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HOW DO SINGULARITIES OF FUNCTIONS AFFECT THE CONVERGENCE OF q–BERNSTEIN POLYNOMIALS?

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Abstract. In this article, the approximation of functions with a singularity at $\alpha \in (0,1)$ by the q-Bernstein polynomials for $q>1$ has been studied. Unlike the situation when $\alpha \in (0,1)\setminus \{q^{-j}\}_{j\in\mathbb{N}}$, in the case when $\alpha = q^{-m}$, $m\in\mathbb{N}$, the type of singularity has a decisive effect on the set where a function can be approximated. In the latter event, depending on the types of singularities, three classes of functions have been examined, and it has been found that the possibility of approximation varies considerably for these classes.

1. Introduction

It is a common practice to consider the Bernstein polynomials and their various analogues only for the continuous functions on $[0,1]$. Despite the fact pointed out by G. Lorentz that, for the Bernstein polynomials, "Remarkable phenomena can occur for unbounded functions" (cf. [6, Ch.1, Sec.1.9]), researchers so far have barely turned their attention to the q-Bernstein polynomials of discontinuous functions. Consequently, this topic has not been explored widely, and only a very few papers are available (see [11] and references therein). The present article, being a continuation of [11], aims to fill this gap.

Let $q>0$. For any non-negative integer n, the *q-integer* $[n]_q$ is defined by

$$[n]_q := 1+q+\cdots+q^{n-1}\,(n=1,2,\ldots),\quad [0]_q:=0;$$

and the *q-factorial* $[n]_q!$ by

$$[n]_q! := [1]_q[2]_q\cdots[n]_q\,(n=1,2,\ldots),\quad [0]_q!:=1.$$

For integers $0\leqslant k\leqslant n$, the *q-binomial* coefficient $\begin{bmatrix} n\\ k\end{bmatrix}_q$ is defined by

$$\begin{bmatrix} n\\ k\end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

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Clearly, for $q = 1$,

$$[n]_1 = n, \quad [n]_1! = n!, \quad \begin{bmatrix} n \\ k \end{bmatrix}_1 = \binom{n}{k}.$$

Also, the following notations

$$(a;q)_0 := 1, \quad (a;q)_k := \prod_{s=0}^{k-1}(1-aq^s), \quad (a;q)_\infty := \prod_{s=0}^{\infty}(1-aq^s)$$

from [2, Ch.10] will be used. Based on the q-integers, the q-Bernstein polynomials have been introduced by G. M. Phillips:

DEFINITION 1. [14] For any $f : [0,1] \to \mathbb{C}$, the *q-Bernstein polynomials* of f are defined by

$$B_{n,q}(f;x) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) p_{nk}(x), \quad n \in \mathbb{N},$$

and the q-Bernstein basis polynomials by

$$p_{nk}(x) := \begin{bmatrix} n \\ k \end{bmatrix}_q x^k (x;q)_{n-k}, \quad k = 0,1,\ldots n. \tag{1}$$

Alternatively – [13, formulae (6) and (7)] – the q-Bernstein polynomials can be expressed as

$$B_{n,q}(f;x) = \sum_{k=0}^{n} c_{kn}x^k, \quad c_{kn} = \lambda_{kn} f[x_0,\ldots,x_k],$$

where

$$\lambda_{0n} = \lambda_{1n} = 1, \quad \lambda_{kn} = \prod_{j=1}^{k-1}\left(1 - \frac{[j]_q}{[n]_q}\right), \quad k = 2,\ldots,n$$

and $f[x_0,\ldots,x_k]$ denotes the k-th order divided difference of f with $k+1$ distinct nodes $x_0,\ldots,x_k$. Correspondingly, for $z \in \mathbb{C}$, $B_{n,q}(f;z) = \sum_{k=0}^{n} c_{kn}z^k$.

The case $q = 1$ corresponds to the classical Bernstein polynomials, and we follow the commonly accepted terminology by using "q-Bernstein polynomials" only for $q \neq 1$.

Owing to their unusual convergence properties, the q-Bernstein polynomials inspired many researchers and led to further investigations of the Bernstein-type operators based on the q-integers. See, for example, [1, 3, 4, 5, 8, 9, 10, 16]. The tribute for the discovery of this fruitful research area must be paid to A. Lupaş, who pioneered the exploration of the subject (cf. [7]).

Throughout the paper, only fixed $q > 1$ is considered. Further, the notations

$$\mathbb{J}_q := \{0\} \cup \{q^{-j}\}_{j=0}^{\infty} \quad \text{and} \quad x_k = [k]_q/[n]_q \quad \text{for} \quad k = 0,\ldots,n \tag{2}$$

adopted in [11], will be used. In addition, let Φ_k and Ψ_k be defined as

$$\Phi_k(x) = \prod_{j=1}^{k}\left(1 - \frac{x_j}{x}\right) \quad \text{and} \quad \Psi_k(x) = \prod_{j=k}^{\infty}\left(1 - \frac{1}{q^j x}\right). \tag{3}$$

In the sequel, for any $a>0$, the open disc $\{z\in\mathbb{C}:|z|<a\}$ and its closure $\{z\in\mathbb{C}:|z|\leqslant a\}$ will be denoted by $\mathscr{D}_a$ and $\overline{\mathscr{D}}_a$, respectively.

Let f be a function defined on $[0,1]$, possessing an analytic continuation from $[0,\alpha)$ into $\mathscr{D}_\alpha$, $0<\alpha<1$, and having a singularity at $x=\alpha$. It is well-known that the Taylor series of f converges to f uniformly on any compact set in $\mathscr{D}_\alpha$ regardless of the type of singularity. In distinction, for the approximation by the q-Bernstein polynomials, the following factors must be taken into account:

- whether or not $\alpha\in\mathbb{J}_q$;
- if affirmative, then the type of singularity.

If $\alpha\in(0,1)\setminus\mathbb{J}_q$, then the investigation of the q-Bernstein polynomials of f can be reduced to that for continuous functions. More precisely, the statement below holds.

THEOREM 1. *Let $\alpha\in(0,1)\setminus\mathbb{J}_q$; that is, $\alpha\in(q^{-(m+1)},q^{-m})$ for some $m\in\mathbb{N}_0$, and let $f:[0,1]\to\mathbb{R}$ be a function possessing an analytic continuation from $[0,\alpha)$ into an open disc $\mathscr{D}_\alpha$. In addition, let f be continuous from the left at $\{q^{-1},\dots,q^{-m}\}\subset\mathbb{J}_q$. Then, f is uniformly approximated by its q-Bernstein polynomials on any compact set in $(-\alpha,\alpha)$.*

Proof. If f is continuous on $[0,1]$, the statement has been proved in [12]. Otherwise, choose any $\rho\in(q^{-(m+1)},\alpha)$ and $\varepsilon\in(0,\alpha-\rho)$. Consider a function $\tilde{f}\in C[0,1]$ such that $\tilde{f}(x)=f(x)$ for $x\in[0,\rho+\varepsilon]$ and in "small" left neighborhoods of $q^{-m},\dots,q^{-1}$. Then, $\tilde{f}$ admits an analytic continuation into the closed disc $\overline{\mathscr{D}}_{\rho+\varepsilon}$ and, by Theorem 2.2 of [12], $B_{n,q}(\tilde{f};x)\to\tilde{f}(x)=f(x)$ on any compact set in $(-\rho-\varepsilon,\rho+\varepsilon)$ and, hence, on any compact set in $(-\alpha,\alpha)$. In addition, for n large enough, $B_{n,q}(f;x)=B_{n,q}(\tilde{f};x)$. This proves the claim. □

REMARK 1. The result is sharp in the sense that there exist functions satisfying the conditions of Theorem 1 which are not approximated by their q-Bernstein polynomials on any interval outside of $(-\alpha,\alpha)$. One such function is $f(x)=\frac{1}{x-\alpha}$ for $x\neq\alpha$ and $f(\alpha)\in\mathbb{R}$.

Although, the result for $\alpha\in\mathbb{J}_q$ given by Theorem 1 is formally new, it is, in fact, merely a direct conclusion of the previously known one. The situation changes if $\alpha\in\mathbb{J}_q$. Previously obtained results (e.g. [11]) reveal that, in this case, the interval of approximation may shrink to $(-\alpha_1,\alpha_1)$, where $0<\alpha_1<\alpha$.

In this work, the impact of the type of singularity at $\alpha=q^{-m}$ for some $m\in\mathbb{N}$ on the behavior of the q-Bernstein polynomials has been studied. To be specific, we discuss the convergence of $B_{n,q}$ for the functions $f:[0,1]\to\mathbb{R}$ continuous on $[0,1]\setminus\{\alpha\}$ and possessing an analytic continuation from $[0,\alpha)$ into $\mathscr{D}_\alpha$. The set of such functions will be denoted by $\mathscr{F}$. Persuant to the type of singularity, the following subsets of $\mathscr{F}$ are considered:

- $\mathscr{A}=\{f\in\mathscr{F}:\text{there exists }\gamma>0\text{ such that }\lim\limits_{x\to\alpha^-}f(x)(\alpha-x)^\gamma=K\in\mathbb{R}\setminus\{0\}\}$.

- $\mathscr{B} = \{f \in \mathscr{F} : \lim_{x \to \alpha^-} f(x)(\alpha - x)^\gamma = \infty \text{ for all } \gamma > 0\}$.
- $\mathscr{C} = \{f \in \mathscr{F} : \lim_{x \to \alpha^-} f(x)(\alpha - x)^\gamma = 0 \text{ for all } \gamma > 0\}$.

Based on this classification, the sets of convergence for the q-Bernstein polynomials have been examined, and it has been shown that these sets depend on whether $f \in \mathscr{A}, \mathscr{B}$, or $\mathscr{C}$.

The rest of the paper is organized as follows. The main outcomes are formulated in Section 2, while Sections 4 and 5 contain their proofs. Section 3 comprises supporting lemmas.

2. Main results

The theorems presented in this section reveal some qualitative characteristics of classes $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ in terms of the behavior of polynomials $B_{n,q}(f;\cdot)$. As a common feature for all these classes, it has been found that $\lim_{n\to\infty} B_{n,q}(f;x) = f(x)$ for $x \in \{1, q^{-1}, \dots, q^{-(m-1)}\} \subset \mathbb{J}_q$. In contrast, the behavior inside of $(-\alpha, \alpha)$ strongly depends on whether f belongs to $\mathscr{A}$, $\mathscr{B}$, or $\mathscr{C}$. More specifically, outside of the points $1, q^{-1}, \dots, q^{-(m-1)}$, the interval of approximation for $f \in \mathscr{C}$ is the same as for the Taylor polynomials, while for $f \in \mathscr{B}$, the polynomials $B_{n,q}$ diverge for all $x \neq 0$. The case $f \in \mathscr{A}$ falls in between these two. Generally speaking, $\mathscr{B}$ is the worst and $\mathscr{C}$ is the best class in terms of the approximation by the q-Bernstein polynomials, whereas $\mathscr{A}$ is an intermediate one.

The first main result is concerned with class $\mathscr{A}$. It turns out that γ, the parameter describing the singularity of f at α, plays a crucial role in finding the interval of convergence for polynomials $B_{n,q}(f;\cdot)$.

THEOREM 2. *Let $f \in \mathscr{A}$. Then*

(i) $B_{n,q}(f;x) \to f(x)$ uniformly on any compact set in $(-\alpha q^{-\gamma}, \alpha q^{-\gamma})$;

(ii) if $|x| > \alpha q^{-\gamma}$ and $x \notin \mathbb{J}_q$, then $|B_{n,q}(f;x)| \to \infty$ as $n \to \infty$.

As it has already been mentioned, for functions belonging to $\mathscr{B}$, the q-Bernstein polynomials have very poor convergence properties. This fact is expressed by

THEOREM 3. *Let $f \in \mathscr{B}$. Then $|B_{n,q}(f;x)| \to \infty$ for $x \notin \{1, q^{-1}, \dots, q^{-(m-1)}, 0\}$ as $n \to \infty$ while, obviously, $B_{n,q}(f;0) = f(0)$.*

Finally comes the case where $f \in \mathscr{C}$. Since, in this case, f possesses a relatively 'mild' singularity at α, it is found that the approximation by the q-Bernstein polynomials occurs on the same interval as the approximation by the Taylor's.

THEOREM 4. *Let $f \in \mathscr{C}$. Then, $B_{n,q}(f;x) \to f(x)$ uniformly on any compact set in $(-\alpha, \alpha)$.*

It can be observed that the statement of Theorem 4 has a form different from the previous ones, as it contains no information on the convergence of $\{B_{n,q}(f;\cdot)\}$ outside of $(-\alpha,\alpha)$. This is because, in general, the functions of class $\mathscr{C}$ may have a wider interval of approximation than $(-\alpha,\alpha)$, such as $f(x)=\sin(x-\alpha)/(x-\alpha)$, $f(\alpha)\neq 1$, which is approximated by $B_{n,q}(f;x)$ on any compact set in $\mathbb{R}\setminus\{\alpha\}$. Therefore, it is worth complementing the last theorem with the demonstration of its sharpness. In essence, it has to be shown that there is $f\in\mathscr{C}$ such that its q-Bernstein polynomials diverge for all $|x|>\alpha$, $x\notin\mathbb{J}_q$. The next assertion fulfills this purpose.

THEOREM 5. *Let $f\in\mathscr{C}$ and $f(x)=g(x)\ln|x-\alpha|$ for $x\in[0,\alpha)$, where the function g admits an analytic continuation from $[0,\alpha)$ into $\overline{\mathscr{D}_\beta}$ with $\beta>\alpha$. Then, $|B_{n,q}(f;x)|\to\infty$ for $|x|>\alpha$ with $x\notin\mathbb{J}_q$.*

3. Auxiliary results

In the forthcoming discussion, C stands for a non-zero constant whose exact value is inconsequential in the context of reasoning and, as such, the same letter may denote different constants. Subscripts are placed to emphasize the dependence on certain parameters whenever such an emphasis is essential. All constants are assumed to be independent of n and k.

LEMMA 1. *For any $\beta>\alpha$,*

$$\frac{1}{\Phi_{n-m-1}(x)}\to\frac{1}{\Psi_{m+1}(x)}\quad as\quad n\to\infty \tag{4}$$

uniformly on $[\alpha,\beta]$, where Φ_{n-m-1} and Ψ_{m+1} are as defined by (3).

Proof. Firstly, notice that when $x\in[\alpha,\beta]$,

$$\Phi_{n-m-1}(x)\geqslant\Psi_{m+1}(x)\geqslant\prod_{j=m+1}^{\infty}\left(1-\frac{1}{q^j\alpha}\right)=\Psi_{m+1}(\alpha)>0. \tag{5}$$

The mean value theorem for $f(t)=\ln(1-t)$ on any interval $[t_1,t_2]\subset[0,1/q]$ leads to

$$\ln(1-t_1)-\ln(1-t_2)\leqslant\frac{q(t_2-t_1)}{q-1}. \tag{6}$$

Now, for any $x\in[\alpha,\beta]$ and $j=m+1,\dots,n-1$, first setting $t_1=x_{n-j}/x,\ t_2=1/(q^jx)$ and then $t_1=0,\ t_2=1/(q^jx)$ in (6) yield

$$\left|\ln\left(1-\frac{x_{n-j}}{x}\right)-\ln\left(1-\frac{1}{q^jx}\right)\right|\leqslant\frac{q}{\alpha(q-1)(q^n-1)}\ \text{and}\ \left|\ln\left(1-\frac{1}{q^jx}\right)\right|\leqslant\frac{q^{1-j}}{\alpha(q-1)},$$

respectively. Therefore,

$$\begin{aligned}\left|\ln\frac{1}{\Phi_{n-m-1}(x)}-\ln\frac{1}{\Psi_{m+1}(x)}\right| &= \left|\sum_{j=m+1}^{n-1}\ln\left(1-\frac{x_{n-j}}{x}\right)-\sum_{j=m+1}^{\infty}\ln\left(1-\frac{1}{q^jx}\right)\right|\\ &\leqslant \sum_{j=m+1}^{n-1}\left|\ln\left(1-\frac{x_{n-j}}{x}\right)-\ln\left(1-\frac{1}{q^jx}\right)\right|+\sum_{j=n}^{\infty}\left|\ln\left(1-\frac{1}{q^jx}\right)\right|\\ &\leqslant \frac{q}{\alpha(q-1)}\frac{n-m-1}{q^n-1}+\frac{q}{\alpha(q-1)}\sum_{j=n}^{\infty}\frac{1}{q^j}\to 0 \quad \text{as} \quad n\to\infty. \quad \square\end{aligned}$$

COROLLARY 1. *Let* $\beta\in(\alpha,q\alpha)$, $\omega_{n,v}(x)=(x-x_{n-m+1})\cdots(x-x_{n-m+v})$, *and* $\omega_v(x)=(x-q\alpha)\cdots(x-q^v\alpha)$ *for* $v=1,2,\ldots$ *with* $\omega_{n,0}(x)=\omega_0(x)=1$. *Set*

$$\tilde{\Phi}_{n,v}(x):=\Phi_{n-m-1}(x)\omega_{n,v}(x) \text{ and } \tilde{\Psi}_v(x):=\Psi_{m+1}(x)\omega_v(x) \text{ for } v=0,1,\ldots,m. \tag{7}$$

Then,

$$\frac{1}{\tilde{\Phi}_{n,v}(x)}\to\frac{1}{\tilde{\Psi}_v(x)} \quad as \quad n\to\infty \quad uniformly\ on \quad [\alpha,\beta].$$

Proof. The statement follows from Lemma 1 since, for $v=0,\ldots,m$, $1/\omega_{n,v}(x)\to 1/\omega_v(x)$ as $n\to\infty$ uniformly on $[\alpha,\beta]$. $\square$

Although, the next lemma is merely a slight modification of Lemma 3.1 from [11] and its proof refers to some properties of polynomials (1) considered in [11, 13], to make the presentation self-explanatory, the adjusted proof is outlined briefly.

LEMMA 2. *If* $f\in\mathscr{F}$, *one has* $\lim_{n\to\infty}B_{n,q}(f;q^{-l})=f(q^{-l})$ *for* $l=0,1,\ldots,m-1$. *Additionally, if* $f\in\mathscr{A}$, *then* $\lim_{n\to\infty}B_{n,q}(f;q^{-l})=f(q^{-l})$ *for integers* $l>m+\gamma$, *and if* $f\in\mathscr{C}$, *then* $\lim_{n\to\infty}B_{n,q}(f;q^{-l})=f(q^{-l})$ *for integers* $l>m$.

Proof. It is easy to see from (1) that $p_{n,n-k}(q;q^{-l})=0$ when $l<k$, implying that

$$B_{n,q}(f;q^{-l})=\sum_{k=0}^{\min\{n,l\}}f(x_{n-k})p_{n,n-k}(q;q^{-l}). \tag{8}$$

On the other hand, for $l\geqslant k$, one has

$$p_{n,n-k}(q;q^{-l})\sim q^{n(k-l)}\cdot\frac{(q^{l-k+1};q)_k}{(q;q)_k}, \quad n\to\infty, \tag{9}$$

and, therefore, $\lim_{n\to\infty}p_{n,n-k}(q;q^{-l})=\delta_{kl}$ for all k and l. Since, $x_{n-k}\to q^{-k}$ as $n\to\infty$, it follows that

$$\lim_{n\to\infty}f(x_{n-k})p_{n,n-k}(q;q^{-l})=f(q^{-k})\delta_{kl},$$

whenever $k\neq m$ and $f\in\mathscr{F}$. Along with (9), the latter shows that

$$\lim_{n\to\infty}B_{n,q}(f;q^{-l})=f(q^{-l}) \quad \text{for} \quad l<m, \quad f\in\mathscr{F}.$$

In addition, for $f \in \mathscr{A}$, we have $f(x_{n-m}) \sim K(\alpha - x_{n-m})^{-\gamma} \sim Cq^{n\gamma}$, $n \to \infty$, and hence, $f(x_{n-m})p_{n,n-m}(q;q^{-l}) \sim Cq^{(\gamma+m-l)n} \to 0$ when $l > m+\gamma$. Consequently, for $f \in \mathscr{F}$ and $l > m+\gamma$,

$$\lim_{n\to\infty} B_{n,q}(f;q^{-l}) = \lim_{n\to\infty} \left[f(x_{n-m})p_{n,n-m}(q;q^{-l}) + f(x_{n-l})p_{n,n-l}(q;q^{-l}) \right] = f(q^{-l}).$$

In the case $f \in \mathscr{C}$, $\lim_{n\to\infty} f(x_{n-m})q^{-n\gamma} = 0$ for all $\gamma > 0$. With this and (9), one gets $f(x_{n-m})p_{n,n-m}(q;q^{-l}) \to 0$ as $n \to \infty$ for all $l > m$. Thus,

$$\lim_{n\to\infty} B_{n,q}(f;q^{-l}) = \lim_{n\to\infty} f(x_{n-l})p_{n,n-l}(q;q^{-l}) = f(q^{-l}),$$

as claimed. $\square$

Notwithstanding the next proof being the same as that of Lemma 3.5 in [11], it has been included here to the reader's convenience.

LEMMA 3. *If f has an analytic continuation from $[0,\alpha)$ into $\mathscr{D}_\alpha$, then for all $\varepsilon > 0$, the following inequalities hold:*

$$|c_{kn}| \leqslant C_{f,\varepsilon} q^{(m+\varepsilon)k}, \quad k = 0,1,\dots,n-m-1.$$

Proof. For any $\varepsilon \in (0,1)$, the nodes x_k, $k = 0,1,\dots,n-m-1$ are inside the circle $\{z : |z| = \rho\}$, where $\rho = q^{-m-\varepsilon}$. Therefore (cf. e.g., [6, Sec. 2.7, p. 44, formula (4)]), one has

$$|f[x_0,\dots,x_k]| = \left| \frac{1}{2\pi i} \oint_{|\zeta|=\rho} \frac{f(\zeta)d\zeta}{\zeta(\zeta - x_1)\cdots(\zeta - x_k)} \right| \leqslant \frac{1}{2\pi} \oint_{|\zeta|=\rho} \frac{|f(\zeta)d\zeta|}{|\zeta^{k+1}\Phi_k(\zeta)|}, \quad (10)$$

where $f(\zeta)$, as stated before, is an analytic continuation of f into $\mathscr{D}_\alpha$. Note that $|1 - \frac{x_k}{\zeta}| \geqslant 1 - \frac{x_k}{\rho}$, $k = 0,1,\dots,n-m-1$ and, employing (5), one obtains that $|\Phi_k(\zeta)| \geqslant \Phi_k(\rho) \geqslant \Psi_{m+1}(\rho)$. So,

$$|f[x_0,\dots,x_k]| \leqslant \frac{\rho^{-k}}{\Psi_{m+1}(\rho)} \max_{|\zeta|=\rho} |f(\zeta)|. \quad (11)$$

Since $\lambda_{kn} \in (0,1]$ for all k and n, it can be readily seen that

$$|c_{kn}| = |\lambda_{kn} f[x_0,x_1,\dots,x_k]| \leqslant C\rho^{-k} = C_{f,\varepsilon} q^{(m+\varepsilon)k}. \quad (12)$$

For any $\varepsilon \geqslant 1$, the inequality (12) implies the required result. $\square$

LEMMA 4. *If $f \in \mathscr{A} \cup \mathscr{B}$, then for $v = 0,1,\dots,m$,*

$$c_{n-m+v,n} \sim C_v f(x_{n-m})\alpha^{-n} \quad \textit{as} \quad n \to \infty.$$

Proof. By the divided difference formula,

$$f[x_0,x_1,\dots,x_{n-m+v}] = \sum_{r=0}^{n-m+v} \frac{f(x_r)}{\prod\limits_{\substack{s=0\\ s\neq r}}^{n-m+v}(x_r-x_s)}$$

$$= \sum_{r=0}^{n-m-1} + \frac{f(x_{n-m})}{\prod\limits_{\substack{s=0\\ s\neq n-m}}^{n-m+v}(x_{n-m}-x_s)} + \sum_{r=n-m+1}^{n-m+v} =: S_1+S_2+S_3. \quad (13)$$

Notice that $S_3=0$ when $v=0$. Next, if $f(\zeta)$ is an analytic continuation of f into $\mathscr{D}_\alpha$, then

$$S_1 = \frac{1}{2\pi i}\oint_{|\zeta|=\rho} \frac{f(\zeta)d\zeta}{\zeta^{n-m}\Phi_{n-m-1}(\zeta)(\zeta-x_{n-m})w_{n,v}(\zeta)}$$

whenever $\varepsilon\in(0,1)$ and $\rho=q^{-(m+\varepsilon)}$. Moreover, since $x_{n-m}\to\alpha$ as $n\to\infty$, it follows that $x_{n-m}>q^{-(m+\varepsilon/2)}$ for n large enough. Thence, for such values of n,

$$|x_{n-m+j}-\zeta| \geqslant |x_{n-m}-\zeta| \geqslant q^{-(m+\varepsilon/2)}-q^{-(m+\varepsilon)}>0,$$

and, consequently,

$$|(\zeta-x_{n-m})\omega_{n,v}(x)| \geqslant \left[q^{-(m+\varepsilon/2)}-q^{-(m+\varepsilon)}\right]^v>0.$$

Applying operations similar to those in the proof of Lemma 3, one obtains that

$$|S_1|\leqslant Cq^{(m+\varepsilon)k}. \quad (14)$$

On the other hand, as it has been proved in [11, formulae (3.6) and (3.7)],

$$\frac{1}{(x_{n-m}-x_0)(x_{n-m}-x_1)\cdots(x_{n-m}-x_{n-m-1})} \sim Cq^{mn}=C\alpha^{-n} \text{ as } n\to\infty \quad (15)$$

implying $S_2\sim Cf(x_{n-m})\alpha^{-n}$ as $n\to\infty$.

To estimate S_3, we refer to Corollary 3.3 of [11], which says that

$$\prod_{\substack{s=0\\ s\neq n-m+u}}^{n-m+v} \frac{1}{x_{n-m+u}-x_s} \sim C_{u,v}q^{n(m-u)}, \quad (16)$$

yielding

$$|S_3|\leqslant Cq^{n(m-1)}. \quad (17)$$

It is worth pointing out that (14), (15), and (17) are valid for all $f\in\mathscr{F}$. Also, if $f\in\mathscr{A}$, then

$$q^{\varepsilon n}=o(f(x_{n-m})) \text{ as } n\to\infty \quad (18)$$

for $\varepsilon<\gamma$, while if $f\in\mathscr{B}$, then (18) holds for all $\varepsilon>0$. As a result, for $f\in\mathscr{A}\cup\mathscr{B}$, by virtue of (14) and (17), one has $S_1=o(S_2)$ and $S_3=o(S_2)$ as $n\to\infty$. Hence,

$$c_{n-m+v,n}\sim\lambda_{n-m+v,n}S_2\sim C_v\lambda_{n-m+v,n}f(x_{n-m})\alpha^{-n}\sim C_vf(x_{n-m})\alpha^{-n} \quad \text{as} \quad n\to\infty. \quad \square$$

COROLLARY 2. *If* $f \in \mathscr{A} \cup \mathscr{B}$, *then* $\lim_{n\to\infty} \frac{c_{n-m+v,n}}{c_{n-m,n}} = C_v \neq 0$, $v = 1, 2, \ldots, m$.

REMARK 2. It can be proved that $\lim_{n\to\infty} \frac{c_{n-m+v,n}}{c_{n-m,n}} = (-1)^v \begin{bmatrix} m \\ v \end{bmatrix} q^{v(v-1)/2}$.

LEMMA 5. *If* $f \in \mathscr{C}$, *then for all* $\varepsilon > 0$, *there exists* $C = C_{f,\varepsilon}$ *such that* $|c_{kn}| \leqslant Cq^{(m+\varepsilon)k}$ *for all* $k = 0, 1, \ldots, n$.

Proof. For $k \leqslant n - m - 1$, the claim is contained in Lemma 3. For the remaining case $k \geqslant n - m$, $f[x_0, \ldots, x_k] = f[x_0, \ldots, x_{n-m+v}] = S_1 + S_2 + S_3$, like in (13). Choose $\varepsilon \in (0,1)$ and $\rho = q^{-(m+\varepsilon)}$. Then, the inequalities for S_1 and S_3 are the same as in (14) and (17), while, by virtue of (15), $S_2 \sim Cf(x_{n-m})\alpha^{-n}$ as $n \to \infty$. Since $f \in \mathscr{C}$, it follows that, for n large enough, $|f(x_{n-m})| \leqslant (\alpha - x_{n-m})^{-\varepsilon} \leqslant Cq^{n\varepsilon}$, whence $|S_2| \leqslant Cq^{(m+\varepsilon)n}$. Collecting the estimates for S_1, S_2, and S_3, one derives $|c_{n-m+v,n}| \leqslant Cq^{(m+\varepsilon)n}$, $v = 0, \ldots, m$, as required. □

4. Proofs of Theorems 2-4

Proof of Theorem 2.

(i) Since, $f(x_{n-m}) \sim Cq^{n\gamma}$ for $f \in \mathscr{A}$, from Lemmas 3 and 4 one has $|c_{kn}| \leqslant C(\alpha^{-1}q^{\gamma})^k$ for all $k = 0, 1, \ldots, n$. Hence, for $|z| \leqslant \rho < \alpha^{-1}q^{\gamma}$, there holds:

$$|B_{n,q}(f;z)| \leqslant \sum_{k=0}^{n} |c_{kn}|\rho^k \leqslant C\sum_{k=0}^{n} (\alpha^{-1}q^{\gamma}\rho)^k \leqslant \frac{C}{1 - \alpha^{-1}q^{\gamma}\rho},$$

that is, polynomials $B_{n,q}(f;z)$ are uniformly bounded in $\overline{\mathscr{D}_\rho}$. Together with Lemma 3, this implies that $\{B_{n,q}(f;z)\}$ satisfies the conditions of Vitali's Convergence Theorem (cf. [15, Ch.5, Th.5.21]) and, thus, is uniformly convergent on any compact set in $\mathscr{D}_\rho$ for any $\rho \in (0, \alpha q^{-\gamma})$.

(ii) Let $B_{n,q}(f;x) = \sum_{k=0}^{n} c_{kn}x^k = \sum_{k=0}^{n-m-1} + \sum_{k=n-m}^{n} := \sigma_1(x) + \sigma_2(x)$. By Lemma 3, for $\varepsilon \in (0, \gamma)$ and $|x| > q^{-(m+\varepsilon)}$,

$$|\sigma_1(x)| \leqslant C_\varepsilon \sum_{k=0}^{n-m-1} q^{(m+\varepsilon)k}|x|^k = C_\varepsilon \frac{(q^{m+\varepsilon}|x|)^{n-m} - 1}{q^{m+\varepsilon}|x| - 1} \leqslant C_{\varepsilon,x}\left(q^{m+\varepsilon}|x|\right)^n. \tag{19}$$

Meanwhile,

$$\sigma_2(x) = \sum_{v=0}^{m} c_{n-m+v,n}x^{n-m+v} = c_{n-m,n}x^{n-m}\left\{1 + \tilde{c}_{1,n}x + \cdots \tilde{c}_{m,n}x^m\right\},$$

which, by Corollary 2, yields

$$\lim_{n\to\infty} (1 + \tilde{c}_{1,n}x + \cdots \tilde{c}_{m,n}x^m) = 1 + d_1x + \cdots d_mx^m. \tag{20}$$

The right-hand side of (20) vanishes for, at most, m points, say $y_1,\dots,y_s,\ s\leqslant m$. Therefore, if $x\notin\{y_1,\dots,y_s\}$, then

$$\sigma_2(x)\sim C_x c_{n-m}x^{n-m}\sim C_x q^{(m+\gamma)n}|x|^n \quad \text{as} \quad n\to\infty$$

and, hence, $\sigma_1(x)=o(\sigma_2(x))$ as $n\to\infty$ for $x\notin\{y_1,\dots,y_s\}$. This shows that $|B_{n,q}(f;x)|\to\infty$ as $n\to\infty$ for $|x|>\alpha q^{-\gamma},\ x\notin\{y_1,\dots,y_s\}$. On the other hand, by Lemma 2, $B_{n,q}(f;x)\to f(x)$ as $n\to\infty$ for $x\in\{1,q^{-1},\dots,q^{-(m-1)}\}$, whence $\{y_1,\dots,y_s\}=\{1,q^{-1},\dots,q^{-(m-1)}\}$. $\square$

Proof of Theorem 3. Similar to the proof of Theorem 2, let

$$B_{n,q}(f;x)=\sum_{k=0}^{n-m-1}c_{kn}x^k+\sum_{k=n-m}^{n}c_{kn}x^k=:\sigma_1(x)+\sigma_2(x).$$

Since Lemma 3 holds for all $f\in\mathscr{F}$, estimate (19) is valid for any $\varepsilon>0$, that is

$$\sigma_1(x)\leqslant C(q^{m+1}|x|)^n \text{ for } x\neq 0. \tag{21}$$

As for $\sigma_2(x)$, by applying the reasoning used in the proof of Theorem 2 (ii), one can derive

$$\sigma_2(x)\sim C_x c_{n-m}x^{n-m} \quad \text{as} \quad n\to\infty \tag{22}$$

for all $x\neq 0$ except, possibly, for at most m points, $y_1,\dots,y_s,\ s\leqslant m$. Take $x\notin\{0,1,q^{-1},\dots,q^{-(m-1)}\}$ and choose $\gamma=\gamma(x)>1$ in such a way that $q^{m+\gamma}|x|>1$. Now, $f(x_{n-m})(\alpha-x_{n-m})^{\gamma}\to\infty$, since $f\in\mathscr{B}$. Bearing in mind that $\alpha-x_{n-m}\sim Cq^{-n}$ as $n\longrightarrow\infty$, one obtains $|f(x_{n-m})|\geqslant Cq^{n\gamma}$ for any $C>0$, when n is large enough. Later, applying Lemma 4 to equation (22) yields $|\sigma_2(x)|\geqslant C(q^{m+\gamma}|x|)^n$ for n large enough. The choice $\gamma>1$ stipulates $\sigma_1(x)=o(\sigma_2(x))$ as $n\to\infty$ due to (21). The stated result follows because $q^{m+\gamma}|x|>1$. $\square$

Proof of Theorem 4. Let $\rho\in(0,\alpha)$, i.e., $\rho=q^{-(m+\delta)}$. By Lemma 4, $|c_{kn}|\leqslant Cq^{(m+\delta/2)}$ for all $k=0,\dots,n$, or, for $|z|\leqslant\rho$,

$$|B_{n,q}(f;z)|\leqslant\sum_{k=0}^{n}|c_{kn}|\rho^k\leqslant C\sum_{k=0}^{n}\left(q^{-\delta/2}\right)^k\leqslant\frac{C}{1-q^{-\delta/2}}.$$

The last inequality demonstrates that polynomials $B_{n,q}(f;z)$ are uniformly bounded in $\mathscr{D}_\rho$. Along with Lemma 5, this means that $\{B_{n,q}(f;z)\}$ satisfies the conditions of Vitali's Theorem and, as a result, converges uniformly to $f(z)$ on any compact set in $\mathscr{D}_\rho$. Due to the fact that $\rho\in(0,\alpha)$ has been chosen arbitrarily, the proof is complete.

5. Proof of Theorem 5

For the ease of presentation, the proof of this theorem has been split into a sequence of lemmas. While Corollary 3, Lemma 8, and Corollary 4 provide estimates for the coefficients of $B_{n,q}(f;x)$ when $f\in\mathscr{C}$, Lemmas 6 and 7 contain the necessary technical background. It should be pointed out that these lemmas are applicable not only for functions in the class $\mathscr{C}$, but also for a more general setting.

LEMMA 6. *Let $f \in \mathscr{F}$ and $f(\zeta)$ be an analytic continuation of f from $[0,\alpha)$ into $\mathscr{D}_\alpha$. If there exist $\gamma > 0$ and $C > 0$ such that*

$$\max_{|\zeta|\leqslant|z|} |f(\zeta)| \leqslant \frac{C}{(\alpha-|z|)^\gamma} \quad \text{for all } z \in \mathscr{D}_\alpha, \tag{23}$$

then, for $k = 0,1,\dots,n-m-1$, one has

$$|c_{kn}| \leqslant C(k+\gamma)^\gamma \alpha^{-k} = C(k+\gamma)^\gamma q^{mk}.$$

Proof. For $\varepsilon \in (0,1)$, set $\rho = q^{-(m+\varepsilon)}$. Incorporating (23) into (11) gives

$$|f[x_0,\dots,x_k]| \leqslant \frac{C\rho^{-k}}{(\alpha-\rho)^\gamma \Psi_{m+1}(\rho)} \leqslant \frac{C}{\rho^k(\alpha-\rho)^\gamma}.$$

Let $k_0 = \lceil \gamma/(q^\varepsilon - 1)\rceil$. If $k \geqslant k_0$, then $\rho_k := \frac{k\alpha}{k+\gamma} \in (\alpha q^{-\varepsilon},\alpha)$. For $k \geqslant k_0$, plugging $\rho = \rho_k$ in the latter inequality yields

$$|c_{kn}| \leqslant C(k+\gamma)^\gamma \alpha^{-k}(1+\gamma/k)^k \leqslant C(k+\gamma)^\gamma \alpha^{-k}.$$

On the other hand, for $k < k_0$, there holds:

$$|c_{kn}| \leqslant |f[x_0,\dots,x_k]| \leqslant \max_{0\leqslant k\leqslant k_0} \left(\max_{0\leqslant \xi\leqslant \alpha q^{-1}} \frac{\left|f^{(k)}(\xi)\right|}{k!} \right) = C,$$

which completes the proof. □

COROLLARY 3. *Let f be as in Theorem 5. Then, for $k = 0,1,\dots,n-m-1$,*

$$|c_{kn}| \leqslant C_\gamma (k+1)^\gamma \alpha^{-k} \quad \text{for all} \quad \gamma > 0.$$

The next Lemma offers asymptotic estimates concerned with the properties of nodes $x_k = [k]_q/[n]_q$, $k = 0,\dots,n$, and functions $\tilde{\Phi}_{n,v}$ and $\tilde{\Psi}_v$ defined by (7).

LEMMA 7. *(i) For all $\delta > 0$,*

$$\int_\alpha^{\alpha+\delta} \frac{dx}{x^{n-m}(x-x_{n-m})} \sim \alpha^m \ln q \cdot n\alpha^{-n} \quad \text{as} \quad n \to \infty. \tag{24}$$

(ii) If $\beta \in (\alpha, q\alpha)$ and φ is any function continuous on $[\alpha,\beta]$ with $\varphi(\alpha) \neq 0$, then, for $v = 0,1,\dots,m$,

$$\int_\alpha^\beta \frac{\varphi(x)dx}{\tilde{\Phi}_{n,v}(x)x^{n-m}(x-x_{n-m})} \sim \frac{\alpha^m \varphi(\alpha)\ln q}{\tilde{\Psi}_v(\alpha)} \cdot n\alpha^{-n} \quad \text{as} \quad n \to \infty. \tag{25}$$

Proof. (i) By partial fraction decomposition,

$$\int_{\alpha}^{\alpha+\delta} \frac{dx}{x^{n-m}(x-x_{n-m})} = \sum_{j=1}^{n-m-1} \frac{1}{j} \cdot \frac{1}{x_{n-m}^{n-m-j}} \left(\frac{1}{(\alpha+\delta)^j} - \frac{1}{\alpha^j} \right) + \frac{1}{x_{n-m}^{n-m}} \left[\ln\left(1 - \frac{x_{n-m}}{\alpha+\delta}\right) - \ln\left(1 - \frac{x_{n-m}}{\alpha}\right) \right].$$

Clearly,

$$\left| \sum_{j=1}^{n-m-1} \right| \leqslant \sum_{j=1}^{n-m-1} \frac{1}{j} \cdot \frac{1}{x_{n-m}^{n-m-j}} \cdot \frac{1}{\alpha^j} \leqslant \frac{1}{x_{n-m}^{n-m}} \sum_{j=1}^{n-m-1} \frac{1}{j} \leqslant \frac{e^{1/\ln q} \ln n}{\alpha^{n-m}}.$$

Meanwhile,

$$\frac{1}{x_{n-m}^{n-m}} \left[\ln\left(1 - \frac{x_{n-m}}{\alpha+\delta}\right) - \ln\left(1 - \frac{x_{n-m}}{\alpha}\right) \right] \sim \frac{1}{\alpha^{n-m}} \left[\ln\left(1 - \frac{\alpha}{\alpha+\delta}\right) - \ln\left(1 - \frac{q^{n-m}-1}{\alpha(q^n-1)}\right) \right] \sim \alpha^m \ln q \cdot n\alpha^{-n} \quad \text{as } n \to \infty.$$

(ii) Without loss of generality, assume that $\varphi(\alpha) > 0$. By Corollary 1, for any $\varepsilon > 0$, there exist $\delta > 0$ and $N \in \mathbb{N}$ such that

$$\left| \frac{\varphi(x)}{\tilde{\Phi}_{n,v}(x)} - \frac{\varphi(\alpha)}{\tilde{\Psi}_v(\alpha)} \right| < \frac{(-1)^v \varphi(\alpha)}{\tilde{\Psi}_v(\alpha)} \cdot \varepsilon \quad \text{for} \quad x \in [\alpha, \alpha+\delta] \quad \text{and} \quad n > N. \tag{26}$$

Consider

$$\frac{\tilde{\Psi}_v(\alpha)}{\varphi(\alpha)} \frac{\alpha^{n-m}}{n \ln q} \int_{\alpha}^{\beta} \frac{\varphi(x)}{\tilde{\Phi}_{n,v}(x)} \frac{dx}{x^{n-m}(x-x_{n-m})} = \frac{\tilde{\Psi}_v(\alpha)}{\varphi(\alpha)} \frac{\alpha^{n-m}}{n \ln q} \left\{ \int_{\alpha}^{\alpha+\delta} + \int_{\alpha+\delta}^{\beta} \right\}.$$

Since, for n large enough, $x_{n-m+1} \geqslant (\beta + \alpha q)/2$, one has $|\omega_{n,v}(x)| \geqslant [(\alpha q - \beta)/2]^v$ for $x \in [\alpha, \beta]$. Therefore,

$$\left| \int_{\alpha+\delta}^{\beta} \right| \leqslant \max_{x \in [\alpha,\beta]} \left| \frac{\varphi(x)}{\tilde{\Phi}_{n,v}(x)} \right| \cdot \left(\frac{2}{\alpha q - \beta} \right)^v \cdot \left(\frac{1}{\alpha+\delta} \right)^{n-m} \cdot \frac{\beta - \alpha}{\delta} = o(\alpha^{-n}), \text{ as } n \to \infty,$$

whence

$$\lim_{n\to\infty} \frac{\tilde{\Psi}_v(\alpha)}{\varphi(\alpha)} \frac{\alpha^{n-m}}{n \ln q} \int_{\alpha+\delta}^{\beta} = 0. \tag{27}$$

It follows from (26) that, for n large enough,

$$(1-\varepsilon) < \frac{\tilde{\Psi}_v(\alpha)}{\varphi(\alpha)} \cdot \frac{\varphi(x)}{\tilde{\Phi}_{n,v}(x)} < (1+\varepsilon), \quad x \in [\alpha, \alpha+\delta],$$

implying

$$(1-\varepsilon) \int_{\alpha}^{\alpha+\delta} \frac{dx}{x^{n-m}(x-x_{n-m})} < \frac{\tilde{\Psi}_v(\alpha)}{\varphi(\alpha)} \int_{\alpha}^{\alpha+\delta} < (1+\varepsilon) \int_{\alpha}^{\alpha+\delta} \frac{dx}{x^{n-m}(x-x_{n-m})}. \tag{28}$$

Then, (27) and (28) along with (24) result in

$$\limsup_{n\to\infty}\frac{\tilde{\Psi}_v(\alpha)}{\varphi(\alpha)}\frac{\alpha^{n-m}}{n\ln q}\int_\alpha^\beta\frac{\varphi(x)}{\tilde{\Phi}_{n,v}(x)}\frac{dx}{x^{n-m}(x-x_{n-m})}=\limsup_{n\to\infty}\frac{\tilde{\Psi}_v(\alpha)}{\varphi(\alpha)}\frac{\alpha^{n-m}}{n\ln q}\int_\alpha^{\alpha+\delta}$$
$$\leqslant\limsup_{n\to\infty}\frac{(1+\varepsilon)\alpha^{n-m}}{n\ln q}\int_\alpha^{\alpha+\delta}\frac{dx}{x^{n-m}(x-x_{n-m})}=(1+\varepsilon).$$

Likewise,

$$\liminf_{n\to\infty}\frac{\tilde{\Psi}_v(\alpha)}{\varphi(\alpha)}\frac{\alpha^{n-m}}{n\ln q}\int_\alpha^\beta\frac{\varphi(x)}{\tilde{\Phi}_{n,v}(x)}\frac{dx}{x^{n-m}(x-x_{n-m})}\geqslant(1-\varepsilon).$$

Finally, since $\varepsilon>0$ has been chosen arbitrarily, the statement is justified. □

LEMMA 8. *If f is as in Theorem 5, then, for $v=0,1,\dots,m$,*

$$c_{n-m+v,n}\sim-\lambda_{n-m+v,n}\frac{\alpha^m g(\alpha)\ln q}{\tilde{\Psi}_v(\alpha)}\cdot n\alpha^{-n}\quad as\quad n\to\infty. \tag{29}$$

Proof. From the divided differences formula,

$$f[x_0,\dots,x_{n-m+v}]=\sum_{r=0}^{n-m+v}\frac{f(x_r)}{x(x-x_1)\cdots(x-x_{n-m+v})}=\sum_{r=0}^{n-m}+\sum_{r=n-m+1}^{n-m+v}.$$

Notice that the second sum is taken as 0 when $v=0$. Using the contour given in Figure 1, with $\rho\in(x_{n-m},\alpha)$ and $\beta\in(\alpha,q\alpha)$, one has:

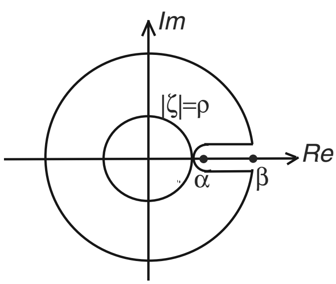

Figure 1: *Contour of integration used in Lemma 8.*

$$\sum_{r=0}^{n-m}=\frac{1}{2\pi i}\oint_{|\zeta|=\rho}\frac{f(\zeta)d\zeta}{\zeta(\zeta-x_1)\cdots(\zeta-x_{n-m+v})}=\frac{1}{2\pi i}\oint_{|\zeta|=\rho}\frac{f(\zeta)d\zeta}{\zeta^{n-m}(\zeta-x_{n-m})\tilde{\Phi}_{n,v}(\zeta)}$$
$$=-\int_\alpha^\beta\frac{f(x)dx}{x^{n-m}(x-x_{n-m})\tilde{\Phi}_{n,v}(x)}+\frac{1}{2\pi i}\oint_{|\zeta|=\beta}\frac{f(\zeta)d\zeta}{\zeta^{n-m}(\zeta-x_{n-m})\tilde{\Phi}_{n,v}(\zeta)}=:I_1+I_2.$$

By Lemma 7 (ii),

$$I_1 \sim -\frac{\alpha^m g(\alpha)\ln q}{\tilde{\Psi}_v(\alpha)} \cdot n\alpha^{-n} \quad \text{as} \quad n \to \infty. \tag{30}$$

To estimate I_2, note that $|\tilde{\Phi}_{n,v}(\zeta)| \geqslant \Psi_{m+1}(\beta) > 0$, $|\zeta - x_{n-m}| \geqslant \beta - \alpha$, and $|\omega_{n,v}(\zeta)| \to |\omega_v(\zeta)| \geqslant (\alpha q - \beta)^v$, as $n \to \infty$ for $v = 0, 1, \dots, m$. Therefore,

$$|I_2| \leqslant \frac{1}{2\pi} \frac{2\pi\beta \max_{|\zeta|=\beta} |f(\zeta)|}{\beta^{n-m}(\beta-\alpha)\tilde{\Phi}_{n,v}(\beta)[(\alpha q - \beta)/2]^v} = O(\beta^{-n}) = o(\alpha^{-n}) \quad \text{as}\, n \to \infty.$$

As a result,

$$\sum_{r=0}^{n-m} \sim I_1 \ \text{ as } \ n \to \infty.$$

In order to estimate $\sum_{r=n-m+1}^{n-m+v}$, recall (16) and obtain

$$\left| \sum_{r=n-m+1}^{n-m+v} \right| \leqslant \sum_{r=1}^{v} |f(x_r)| \prod_{\substack{s=0 \\ s \neq n-m+r}}^{n-m+v} \frac{1}{|x_{n-m+r} - x_s|} \leqslant C q^{n(m-1)} = o(\alpha^{-n}), \ \ n \to \infty.$$

Thus, $c_{n-m+v,n} = \lambda_{n-m+v,n} f[x_0, \dots, x_{n-m+v}] \sim \lambda_{n-m+v,n} I_1$ as $n \to \infty$ and, by means of (30), estimate (29) comes out. □

COROLLARY 4. *If f is as in Theorem 5, then, for $v = 1, 2, \dots, m$,*

$$\lim_{n\to\infty} \frac{c_{n-m+v,n}}{c_{n-m,n}} = (-1)^v \begin{bmatrix} m \\ v \end{bmatrix}_q q^{v(v-1)/2}. \tag{31}$$

Proof. Taking into account that

$$\begin{bmatrix} m \\ v \end{bmatrix}_q = \frac{(q^m - 1)\cdots(q^{m-v+1} - 1)}{(q^v - 1)\cdots(q - 1)},$$

one has

$$\lim_{n\to\infty} \frac{c_{n-m+v,n}}{c_{n-m,n}} = \frac{\tilde{\Psi}_0(\alpha)}{\tilde{\Psi}_v(\alpha)} \lim_{n\to\infty} \frac{\lambda_{n-m+v,n}}{\lambda_{n-m,n}} = \frac{1}{\omega_v(\alpha)}(1 - q^{-m})\cdots(1 - q^{-m+v-1})$$
$$= \frac{q^{mv}}{(1-q)\cdots(1-q^v)} \frac{(q^m - 1)\cdots(q^{m-v+1} - 1)}{q^{mv - v(v-1)/2}} = (-1)^v \begin{bmatrix} m \\ v \end{bmatrix}_q q^{v(v-1)/2},$$

as claimed. □

At this stage, all preliminary work has been completed and the way to reach the proof, which is presented below, has been paved.

Proof of Theorem 5. Consider

$$B_{n,q}(f;x) = \sum_{k=0}^{n} c_{kn} x^k = \sum_{k=0}^{n-m-1} c_{kn} x^k + \sum_{k=n-m}^{n} c_{kn} x^k =: S_1(x) + S_2(x).$$

Select $\gamma \in (0,1)$. By Corollary 3, there exists $C > 0$ so that $|S_1(x)| \leqslant C\sum_{k=0}^{n-m-1}(k+1)^{\gamma}\alpha^{-k}x^k$. Then, for $|x| > \alpha$,

$$|S_1(x)| \leqslant Cn^{\gamma}\sum_{k=0}^{n-m-1}\left(\alpha^{-1}|x|\right)^k \leqslant C_x n^{\gamma}(\alpha^{-1}|x|)^n = o\left(n(\alpha^{-1}|x|)^n\right)$$

as $n \to \infty$. Meanwhile, by the Rothe Identity ([2, Ch. 10])

$$(x;q)_m = \sum_{\nu=0}^{m}(-1)^{\nu}\begin{bmatrix} m \\ \nu \end{bmatrix}_q q^{\nu(\nu-1)/2}x^{\nu},$$

and by virtue of Corollary 4,

$$S_2(x) \sim c_{n-m,n}x^{n-m}(x;q)_m \quad \text{as} \quad n \to \infty.$$

If $x \neq 1, q^{-1}, \ldots, q^{-(m-1)}$, then $(x;q)_m \neq 0$ and

$$S_2(x) \sim \lambda_{n-m,n}(x;q)_m \cdot \frac{\alpha^{-m}g(\alpha)\ln q}{\tilde{\Psi}_0(\alpha)} \cdot n\alpha^{-n}x^{n-m} \sim C_x n\alpha^{-n}x^n \quad \text{as} \quad n \to \infty.$$

Hence, $S_1(x) = o(S_2(x))$, $n \to \infty$. Thus, when $x \notin \{1, q^{-1}, \ldots, q^{-(m-1)}\}$, we arrive at

$$|B_{n,q}(f;x)| \sim |S_2(x)| \to \infty \text{ as } n \to \infty \text{ since } |x| > \alpha. \quad \square$$

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