Journal of
Mathematical
Inequalities
Volume 9, Number 1 (2015), 113–120 doi:10.7153/jmi-09-11

REVERSES OF YOUNG TYPE INEQUALITIES

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(*Communicated by J. Liu*)

Abstract. This paper aims to give reverses of Young type inequalities which were established by Kai [4]. Then we use these inequalities to establish corresponding inequalities for matrices. Also we present a refinement of the trace version of Young's inequality.

1. Introduction

Let M_n be the space of $n \times n$ complex matrices. For $A = (a_{ij}) \in M_n$, the Hilbert-Schmidt norm and the trace norm of A are defined by $\|A\|_2 = \left(\sum_{j=1}^{n} s_j^2(A)\right)^{\frac{1}{2}}$, $\|A\|_1 = \sum_{j=1}^{n} s_j(A)$, respectively, where $s_1(A) \geqslant s_2(A) \geqslant .... \geqslant s_n(A)$ are the singular values of A, that is, the eigenvalues of the positive matrix $|A| = (A^*A)^{\frac{1}{2}}$, arranged in decreasing order and repeated according to multiplicity. Note that $\|A\|_2 = (\operatorname{tr}(AA^\star))^{\frac{1}{2}}$, where tr is the usual trace functional. It is known that the Hilbert-Schmidt norm is unitarily invariant, and it is evident that each unitarily invariant norm is symmetric gauge function of singular values.

The scalar Young inequality says that if $a, b \geqslant 0$ and $0 \leqslant \nu \leqslant 1$, then

$$a^\nu b^{1-\nu} \leqslant \nu a + (1-\nu) b, \tag{1}$$

with equality if and only if $a = b$, or equivalently

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q},$$

where $p,\ q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. A fundamental inequality between positive real numbers $a,\ b$ is the arithmetic-geometric mean inequality, which is

$$\sqrt{ab} \leqslant \frac{a+b}{2},$$

with equality if and only if $a = b$. It is a particular case of (1) when $\nu = \frac{1}{2}$.

Mathematics subject classification (2010): 15A45, 15A60.
Keywords and phrases: Unitarily invariant norms, Young inequality, positive matrices, singular values.

The inequality (1) was refined by Kittaneh and Manasrah [5] in the following form
$$a^{\nu}b^{1-\nu}+r_0\left(\sqrt{a}-\sqrt{b}\right)^2\leqslant \nu a+(1-\nu)b,$$
where $r_0=\min\{\nu,\ (1-\nu)\}$.

Kittaneh and Manasrah [6] gave the following reverse of Young's inequality
$$\nu a+(1-\nu)b\leqslant a^{\nu}b^{1-\nu}+R_0\left(\sqrt{a}-\sqrt{b}\right)^2,$$
where $R_0=\max\{\nu,\ (1-\nu)\}$.

In a recent work, Kai [4] gave the following Young type inequalities
$$\left(\nu^2a\right)^{\nu}b^{1-\nu}+\nu^2\left(\sqrt{a}-\sqrt{b}\right)^2\leqslant \nu^2a+(1-\nu)^2b,\ \ 0\leqslant\nu\leqslant\frac{1}{2} \tag{2}$$
and
$$a^{\nu}\left((1-\nu)^2b\right)^{1-\nu}+(1-\nu)^2\left(\sqrt{a}-\sqrt{b}\right)^2\leqslant \nu^2a+(1-\nu)^2b,\ \ \frac{1}{2}\leqslant\nu\leqslant 1. \tag{3}$$
Also, important inequalities were obtained by Cartwright and Field [3]. These inequalities can be written as
$$a^{\nu}b^{1-\nu}+\frac{\nu(1-\nu)}{2M}(a-b)^2\leqslant \nu a+(1-\nu)b\leqslant a^{\nu}b^{1-\nu}+\frac{\nu(1-\nu)}{2m}(a-b)^2, \tag{4}$$
where $a,\ b>0$, $m=\min\{a,b\}$, $M=\max\{a,b\}$ and $0\leqslant\nu\leqslant 1$.

A matrix version of (1) proved in [1] says that if $A,\ B\in M_n$ are positive and $0\leqslant\nu\leqslant 1$, then
$$s_j\left(A^{\nu}B^{1-\nu}\right)\leqslant s_j\left(\nu A+(1-\nu)B\right), \tag{5}$$
for $j=1,2,....,n$. It follows from inequality (5) that if $A,\ B\in M_n$ are positive and $0\leqslant\nu\leqslant 1$, then
$$\operatorname{tr}\left|A^{\nu}B^{1-\nu}\right|\leqslant \operatorname{tr}\left(\nu A+(1-\nu)B\right), \tag{6}$$
which is a trace version of Young's inequality.

Kai [4] obtain the matrix versions of the inequalities (2) and (3), where he proved that if $A,\ B,\ X\in M_n$ such that $A,\ B$ are positive, then
$$\begin{aligned}\nu^{2\nu}\left\|A^{\nu}XB^{1-\nu}\right\|_2^2+2\nu(1-\nu)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2+\nu^2\left\|AX-XB\right\|_2^2&\\ \leqslant\left\|\nu AX+(1-\nu)XB\right\|_2^2\qquad\text{for }0\leqslant\nu\leqslant\frac{1}{2}&\end{aligned} \tag{7}$$
and
$$\begin{aligned}(1-\nu)^{2(1-\nu)}\left\|A^{\nu}XB^{1-\nu}\right\|_2^2+2\nu(1-\nu)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2+(1-\nu)^2\left\|AX-XB\right\|_2^2&\\ \leqslant\left\|\nu AX+(1-\nu)XB\right\|_2^2\qquad\text{for }\frac{1}{2}\leqslant\nu\leqslant 1&\end{aligned} \tag{8}$$

In this paper, we present reverses of the inequalities (2) and (3) and based on the spectral theorem for positive matrices, we use these inequalities to establish corresponding inequalities for matrices. We also obtain a matrix version of the second inequality in (4) and give some refinements of the trace version of Young's inequality and its reverse.

2. Reverses of the scalar Young type inequalities

In this section, we present reverses of the inequalities (2) and (3).

THEOREM 1. *Let* $a,b \geqslant 0$. *If* $0 \leqslant \nu \leqslant \frac{1}{2}$, *then*

$$\nu^2 a + (1-\nu)^2 b \leqslant (1-\nu)^2 \left(\sqrt{a}-\sqrt{b}\right)^2 + a^\nu \left[(1-\nu)^2 b\right]^{1-\nu}. \tag{9}$$

If $\frac{1}{2} \leqslant \nu \leqslant 1$, *then*

$$\nu^2 a + (1-\nu)^2 b \leqslant \nu^2 \left(\sqrt{a}-\sqrt{b}\right)^2 + \left(\nu^2 a\right)^\nu b^{1-\nu}. \tag{10}$$

Proof. If $0 \leqslant \nu \leqslant \frac{1}{2}$, then by the inequality (1), we have

$$\begin{aligned}
&(1-\nu)^2 \left(\sqrt{a}-\sqrt{b}\right)^2 - \nu^2 a - (1-\nu)^2 b + a^\nu \left[(1-\nu)^2 b\right]^{1-\nu} \\
&= a^{\frac{1}{2}} \left[(1-2\nu) a^{\frac{1}{2}} + 2\nu(1-\nu) b^{\frac{1}{2}}\right] - 2(1-\nu)\sqrt{ab} + a^\nu \left[(1-\nu)^2 b\right]^{1-\nu} \\
&\geqslant a^{\frac{1}{2}} \left[\left(a^{\frac{1}{2}}\right)^{1-2\nu} \left\{(1-\nu) b^{\frac{1}{2}}\right\}^{2\nu}\right] - 2(1-\nu)\sqrt{ab} + a^\nu \left[(1-\nu)^2 b\right]^{1-\nu} \\
&= a^{1-\nu} \left[(1-\nu)^2 b\right]^\nu + a^\nu \left[(1-\nu)^2 b\right]^{1-\nu} - 2(1-\nu)\sqrt{ab} \\
&= \left[a^{\frac{1-\nu}{2}} (1-\nu)^\nu b^{\frac{\nu}{2}} - a^{\frac{\nu}{2}} (1-\nu)^{1-\nu} b^{\frac{1-\nu}{2}}\right]^2 \geqslant 0.
\end{aligned}$$

Thus, we get (9).

On the other hand, if $\frac{1}{2} \leqslant \nu \leqslant 1$, then

$$\begin{aligned}
&\nu^2 \left(\sqrt{a}-\sqrt{b}\right)^2 - \nu^2 a - (1-\nu)^2 b + b^{1-\nu} \left[\nu^2 a\right]^\nu \\
&= (2\nu-1) b + \nu(2-2\nu)\sqrt{ab} - 2\nu\sqrt{ab} + b^{1-\nu} \left[\nu^2 a\right]^\nu \\
&\geqslant b^{\frac{1}{2}} \left[(2\nu-1) b^{\frac{1}{2}} + (2-2\nu)\nu a^{\frac{1}{2}}\right] - 2\nu\sqrt{ab} + b^{1-\nu} \left[\nu^2 a\right]^\nu \\
&\geqslant \left[b^\nu \left(\nu^2 a\right)^{1-\nu}\right] - 2\nu\sqrt{ab} + b^{1-\nu} \left[\nu^2 a\right]^\nu \geqslant 0.
\end{aligned}$$

Hence, for $\frac{1}{2} \leqslant \nu \leqslant 1$, we have

$$\nu^2 a + (1-\nu)^2 b \leqslant \nu^2 \left(\sqrt{a}-\sqrt{b}\right)^2 + \left(\nu^2 a\right)^\nu b^{1-\nu}.$$

This completes the proof. □

3. Reverses of Young type inequalities for matrices

In this section, we first establish matrix versions of the inequalities (9) and (10), whose proof is based on the spectral theorem for positive matrices.

THEOREM 2. *Let* A, B, $X \in M_n$ *such that* A *and* B *are positive. If* $0 \leqslant \nu \leqslant \frac{1}{2}$, *then*

$$\|\nu AX + (1-\nu)XB\|_2^2 \leqslant (1-\nu)^2 \|AX - XB\|_2^2 + 2\nu(1-\nu)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2 + (1-\nu)^{2(1-\nu)}\left\|A^{\nu}XB^{1-\nu}\right\|_2^2. \tag{11}$$

If $\frac{1}{2} \leqslant \nu \leqslant 1$, *then*

$$\|\nu AX + (1-\nu)XB\|_2^2 \leqslant \nu^2 \|AX - XB\|_2^2 + 2\nu(1-\nu)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2 + \nu^{2\nu}\left\|A^{\nu}XB^{1-\nu}\right\|_2^2. \tag{12}$$

Proof. Since every positive matrix is unitarily diagonalizable, it follows that there are unitary matrices U, $V \in M_n$ such that $A = UDU^*$ and $B = VEV^*$, where

$$D = \operatorname{diag}(\lambda_1, ..., \lambda_n) \text{ and } E = \operatorname{diag}(\mu_1, ..., \mu_n)$$

with $\lambda_i, \mu_i \geqslant 0$ for $i = 1, ..., n$.

Let $Y = U^*XV = [y_{ij}]$, then we have

$$\begin{aligned} \nu AX + (1-\nu)XB &= U[(\nu\lambda_i + (1-\nu)\mu_i)y_{ij}]V^*, \\ AX - XB &= U[(\lambda_i - \mu_i)y_{ij}]V^*, \\ A^{\nu}XB^{1-\nu} &= U\left[\lambda_i^{\nu}\mu_i^{1-\nu}y_{ij}\right]V^*, \end{aligned}$$

and

$$A^2X + XB^2 = U\left[\left(\lambda_i^2 + \mu_i^2\right)y_{ij}\right]V^*.$$

If $0 \leqslant \nu \leqslant \frac{1}{2}$, then by inequality (9) and the unitary invariance of the Hilbert-Schmidt norm, we have

$$\begin{aligned} \|\nu AX + (1-\nu)XB\|_2^2 &= \sum_{i,j=1}^{n} (\nu\lambda_i + (1-\nu)\mu_i)^2 |y_{ij}|^2 \\ &\leqslant (1-\nu)^2 \sum_{i,j=1}^{n} (\lambda_i - \mu_i)^2 |y_{ij}|^2 \\ &\quad + (1-\nu)^{2(1-\nu)} \sum_{i,j=1}^{n} \left(\lambda_i^{\nu}\mu_i^{1-\nu}\right)^2 |y_{ij}|^2 \\ &\quad + 2\nu(1-\nu) \sum_{i,j=1}^{n} \left(\lambda_i^{\frac{1}{2}}\mu_i^{\frac{1}{2}}\right)^2 |y_{ij}|^2 \end{aligned}$$

$$= (1-\nu)^2 \|AX - XB\|_2^2 + 2\nu(1-\nu)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2 + (1-\nu)^{2(1-\nu)}\left\|A^{\nu}XB^{1-\nu}\right\|_2^2.$$

If $\frac{1}{2} \leqslant \nu \leqslant 1$, then by the inequality (10) and using the same technique in the first part we get (12).

This completes the proof. □

Now, we obtain a matrix version of the second inequality in (4). To achieve this, we need the following result:

LEMMA 3. [2] (Minkowski Inequality). Let $a_i \geqslant 0$, $b_i \geqslant 0$ for $i = 1,..,n$ and $p \geqslant 1$. Then

$$\left(\sum_{i=1}^{n}(a_i+b_i)^p\right)^{\frac{1}{p}} \leqslant \left(\sum_{i=1}^{n}a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n}b_i^p\right)^{\frac{1}{p}}.$$

Now, we introduce the following theorem:

THEOREM 4. *Let* A, B, $X \in M_n$ *such that* A *and* B *are positive definite. Then, for* $0 \leqslant \nu \leqslant 1$, *we have*

$$\|\nu AX + (1-\nu)XB\|_2 \leqslant \left\|A^{\nu}XB^{1-\nu}\right\|_2 + \frac{\nu(1-\nu)M_0}{2}\left\|A^2X + XB^2\right\|_2,$$

where $M_0 = \max\left(\left\|A^{-1}\right\|, \left\|B^{-1}\right\|\right)$.

Proof. Using inequality (4) and the same argument in the proof of Theorem 2, we have

$$\begin{aligned}
&\|\nu AX + (1-\nu)XB\|_2 \\
&= \sqrt{\sum_{i,j=1}^{n}(\nu\lambda_i + (1-\nu)\mu_i)^2\left|y_{ij}\right|^2} \\
&\leqslant \sqrt{\sum_{i,j=1}^{n}\left(\frac{\nu(1-\nu)M_0}{2}(\lambda_i-\mu_i)^2 + \lambda_i^{\nu}\mu_i^{1-\nu}\right)^2\left|y_{ij}\right|^2} \\
&\leqslant \frac{\nu(1-\nu)M_0}{2}\sqrt{\sum_{i,j=1}^{n}\left((\lambda_i-\mu_i)^2\left|y_{ij}\right|\right)^2} + \sqrt{\sum_{i,j=1}^{n}\left(\lambda_i^{\nu}\mu_i^{1-\nu}\left|y_{ij}\right|\right)^2} \\
&\leqslant \frac{\nu(1-\nu)M_0}{2}\sqrt{\sum_{i,j=1}^{n}\left((\lambda_i^2+\mu_i^2)\left|y_{ij}\right|\right)^2} + \sqrt{\sum_{i,j=1}^{n}\left(\lambda_i^{\nu}\mu_i^{1-\nu}\left|y_{ij}\right|\right)^2} \\
&= \frac{\nu(1-\nu)M_0}{2}\left\|A^2X + XB^2\right\|_2 + \left\|A^{\nu}XB^{1-\nu}\right\|_2.
\end{aligned}$$

The second inequality is obtained by the Minkowski inequality.

This completes the proof. □

4. Trace versions of Young inequalities

To obtain refinements of the trace version of Young's inequality and its reverse based on the inequality (4), we need the following Lemmas

LEMMA 5. [2] (Cauchy-Schwarz Inequality). Let $a_i \geqslant 0,\ b_i \geqslant 0$ for $i = 1, .., n$. Then

$$\sum_{i=1}^{n} a_i b_i \leqslant \left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} b_i^2\right)^{\frac{1}{2}}$$

LEMMA 6. [2] If $A,\ B \in M_n$, then

$$\sum_{j=1}^{n} s_j(AB) \leqslant \sum_{j=1}^{n} s_j(A)\ s_j(B).$$

The following result gives refinements of the trace version of Young's inequality and its reverse based on the inequality (4)

THEOREM 7. *Let $A,\ B \in M_n$ such that A and B are positive definite. Then, for $0 \leqslant \nu \leqslant 1$, we have*

a) $\operatorname{tr}\left|A^\nu B^{1-\nu}\right| + \frac{\nu(1-\nu)}{2M_0}\left(\sqrt{\operatorname{tr}A^2} - \sqrt{\operatorname{tr}B^2}\right)^2 \leqslant \operatorname{tr}(\nu A + (1-\nu)B)$,
where $M_0 = \max(\|A\|, \|B\|)$.

b) $\operatorname{tr}(\nu A + (1-\nu)B) \leqslant \operatorname{tr}(A^\nu)\operatorname{tr}\left(B^{1-\nu}\right) + \frac{\nu(1-\nu)R_0}{2}\left(\operatorname{tr}A^2 - 2\operatorname{tr}(AB) + \operatorname{tr}B^2\right)$,
where $R_0 = \max\left(\left\|A^{-1}\right\|, \left\|B^{-1}\right\|\right)$.

Proof. a) By Lemma 5, Lemma 6 and the inequality (4), we have

$$\begin{aligned}
&\operatorname{tr}(\nu A + (1-\nu)B) \\
&= \nu \operatorname{tr}A + (1-\nu)\operatorname{tr}B \\
&= \sum_{j=1}^{n} (\nu\, s_j(A) + (1-\nu)s_j(B)) \\
&\geqslant \sum_{j=1}^{n} s_j(A^\nu)s_j\left(B^{1-\nu}\right) + \frac{\nu(1-\nu)}{2M_0}\left(\sum_{j=1}^{n} s_j^2(A) - 2\sum_{j=1}^{n} s_j(A)s_j(B) + \sum_{j=1}^{n} s_j^2(B)\right) \\
&\geqslant \sum_{j=1}^{n} s_j\left(A^\nu B^{1-\nu}\right) + \frac{\nu(1-\nu)}{2M_0}\left(\operatorname{tr}A^2 + \operatorname{tr}B^2 - 2\sum_{j=1}^{n} s_j(A)s_j(B)\right) \\
&\geqslant \sum_{j=1}^{n} s_j\left(A^\nu B^{1-\nu}\right) + \frac{\nu(1-\nu)}{2M_0}\left(\operatorname{tr}A^2 + \operatorname{tr}B^2 - 2\sqrt{\sum_{j=1}^{n} s_j^2(A)}\sqrt{\sum_{j=1}^{n} s_j^2(B)}\right) \\
&= \operatorname{tr}\left|A^\nu B^{1-\nu}\right| + \frac{\nu(1-\nu)}{2M_0}\left(\operatorname{tr}A^2 + \operatorname{tr}B^2 - 2\sqrt{\operatorname{tr}A^2}\sqrt{\operatorname{tr}B^2}\right) \\
&= \operatorname{tr}\left|A^\nu B^{1-\nu}\right| + \frac{\nu(1-\nu)}{2M_0}\left(\sqrt{\operatorname{tr}A^2} - \sqrt{\operatorname{tr}B^2}\right)^2.
\end{aligned}$$

b) By Lemma 6 and the inequality (4), we have

$$\begin{aligned}
&\operatorname{tr}(\nu A+(1-\nu)B)\\
&\leqslant \sum_{j=1}^{n} s_j(A^{\nu})s_j\left(B^{1-\nu}\right)+\frac{\nu(1-\nu)R_0}{2}\left(\sum_{j=1}^{n} s_j^2(A)-2\sum_{j=1}^{n} s_j(A)s_j(B)+\sum_{j=1}^{n} s_j^2(B)\right)\\
&\leqslant \sum_{j=1}^{n} s_j(A^{\nu})\sum_{j=1}^{n} s_j\left(B^{1-\nu}\right)+\frac{\nu(1-\nu)R_0}{2}\left(\sum_{j=1}^{n} s_j^2(A)-2\sum_{j=1}^{n} s_j(AB)+\sum_{j=1}^{n} s_j^2(B)\right)\\
&=\operatorname{tr}(A^{\nu})\operatorname{tr}\left(B^{1-\nu}\right)+\frac{\nu(1-\nu)R_0}{2}\left(\operatorname{tr}A^2-2\operatorname{tr}(AB)+\operatorname{tr}B^2\right).
\end{aligned}$$

This completes the proof. □

Based on the refined Young type inequalities (2) and (3), it has been shown in [4] that if $A,\ B\in M_n$ such that A and B are positive, then
for $0\leqslant\nu\leqslant\frac{1}{2}$

$$\nu^{\nu}\left\|A^{\nu}B^{1-\nu}\right\|_1\leqslant\sqrt{\nu^2\|A\|_2^2+(1-\nu^2)\|B\|_2^2-\nu^2(\|A\|_2-\|B\|_2)^2} \tag{13}$$

and for $\frac{1}{2}\leqslant\nu\leqslant 1$.

$$(1-\nu)^{1-\nu}\left\|A^{\nu}B^{1-\nu}\right\|_1\leqslant\sqrt{\nu^2\|A\|_2^2+(1-\nu^2)\|B\|_2^2-(1-\nu)^2(\|A\|_2-\|B\|_2)^2}. \tag{14}$$

Now, we introduce reverses of inequalities (13) and (14).

THEOREM 8. *Let A, B, $X\in M_n$ such that A and B are positive. If $0\leqslant\nu\leqslant\frac{1}{2}$, then*

$$\begin{aligned}
&(1-\nu)^{1-\nu}\|A^{\nu}\|_2\left\|B^{1-\nu}\right\|_2\\
&\geqslant\sqrt{\nu^2\|A\|_2^2+(1-\nu)^2\|B\|_2^2-(1-\nu)^2\left(\|A\|_2^2+\|B\|_2^2-2\|AB\|_1\right)}
\end{aligned}$$

If $\frac{1}{2}\leqslant\nu\leqslant 1$, then

$$\begin{aligned}
&\nu^{\nu}\|A^{\nu}\|_2\left\|B^{1-\nu}\right\|_2\\
&\geqslant\sqrt{\nu^2\|A\|_2^2+(1-\nu)^2\|B\|_2^2-\nu^2\left(\|A\|_2^2+\|B\|_2^2-2\|AB\|_1\right)}
\end{aligned}$$

Proof. If $0\leqslant\nu\leqslant\frac{1}{2}$, then by Lemma 5, Lemma 6 and the inequality (9), we have

$$\begin{aligned}
\operatorname{tr}\left(\nu^2A^2+(1-\nu)^2B^2\right)&=\nu^2\operatorname{tr}A^2+(1-\nu)^2\operatorname{tr}B^2\\
&=\sum_{j=1}^{n}\left(\nu^2 s_j^2(A)+(1-\nu)^2 s_j^2(B)\right)
\end{aligned}$$

$$
\begin{aligned}
&\leqslant (1-\nu)^2 \left(\sum_{j=1}^{n} s_j^2(A) + \sum_{j=1}^{n} s_j^2(B) - 2\sum_{j=1}^{n} s_j(A)s_j(B) \right) \\
&\quad + (1-\nu)^{2(1-\nu)} \left(\sum_{j=1}^{n} s_j(A^\nu)\ s_j\left(B^{1-\nu}\right) \right)^2 \\
&\leqslant (1-\nu)^2 \left(\|A\|_2^2 + \|B\|_2^2 - 2\sum_{j=1}^{n} s_j(AB) \right) \\
&\quad + (1-\nu)^{2(1-\nu)} \left(\sum_{j=1}^{n} s_j^2(A^\nu) \sum_{j=1}^{n} s_j^2\left(B^{1-\nu}\right) \right).
\end{aligned}
$$

Thus,

$$
\nu^2 \|A\|_2^2 + (1-\nu)^2 \|B\|_2^2
$$
$$
\leqslant (1-\nu)^2 \left(\|A\|_2^2 - \|B\|_2^2 - 2\|AB\|_1 \right) + (1-\nu)^{2(1-\nu)} \|A^\nu\|_2^2 \left\|B^{1-\nu}\right\|_2^2.
$$

If $0 \leqslant \nu \leqslant \frac{1}{2}$, then by the inequality (10) and the same method above, we have the required inequality.

This completes the proof. □

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(Received February 7, 2014)

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