\right) &= \kappa \sum_{i=1}^n L_i(g_i) + \lambda \sum_{i=1}^n L_i(I) = \sum_{i=1}^n L_i(\kappa g_i + \lambda I) \\ &= \sum_{i=1}^n L_i(f(g_i)). \end{aligned} \tag{55}$$

Respecting the requirement of unit equality in equation (54), the sum  $\sum_{i=1}^n L_i(g_i)$  could be called the functional convex combination. In the case  $n = 1$ , the functional  $L = L_1$  must be unital by the unit equality in equation (54).

In 1931, Jessen stated the functional form of Jensen's inequality for convex functions of one variable, see [3]. Adapted to our purposes, that statement is as follows.

**THEOREM 2.** *Let  $\mathcal{I} \subseteq \mathbb{R}$  be a closed interval, and let  $g \in \mathbb{X}_{\mathcal{I}}$  be a function. Then a unital positive functional  $L \in \mathbb{L}(\mathbb{X})$  ensures the inclusion*

$$L(g) \in \mathcal{I}, \quad (56)$$

and satisfies the inequality

$$f(L(g)) \leq L(f(g)) \quad (57)$$

for every continuous convex function  $f : \mathcal{I} \rightarrow \mathbb{R}$  providing that  $f(g) \in \mathbb{X}$ .

The interval  $\mathcal{I}$  must be closed, otherwise it could happen that  $L(g) \notin \mathcal{I}$ . The function  $f$  must be continuous, otherwise it could happen that the inequality in (57) does not apply. Such boundary cases are presented in [14].

In 1937, McShane extended the functional form of Jensen's inequality to convex functions of several variables. He has covered the generalization in two steps, calling them the geometric (the inclusion in (58)) and analytic (the inequality in (59)) formulation of Jensen's inequality, see [15, Theorem 1 and Theorem 2]. Summarized in a theorem, that generalization is as follows.

**THEOREM 3.** *Let  $\mathcal{C} \subseteq \mathbb{R}^r$  be a closed convex set, and let  $g = (g_1, \dots, g_r) \in (\mathbb{X}^r)_{\mathcal{C}}$  be a function.*

*Then a unital positive functional  $L \in \mathbb{L}(\mathbb{X})$  ensures the inclusion*

$$(L(g_1), \dots, L(g_r)) \in \mathcal{C}, \quad (58)$$

and satisfies the inequality

$$f(L(g_1), \dots, L(g_r)) \leq L(f(g_1, \dots, g_r)). \quad (59)$$

for every continuous convex function  $f : \mathcal{C} \rightarrow \mathbb{R}$  providing that  $f(g_1, \dots, g_r) \in \mathbb{X}$ .

### 5.1. Functions of one variable

Through this subsection we will use a closed interval  $\mathcal{I} \subseteq \mathbb{R}$ , and a bounded closed subinterval  $[a, b] \subseteq \mathcal{I}$  with endpoints  $a < b$ .

A convex function  $f : \mathcal{I} \rightarrow \mathbb{R}$  satisfies the inequality

$$f(x) \leq f_{\{a,b\}}^{\text{line}}(x) \text{ if } x \in [a, b], \quad (60)$$

and the reverse inequality

$$f(x) \geq f_{\{a,b\}}^{\text{line}}(x) \text{ if } x \in \mathcal{I} \setminus (a, b). \quad (61)$$

In the following consideration, we use continuous functions satisfying the inequalities in equations (60)-(61).

**THEOREM 4.** *Let  $\mathcal{I} \subseteq \mathbb{R}$  be a closed interval, let  $[a, b] \subseteq \mathcal{I}$  be a bounded closed subinterval, and let  $g \in \mathbb{X}_{[a,b]}$  and  $h \in \mathbb{X}_{\mathcal{I} \setminus (a,b)}$  be functions.*

*Then a pair of unital positive functionals  $L, H \in \mathbb{L}(\mathbb{X})$  such that*

$$L(g) = H(h), \tag{62}$$

*satisfies the inequality*

$$L(f(g)) \leq H(f(h)) \tag{63}$$

*for every continuous function  $f : \mathcal{I} \rightarrow \mathbb{R}$  satisfying equations (60)-(61), and providing that  $f(g), f(h) \in \mathbb{X}$ .*

*Proof.* The number  $L(g)$  belongs to the interval  $[a, b]$  by the inclusion in equation (56). Using the features of the function  $f$ , and applying the affinity of the function  $f_{\{a,b\}}^{\text{line}}$ , we get

$$\begin{aligned} L(f(g)) &\leq L(f_{\{a,b\}}^{\text{line}}(g)) = f_{\{a,b\}}^{\text{line}}(L(g)) \\ &= f_{\{a,b\}}^{\text{line}}(H(h)) = H(f_{\{a,b\}}^{\text{line}}(h)) \\ &\leq H(f(h)) \end{aligned} \tag{64}$$

because  $f_{\{a,b\}}^{\text{line}}(h(x)) \leq f(h(x))$  for every  $x \in \mathcal{I}$ .

Involving the binomial convex combination  $\alpha a + \beta b$  with the equality in equation (62) by assuming that

$$L(g) = \alpha a + \beta b = H(h), \tag{65}$$

and inserting the term  $\alpha f(a) + \beta f(b)$  in equation (64) via the double equality

$$f_{\{a,b\}}^{\text{line}}(L(g)) = \alpha f(a) + \beta f(b) = f_{\{a,b\}}^{\text{line}}(H(h)) \tag{66}$$

which is true because  $f_{\{a,b\}}^{\text{line}}(\alpha a + \beta b) = \alpha f(a) + \beta f(b)$ , we achieve the double inequality

$$L(f(g)) \leq \alpha f(a) + \beta f(b) \leq H(f(h)). \tag{67}$$

The functions used in Theorem 4 satisfy the functional form of Jensen’s inequality in the following case.

**COROLLARY 9.** *Let  $\mathcal{I} \subseteq \mathbb{R}$  be a closed interval, let  $[a, b] \subseteq \mathcal{I}$  be a bounded closed subinterval, and let  $h \in \mathbb{X}_{\mathcal{I} \setminus (a,b)}$  be a function.*

*Then a unital positive functional  $H \in \mathbb{L}(\mathbb{X})$  such that*

$$H(h) \in [a, b], \tag{68}$$

*satisfies the inequality*

$$f(H(h)) \leq H(f(h)) \tag{69}$$

*for every continuous function  $f : \mathcal{I} \rightarrow \mathbb{R}$  satisfying equations (60)-(61), and providing that  $f(h) \in \mathbb{X}$ .*

*Proof.* Putting  $\alpha a + \beta b = H(h)$ , it follows that

$$\begin{aligned} f(H(h)) &= f(\alpha a + \beta b) \leq f_{\{a,b\}}^{\text{line}}(\alpha a + \beta b) \\ &= \alpha f(a) + \beta f(b) \leq H(f(h)) \end{aligned} \tag{70}$$

by the right inequality in equation (67).

**5.2. Functions of several variables**

We want to transfer the results of the previous subsection to higher dimensions.

Let  $\mathcal{C} \subseteq \mathbb{R}^2$  be a convex set, let  $\Delta \subseteq \mathcal{C}$  be a triangle with vertices  $A, B$  and  $C$ , and let  $\Delta^\circ$  be the triangle interior. In the following observation, we assume that  $f : \mathcal{C} \rightarrow \mathbb{R}$  is a continuous function satisfying the inequality

$$f(P) \leq f_{\{A,B,C\}}^{\text{plane}}(P) \text{ if } P \in \Delta, \tag{71}$$

and the reverse inequality

$$f(P) \geq f_{\{A,B,C\}}^{\text{plane}}(P) \text{ if } P \in \mathcal{C} \setminus \Delta^\circ, \tag{72}$$

where  $f_{\{A,B,C\}}^{\text{plane}}$  is the function of the plane passing through the corresponding points of the graph of  $f$ .

It should be noted that convex functions of two variables do not generally satisfy equation (72). The next example confirms this claim.

EXAMPLE 3. We take the convex function  $f(x,y) = x^2 + y^2$ , the triangle with vertices  $A(0,0), B(1,0)$  and  $C(0,2)$ , and the outside point  $P(1,1)$ .

The valuation of functions  $f$  and  $f_{\{A,B,C\}}^{\text{plane}}(x,y) = x + 2y$  at the point  $P$  is

$$2 = f(P) < f_{\{A,B,C\}}^{\text{plane}}(P) = 3 \tag{73}$$

as opposed to equation (72).

The generalization of Theorem 4 to two dimensions is as follows.

LEMMA 5. Let  $\mathcal{C} \subseteq \mathbb{R}^2$  be a closed convex set, let  $\Delta \subseteq \mathcal{C}$  be a triangle, and let  $g = (g_1, g_2) \in (\mathbb{X}^2)_\Delta$  and  $h = (h_1, h_2) \in (\mathbb{X}^2)_{\mathcal{C} \setminus \Delta^\circ}$  be functions.

Then a pair of unital positive functionals  $L, H \in \mathbb{L}(\mathbb{X})$  such that

$$(L(g_1), L(g_2)) = (H(h_1), H(h_2)), \tag{74}$$

satisfies the inequality

$$L(f(g_1, g_2)) \leq H(f(h_1, h_2)) \tag{75}$$

for every continuous function  $f : \mathcal{C} \rightarrow \mathbb{R}$  satisfying equations (71)-(72), and providing that  $f(g_1, g_2), f(h_1, h_2) \in \mathbb{X}$ .

*Proof.* The proof is similar to that of Theorem 4. Using the triangle vertices  $A, B$  and  $C$ , we apply the plane function  $f_{\{A,B,C\}}^{\text{plane}}$  instead of the line function  $f_{\{a,b\}}^{\text{line}}$ .

The previous lemma suggests how the results of the previous subsection can be transferred to higher dimensions.

Let  $\mathcal{C} \subseteq \mathbb{R}^r$  be a convex set, and let  $\mathcal{S} \subseteq \mathcal{C}$  be a  $r$ -simplex with vertices  $S_1, \dots, S_{r+1}$ . In the consideration that follows, we use a function  $f : \mathcal{C} \rightarrow \mathbb{R}$  satisfying the inequality

$$f(P) \leq f_{\{S_1, \dots, S_{r+1}\}}^{\text{hyperplane}}(P) \text{ if } P \in \mathcal{S}, \tag{76}$$

and the reverse inequality

$$f(P) \geq f_{\{S_1, \dots, S_{r+1}\}}^{\text{hyperplane}}(P) \text{ if } P \in \mathcal{C} \setminus \mathcal{S}^o, \tag{77}$$

where  $f_{\{S_1, \dots, S_{r+1}\}}^{\text{hyperplane}}$  is the function of the hyperplane passing through the corresponding points of the graph of  $f$ .

**THEOREM 5.** *Let  $\mathcal{C} \subseteq \mathbb{R}^r$  be a closed convex set, let  $\mathcal{S} \subseteq \mathcal{C}$  be an  $r$ -simplex, and let  $g = (g_1, \dots, g_r) \in (\mathbb{X}^k)_{\mathcal{S}}$  and  $h = (h_1, \dots, h_r) \in (\mathbb{X}^r)_{\mathcal{C} \setminus \mathcal{S}^o}$  be functions.*

*Then a pair of unital positive functionals  $L, H \in \mathbb{L}(\mathbb{X})$  such that*

$$(L(g_1), \dots, L(g_r)) = (H(h_1), \dots, H(h_r)), \tag{78}$$

*satisfies the inequality*

$$L(f(g_1, \dots, g_r)) \leq H(f(h_1, \dots, h_r)) \tag{79}$$

*for every continuous function  $f : \mathcal{C} \rightarrow \mathbb{R}$  satisfying equations (76)-(77), and providing that  $f(g_1, \dots, g_r), f(h_1, \dots, h_r) \in \mathbb{X}$ .*

*Proof.* Relying on the hyperplane function  $f_{\{S_1, \dots, S_{r+1}\}}^{\text{hyperplane}}$  where  $S_1, \dots, S_{r+1}$  are the simplex vertices, we can apply the proof similar to that of Theorem 4.

Including the  $(r + 1)$ -membered convex combination  $\sum_{k=1}^{r+1} \gamma_k S_k$  to the equality in equation (78) in a way that

$$(L(g_1), \dots, L(g_r)) = \sum_{k=1}^{r+1} \gamma_k S_k = (H(h_1), \dots, H(h_r)), \tag{80}$$

and using the double equality

$$f_{\{S_1, \dots, S_{r+1}\}}^{\text{hyperplane}}(L(g_1), \dots, L(g_r)) = \sum_{k=1}^{r+1} \gamma_k f(S_k) = f_{\{S_1, \dots, S_{r+1}\}}^{\text{hyperplane}}(H(h_1), \dots, H(h_r)), \tag{81}$$

we can derive the double inequality

$$L(f(g_1, \dots, g_r)) \leq \sum_{k=1}^{r+1} \gamma_k f(S_k) \leq H(f(h_1, \dots, h_r)). \tag{82}$$

The following functional form of Jensen’s inequality is true for functions of several variables.

COROLLARY 10. Let  $\mathcal{C} \subseteq \mathbb{R}^r$  be a closed convex set, let  $\mathcal{S} \subseteq \mathcal{C}$  be an  $r$ -simplex, and let  $h = (h_1, \dots, h_r) \in (\mathbb{X}^r)_{\mathcal{C} \setminus \mathcal{S}^o}$  be a function.

Then a unital positive functional  $H \in \mathbb{L}(\mathbb{X})$  such that

$$(H(h_1), \dots, H(h_r)) \in \mathcal{S}, \quad (83)$$

satisfies the inequality

$$f(H(h_1), \dots, H(h_r)) \leq H(f(h_1, \dots, h_r)) \quad (84)$$

for every continuous function  $f : \mathcal{C} \rightarrow \mathbb{R}$  satisfying equations (76)-(77), and providing that  $f(h_1, \dots, h_r) \in \mathbb{X}$ .

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