Journal of Mathematical Inequalities
Volume 9, Number 1 (2015), 137–144 doi:10.7153/jmi-09-13

CONTRACTIONS AND THE SPECTRAL CONTINUITY FOR k–QUASI–PARANORMAL OPERATORS

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(*Communicated by J. Pečarić*)

Abstract. For a positive integer k, an operator $T \in B(\mathscr{H})$ is called k-quasi-paranormal if $\|T^{k+1}x\|^2 \leqslant \|T^{k+2}x\|\|T^k x\|$ for all $x \in \mathscr{H}$, which is a common generalization of paranormal and quasi-paranormal. In this paper, firstly we prove that if T is a contraction of k-quasi-paranormal operators, then either T has a nontrivial invariant subspace or T is a proper contraction and the nonnegative operator $D_\lambda = T^{*k}(|T^2|^2 - 2\lambda|T|^2 + \lambda^2 I)T^k$ for $0 < \lambda \leqslant 1$ is a strongly stable contraction; secondly we prove that k-quasi-paranormal operators are not supercyclic; at last we prove that the spectrum is continuous on the class of all k-quasi-paranormal operators.

1. Introduction

Throughout this paper let $\mathscr{H}$ be a separable complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$. Let $B(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators on $\mathscr{H}$. If $T \in B(\mathscr{H})$, we shall write $\ker T$ and $\operatorname{ran} T$ for the null space and range of T respectively. Also let $\alpha(T) = \dim \ker T$, $\beta(T) = \dim \ker T^*$ and let $\sigma(T)$ and $\sigma_a(T)$ denote the spectrum and the approximate point spectrum of T. Let $p = p(T)$ and $q = q(T)$ be the ascent and descent of T respectively. An operator $T \in B(\mathscr{H})$ is called upper (resp. lower) semi-Fredholm if $\operatorname{ran} T$ is closed and $\alpha(T) < \infty$ (resp. $\beta(T) < \infty$). If $T \in B(\mathscr{H})$ is either an upper semi-Fredholm operator or a lower semi-Fredholm operator, then T is called a semi-Fredholm operator, and the index of a semi-Fredholm operator $T \in B(\mathscr{H})$, denoted by $\operatorname{ind}(T)$, is given by the integer $\operatorname{ind}(T) = \alpha(T) - \beta(T)$. If both $\alpha(T)$ and $\beta(T)$ are finite, then T is called a Fredholm operator. An operator $T \in B(\mathscr{H})$ is called Weyl if it is Fredholm of index zero and Browder if it is Fredholm of finite ascent and descent. The essential spectrum $\sigma_e(T)$, the Weyl spectrum $\sigma_w(T)$ and the Browder spectrum $\sigma_b(T)$ of $T \in B(\mathscr{H})$ are defined by $\sigma_e(T) = \{\lambda \in \mathscr{C} : T - \lambda$ is not Fredholm$\}$, $\sigma_w(T) = \{\lambda \in \mathscr{C} : T - \lambda$ is not Weyl$\}$, and $\sigma_b(T) = \{\lambda \in \mathscr{C} : T - \lambda$ is not Browder$\}$.

Recall that $T \in B(\mathscr{H})$ is called p-hyponormal for $p > 0$ if $(T^*T)^p - (TT^*)^p \geqslant 0$ [1]; when $p = 1$, T is called hyponormal. An operator is called paranormal if $\|Tx\|^2 \leqslant \|T^2x\|\|x\|$ for each $x \in \mathscr{H}$, [10, 11]. And T is called normaloid if $\|T^n\| = \|T\|^n$ for all $n \in \mathbb{N}$(equivalently, $\|T\| = r(T)$, the spectral radius of T). In order to discuss

Mathematics subject classification (2010): 47B20, 47A63.
Keywords and phrases: Contractions, k-quasi-paranormal operators, spectral continuity.

Paper JMI-09-13

the relations between paranormal and p-hyponormal and log-hyponormal operators, Furuta, Ito and Yamazaki [12] introduced class A operators defined by $|T^2|-|T|^2 \geqslant 0$, and they showed that class A is a subclass of paranormal and contains p-hyponormal and log-hyponormal operators.

DEFINITION 1.1. $T \in B(\mathscr{H})$ is called k-quasi-paranormal if for a positive integer k

$$\|T^{k+1}x\|^2 \leqslant \|T^{k+2}x\|\|T^kx\|$$

for all $x \in \mathscr{H}$.

When $k=1$, T is called quasi-paranormal operators; quasi-paranormal and k-quasi-paranormal operators have been studied in [13, 19, 21].

It is clear that

$$\begin{aligned} \text{the class of paranormal operators} &\subseteq \text{the class of quasi-paranormal operators} \\ &\subseteq \text{the class of k-quasi-paranormal operators} \\ &\subseteq \text{the class of (k+1)-quasi-paranormal operators.} \end{aligned} \tag{1.1}$$

We have shown that the inclusion relations in (1.1) is strict in [13].

A contraction is an operator T such that $\|Tx\| \leqslant \|x\|$ for every $x \in \mathscr{H}$. A contraction T is said to be a proper contraction if $\|Tx\| < \|x\|$ for every nonzero $x \in \mathscr{H}$. A strict contraction is an operator T such that $\|T\| < 1$. A strict contraction is a proper contraction, but a proper contraction is not necessary a strict contraction, although the concepts of strict and proper contractions coincide for compact operators. A contraction T is of class $C_{0.}$ if $\|T^nx\| \to 0$ when $n \to \infty$ for every $x \in \mathscr{H}$ (i.e., T is a strongly stable contraction) and it is said to be of class $C_{1.}$ if $\lim_{n\to\infty}\|T^nx\| > 0$ for every nonzero $x \in \mathscr{H}$. Classes $C_{.0}$ and $C_{.1}$ are defined by considering T^* instead of T and we define the class $C_{\alpha\beta}$ for α, $\beta = 0, 1$ by $C_{\alpha\beta} = C_{\alpha.} \bigcap C_{.\beta}$.

In this paper, firstly we prove that if T is a contraction of k-quasi-paranormal operators, then either T has a nontrivial invariant subspace or T is a proper contraction and the nonnegative operator $D_\lambda = T^{*k}(|T^2|^2 - 2\lambda|T|^2 + \lambda^2 I)T^k$ for $0 < \lambda \leqslant 1$ is a strongly stable contraction; secondly we prove that k-quasi-paranormal operators are not supercyclic; at last we prove that the spectrum is continuous on the class of all k-quasi-paranormal operators.

2. Contractions of k-quasi-paranormal operators

It is well known that T is paranormal if and only if $T^{2*}T^2 - 2\lambda T^*T + \lambda^2 I \geqslant 0$ for all $\lambda > 0$. Similarly, we have the following result.

LEMMA 2.1. (see [13, 19]) *Let $T \in B(\mathscr{H})$. Then T is k-quasi-paranormal for a positive integer k if and only if*

$$T^{k*}(T^{2*}T^2 - 2\lambda T^*T + \lambda^2 I)T^k \geqslant 0$$

for all $\lambda > 0$.

For each $\lambda > 0$, let

$$\mathscr{D}_\lambda = \{T \in B(\mathscr{H}) : T^{k*}(T^{2*}T^2 - 2\lambda T^*T + \lambda^2 I)T^k \geqslant 0\}$$

for a positive integer k. Thus T is k-quasi-paranormal if and only if $T \in \bigcap_{\lambda>0} \mathscr{D}_\lambda$.

THEOREM 2.2. *If T is a contraction of class $\mathscr{D}_\lambda$ for $0 < \lambda \leqslant 1$, then the nonnegative operator $D_\lambda = T^{k*}(|T^2|^2 - 2\lambda|T|^2 + \lambda^2 I)T^k$ is a contraction whose power sequence $\{D_\lambda^n\}$ converges strongly to a projection P, and $T^kP = 0$ when $\lambda \in (\frac{1}{2}, 1]$.*

Proof. Suppose that T is a contraction of class $\mathscr{D}_\lambda$, then $D_\lambda = T^{k*}(|T^2|^2 - 2\lambda|T|^2 + \lambda^2 I)T^k \geqslant 0$. Let $R_\lambda = D_\lambda^{\frac{1}{2}}$. Then for every $x \in \mathscr{H}$,

$$\begin{aligned}
\langle D_\lambda^{n+1}x, x\rangle &= \|R_\lambda^{n+1}x\|^2 = \langle D_\lambda R_\lambda^n x, R_\lambda^n x\rangle \\
&= \langle T^{k*}|T^2|^2 T^k R_\lambda^n x, R_\lambda^n x\rangle - 2\lambda\langle T^{k*}|T|^2 T^k R_\lambda^n x, R_\lambda^n x\rangle + \lambda^2\langle T^{k*}T^k R_\lambda^n x, R_\lambda^n x\rangle \\
&= \|T^2 T^k R_\lambda^n x\|^2 - 2\lambda\|T T^k R_\lambda^n x\|^2 + \lambda^2\|T^k R_\lambda^n x\|^2 \\
&\leqslant \lambda^2\|T^k R_\lambda^n x\|^2 + (1 - 2\lambda)\|T^{k+1}R_\lambda^n x\|^2,
\end{aligned}$$

when $\lambda \in (0, \frac{1}{2}]$, we have

$$\|R_\lambda^{n+1}x\|^2 \leqslant (\lambda - 1)^2\|R_\lambda^n x\|^2 \leqslant \|R_\lambda^n x\|^2;$$

when $\lambda \in (\frac{1}{2}, 1]$, we have

$$\|R_\lambda^{n+1}x\|^2 \leqslant \lambda^2\|R_\lambda^n x\|^2 \leqslant \|R_\lambda^n x\|^2.$$

Thus R_λ (and so D_λ) is a contraction and $\{D_\lambda^n\}$ is a decreasing sequence of nonnegative contractions. Hence $\{D_\lambda^n\}$ converges strongly to a projection P. Moreover when $\lambda \in (\frac{1}{2}, 1]$,

$$\begin{aligned}
\Sigma_{n=0}^m (2\lambda - 1)\|T^{k+1}R_\lambda^n x\|^2 &\leqslant \Sigma_{n=0}^m (\|R_\lambda^n x\|^2 - \|R_\lambda^{n+1}x\|^2) \\
&= \|x\|^2 - \|R_\lambda^{m+1}x\|^2 \leqslant \|x\|^2
\end{aligned}$$

for all nonnegative integers m and every $x \in \mathscr{H}$. Therefore $\|T^{k+1}R_\lambda^n x\| \to 0$ as $n \to \infty$, hence we have

$$T^{k+1}Px = T^{k+1}\lim_{n\to\infty} D_\lambda^n x = \lim_{n\to\infty} T^{k+1}R_\lambda^{2n}x = 0$$

for every $x \in \mathscr{H}$. So that $T^{k+1}P = 0$. □

REMARK. When $0 < \lambda < 1$, by the proof of Theorem 2.2, we have that D_λ is a proper contraction.

COROLLARY 2.3. *If T is a contraction of k-quasi-paranormal for a positive integer k, then the nonnegative operator $D_\lambda = T^{k*}(|T^2|^2 - 2\lambda|T|^2 + \lambda^2 I)T^k$ for $0 < \lambda \leqslant 1$ is a contraction whose power sequence $\{D_\lambda^n\}$ converges strongly to a projection P, and $T^k P = 0$ when $\lambda \in (\frac{1}{2}, 1]$.*

THEOREM 2.4. *Let T be a contraction of k-quasi-paranormal operators for a positive integer k with no nontrivial invariant subspace, then*

(a) T is a proper contraction;

(b) the nonnegative operator $D_\lambda = T^{*k}(|T^2|^2 - 2\lambda|T|^2 + \lambda^2 I)T^k$ is a strongly stable contraction for $\lambda \in (0,1]$ (so that $D_\lambda \in C_{00}$).

Proof. We may assume that $T \neq 0$. If either $\ker(T)$ or $\overline{\operatorname{ran}(T^k)}$ is non-trivial, then T has a nontrivial invariant subspace. So we have that $\ker(T) = \{0\}$ and $\overline{\operatorname{ran}(T^k)} = \mathscr{H}$. (a) Hence if T is a k-quasi-paranormal operator with no nontrivial invariant subspace, then T is injective and T^k has dense range. Consequently, T is a paranormal operator. So that T is a proper contraction by [7] Theorem 1.

(b) Let T be a contraction of k-quasi-paranormal operators. When $0 < \lambda < 1$, D_λ is a proper contraction, so a strongly stable contraction because D_λ is a a self-adjoint operator. When $\lambda = 1$, by Corollary 2.3 we have D_λ is a contraction, $\{D_\lambda^n\}$ converges strongly to a projection P, and $T^{k+1}P = 0$. Since $\ker(T) = \{0\}$, we have that $\ker(T^{k+1}) = \{0\}$. Hence $P = 0$ and so D_λ^n converges strongly to 0, that is, $D_\lambda = T^{*k}(|T^2|^2 - 2\lambda|T|^2 + \lambda^2 I)T^k$ is a strongly stable contraction. D_λ is self-adjoint, so that $D_\lambda \in C_{00}$. $\square$

Since a self-adjoint operator T is a proper contraction if and only if T is a C_{00}-contraction, we have the following corollary by Theorem 2.4.

COROLLARY 2.5. *Let T be a contraction of k-quasi-paranormal operators for a positive integer k. If T has no nontrivial invariant subspace, then both T and the nonnegative operator $D_\lambda = T^{*k}(|T^2|^2 - 2\lambda|T|^2 + \lambda^2 I)T^k$ for $\lambda \in (0,1]$ are proper contractions.*

An operator $T \in B(\mathscr{H})$ is said to be supercyclic if for some $x \in \mathscr{H}$, the homogeneous orbit $\{\lambda T^n x : \lambda \in \mathbb{C}, n \in \mathbb{N} \bigcup \{0\}\}$ is dense in $\mathscr{H}$. It is known that paranormal operators are not supercyclic. In the following, we shall prove that k-$*$-class A operators have a similar property.

THEOREM 2.6. *Let T be a k-quasi-paranormal operators for a positive integer k. Then T is not supercyclic.*

Proof. If $r(T) = 0$, then T is quasinilpotent. By [21] Lemma 5.3, we have that T is nilpotent. So that T is not supercyclic. We may assume that $r(T) > 0$ and $\|T\| = 1$. If T is supercyclic, then $\sigma(T) \subseteq \{\lambda : |\lambda| = r(T)\}$ by [18] Proposition 3.3.18. Hence $0 \notin \sigma(T)$, which implies that T is invertible. So we have that T is a paranormal operators. We have a contradiction since paranormal operators are not supercyclic by [3] Theorem 3.1. So we have that T is not supercyclic. $\square$

3. Spectral continuity for k-quasi-paranormal operators

Let $\{\tau_n\}$ be a sequence of compact subsets of $\mathscr{C}$. Then its limit inferior is defined by

$$\lim\inf\{\tau_n\}=\{\lambda\in\mathscr{C} : \text{there exists } \lambda_n\in\tau_n \text{ such that } \lambda_n\longrightarrow\lambda\}$$

and its limit superior is defined by

$$\lim\sup\{\tau_n\}=\{\lambda\in\mathscr{C} : \text{there exists } \lambda_{n_k}\in\tau_{n_k} \text{ such that } \lambda_{n_k}\longrightarrow\lambda\}.$$

If lim inf$\{\tau_n\}$=lim sup$\{\tau_n\}$, then lim $\{\tau_n\}$ is defined by this common limit. A map p, defined on $B(\mathscr{H})$, whose values are compact subsets of $\mathscr{C}$, is said to be upper (resp. lower) semi-continuous at T, if $T_n\longrightarrow T$ then lim sup$\,p(T_n)\subset p(T)$ (resp. $p(T)\subset$ lim inf$\,p(T_n)$. If p is both upper and lower semi-continuous at T, then it is said to be continuous at T and in this case lim $p(T_n)=p(T)$.

For every $T\in B(\mathscr{H})$, $\sigma(T)$ is a compact subset of $\mathscr{C}$. The function σ viewed as a function from $B(\mathscr{H})$ into the set of all compact subsets of $\mathscr{C}$, equipped with the Hausdorff metric, is well known to be upper semi-continuous, but fails to be continuous in general. J. B. Conway and B. B. Morrel [4] have carried out a detailed study of spectral continuity in $B(\mathscr{H})$. Recently, the continuity of spectrum was considered when restricted to certain subsets of the entire manifold of Toeplitz operators in [9, 16]. It has been proved that σ is continuous in the set of normal operators and hyponormal operators in [15]. And this result has been extended to quasihyponormal operators by S. V. Djordjević in [5], to p-hyponormal operators by I. S. Hwang and W. Y. Lee in [17], to (p,k)-quasihyponormal, M-hyponormal, $*$-paranormal and paranormal operators by B. P. Duggal, I. H. Jeon and I. H. Kim in [8], and to quasi-class (A,k) operators by F. Gao and X. Fang in [14]. In the following, we extend this result to k-quasi-paranormal operators.

In the following, we prove that spectrum σ is continuous on the set of all k-quasi-paranormal operators.

LEMMA 3.1. *Let T be a k-quasi-paranormal operator for a positive integer k. Then the following assertions hold:*

(1) If T is quasinilpotent, then T is nilpotent.

(2) For every non-zero $\lambda\in\sigma_p(T)$, the matrix representation of T with respect to the decomposition $\mathscr{H}=\ker(T-\lambda)\bigoplus(\ker(T-\lambda))^\perp$ is: $T=\begin{pmatrix}\lambda & A\\ 0 & B\end{pmatrix}$ for some operator B satisfying $\lambda\notin\sigma_p(B)$ and $\sigma(T)=\{\lambda\}\cup\sigma(B)$.

Proof. Suppose T is a k-quasi-paranormal operator for a positive integer k. (1) holds by [21] Lemma 5.3 and (2) holds by [21] Corollary 3.2. For the convenience of the reader, we give a proof of (2). If $\lambda\neq 0$ and $\lambda\in\sigma_p(T)$, we have that $T=\begin{pmatrix}\lambda & A\\ 0 & B\end{pmatrix}$ on $\mathscr{H}=\ker(T-\lambda)\bigoplus(\ker(T-\lambda))^\perp$ for operators A and B satisfying $A(B-\lambda)B^k=0$ and $\|B^{k+2}x\|\|B^kx\|\geqslant\|AB^kx\|^2+\|B^{k+1}x\|^2$ for any $x\in(\ker(T-\lambda))^\perp$ by [21] Theorem

3.1. Let $x \in (\ker(T-\lambda))^{\perp}$. Suppose $(B-\lambda)x = 0$, we shall prove that $x = 0$. We have that

$$\begin{aligned}\|AB^kx\|^2 &\leqslant \|B^{k+2}x\|\|B^kx\| - \|B^{k+1}x\|^2\\ &= |\lambda|^{k+2}|\lambda|^k\|x\|^2 - |\lambda|^{2(k+1)}\|x\|^2\\ &= 0.\end{aligned}$$

So we have that $|\lambda|^k\|Ax\|^2 = 0$. Since $\lambda \neq 0$, we have that $\|Ax\|^2 = 0$. hence we have that $\|(T-\lambda)x\|^2 = \|Ax\|^2 = 0$. So, $x \in \ker(T-\lambda)$ and so $x = 0$. Hence we have that (2) holds. □

The Berberian extension theorem shows that given an operator $T \in B(\mathscr{H})$, there exists a Hilbert space $\mathscr{K} \supseteq \mathscr{H}$ and an isometric $*$-isomorphism $T \to T^{\circ} \in B(\mathscr{K})$ preserving order such that $\sigma(T) = \sigma(T^{\circ})$ and $\sigma_p(T^{\circ}) = \sigma_a(T^{\circ}) = \sigma_a(T)$. See details in the following lemma:

LEMMA 3.2. (see [2]) *Let $\mathscr{H}$ be a complex Hilbert space. Then there exists a Hilbert space $\mathscr{K}$ such that $\mathscr{H} \subset \mathscr{K}$ and a map $\varphi: B(\mathscr{H}) \longrightarrow B(\mathscr{K})$ such that*

(1) φ is a faithful $*$-representation of the algebra $B(\mathscr{H})$ on $\mathscr{K}$, i.e., $\varphi(S+T) = \varphi(S)+\varphi(T)$, $\varphi(\lambda T) = \lambda\varphi(T)$, $\varphi(ST) = \varphi(S)\varphi(T)$, $\varphi(T^*) = (\varphi(T))^*$, $\varphi(I) = I$ and $\|\varphi(T)\| = \|T\|$ for any S, $T \in B(\mathscr{H})$ and $\lambda \in \mathscr{C}$.

(2) $\varphi(A) \geqslant 0$ for any $A \geqslant 0$ in $B(\mathscr{H})$.

(3) $\sigma_a(T) = \sigma_a(\varphi(T)) = \sigma_p(\varphi(T))$ for any $T \in B(\mathscr{H})$.

DEFINITION 3.3. (see [8]) The set $\mathscr{C}(i)$ consists of (all) the operators $T \in B(\mathscr{H})$ for which $\sigma(T) = \{0\}$ implies T is nilpotent (possibly, the 0 operator) and T° (the Berberian extension of T) satisfies the property:

$$T^{\circ} = \begin{pmatrix} \lambda & X \\ 0 & B \end{pmatrix} \begin{pmatrix} (T^{\circ}-\lambda)^{-1}(0) \\ ((T^{\circ}-\lambda)^{-1}(0))^{\perp} \end{pmatrix}$$

at every non-zero $\lambda \in \sigma_p(T^{\circ})$ for some operators X and B such that $\lambda \notin \sigma_p(B)$ and $\sigma(T^{\circ}) = \{\lambda\} \cup \sigma(B)$.

THEOREM 3.4. *The spectrum σ is continuous on the set of k-quasi-paranormal operators for a positive integer k.*

Proof. Suppose T is a k-quasi-paranormal operator for a positive integer k. Let $\varphi: B(\mathscr{H}) \longrightarrow B(\mathscr{K})$ be Berberian's faithful $*$-representation of Lemma 3.2. In the following, we shall show that $\varphi(T)$ is also a k-quasi-paranormal operator for a positive integer k. In fact, since T is a k-quasi-paranormal operator, we have $T^{k*}(T^{2*}T^2 - 2\lambda T^*T + \lambda^2)T^k \geqslant 0$ for all $\lambda > 0$. Hence we have

$$\begin{aligned}&(\varphi(T))^{*k}(|(\varphi(T))^2|^2 - 2\lambda|\varphi(T)|^2 + \lambda^2\varphi(I))(\varphi(T))^k\\ &= \varphi(T^{*k}(|T^2|^2 - 2\lambda|T|^2 + \lambda^2)T^k) \text{ by Lemma 3.2 (1)}\\ &\geqslant 0 \text{ by Lemma 3.2 (2)}\end{aligned}$$

for all $\lambda > 0$. So we have that its Berberian extension $T^{\circ} = \varphi(T)$ is also a k-quasi-paranormal operator for a positive integer k. By Lemma 3.1 we have that T belongs to the set $\mathscr{C}(i)$. So we have that the spectrum σ is continuous on the set of k-quasi-paranormal operators for a positive integer k by [8, Theorem 1.1]. This completes the proof. □

COROLLARY 3.5. *The Weyl spectrum σ_w is continuous if and only if the Browder spectrum σ_b is continuous on the set of k-quasi-paranormal operators for a positive integer k.*

Proof. Suppose T is a k-quasi-paranormal operator for a positive integer k. By [21] Corollary 5.5, we have that Weyl's theorem holds for T. So we have that Browder's theorem holds for T. Hence Corollary 3.4 holds by the remark of [20] or the equivalence between (ii) and (iii) of [6] Theorem 2.2. □

Acknowledgement. This research is supported by the National Natural Science Foundation of China(11271112)), the Natural Science Foundation of the Department of Education, Henan Province(2011A110009)(13B110077), the Youth Science Foundation of Henan Normal University and the New Teacher Science Foundation of Henan Normal University (No. qd12102).

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(Received June 28, 2013)

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