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ON SCHUR m–POWER CONVEXITY FOR RATIOS OF SOME MEANS

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Abstract. In the paper, the authors discuss the Schur m-power convexity on $(0,\infty)\times(0,\infty)$ for ratios of some famous means, such as the arithmetic, geometric, harmonic, root-square means, and the like, and obtain some inequalities related to ratios of means.

1. Introduction

In [7], a chain of inequalities for several means of two positive numbers is given as follows.

THEOREM 1.1. ([7]) *Let* $a,b\in\mathbb{R}_+=(0,\infty)$. *Then*

$$H(a,b)\leqslant G(a,b)\leqslant N_1(a,b)\leqslant N_3(a,b)\leqslant N_2(a,b)\leqslant A(a,b)\leqslant S(a,b), \tag{1.1}$$

where

$$A(a,b)=\frac{a+b}{2},\quad G(a,b)=\sqrt{ab},\quad H(a,b)=\frac{2ab}{a+b},$$
$$S(a,b)=\sqrt{\frac{a^2+b^2}{2}},\quad N_1(a,b)=\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2,$$
$$N_2(a,b)=\frac{\sqrt{a}+\sqrt{b}}{2}\sqrt{\frac{a+b}{2}},\quad N_3(a,b)=\frac{a+\sqrt{ab}+b}{3}.$$

The means $A(a,b)$, $G(a,b)$, $H(a,b)$, $S(a,b)$, $N_1(a,b)$, and $N_3(a,b)$ are called the arithmetic, geometric, harmonic, root-square, square-root, and Heron means respectively. The mean $N_2(a,b)$ can be found in [6].

Furthermore, the differences of means

$$M_{SA}(a,b)=S(a,b)-A(a,b),\qquad M_{SN_2}(a,b)=S(a,b)-N_2(a,b), \tag{1.2}$$
$$M_{SN_3}(a,b)=S(a,b)-N_3(a,b),\qquad M_{SN_1}(a,b)=S(a,b)-N_1(a,b), \tag{1.3}$$
$$M_{SG}(a,b)=S(a,b)-G(a,b),\qquad M_{SH}(a,b)=S(a,b)-H(a,b), \tag{1.4}$$
$$M_{AN_2}(a,b)=A(a,b)-N_2(a,b),\qquad M_{AG}(a,b)=A(a,b)-G(a,b), \tag{1.5}$$

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Paper JMI-09-14

$$M_{AH}(a,b) = A(a,b) - H(a,b), \qquad M_{N_2N_1}(a,b) = N_2(a,b) - N_1(a,b), \tag{1.6}$$
$$M_{N_2G}(a,b) = N_2(a,b) - G(a,b) \tag{1.7}$$

were considered in [7] and obtained the following theorem.

THEOREM 1.2. ([7]) *The differences of means defined by* (1.2) *to* (1.7) *are non-negative and convex in* $\mathbb{R}^2_+$.

Hereafter, the above differences were investigated once again [4].

THEOREM 1.3. ([4]) *The differences given in* (1.2) *to* (1.7) *are Schur-geometrically convex in* $\mathbb{R}^2_+$.

Recently, the differences of means

$$M_{AN_3}(a,b) = A(a,b) - N_3(a,b), \qquad M_{AN_1}(a,b) = A(a,b) - N_1(a,b), \tag{1.8}$$
$$M_{N_2N_3}(a,b) = N_2(a,b) - N_3(a,b), \qquad M_{N_2H}(a,b) = N_2(a,b) - H(a,b), \tag{1.9}$$
$$M_{N_3N_1}(a,b) = N_3(a,b) - N_1(a,b), \qquad M_{N_3G}(a,b) = N_3(a,b) - G(a,b), \tag{1.10}$$
$$M_{N_3H}(a,b) = N_3(a,b) - H(a,b), \qquad M_{N_1G}(a,b) = N_1(a,b) - G(a,b), \tag{1.11}$$
$$M_{N_1H}(a,b) = N_1(a,b) - H(a,b), \qquad M_{GH}(a,b) = G(a,b) - H(a,b), \tag{1.12}$$

and

$$D_{SH-SA}(a,b) = \frac{M_{SH}(a,b)}{3} - M_{SA}(a,b), \tag{1.13}$$
$$D_{AH-SH}(a,b) = \frac{M_{AH}(a,b)}{2} - \frac{M_{SH}(a,b)}{3}, \tag{1.14}$$
$$D_{SG-AH}(a,b) = M_{SG}(a,b) - M_{AH}(a,b), \tag{1.15}$$
$$D_{AG-SG}(a,b) = M_{AG}(a,b) - \frac{M_{SG}(a,b)}{2}, \tag{1.16}$$
$$D_{N_2N_1-AH}(a,b) = M_{N_2N_1}(a,b) - \frac{M_{AH}(a,b)}{8}, \tag{1.17}$$
$$D_{N_2G-N_2N_1}(a,b) = \frac{M_{N_2G}(a,b)}{3} - M_{N_2N_1}(a,b), \tag{1.18}$$
$$D_{AG-N_2G}(a,b) = \frac{M_{AG}(a,b)}{4} - \frac{M_{N_2G}(a,b)}{3}, \tag{1.19}$$
$$D_{AN_2-AG}(a,b) = M_{AN_2}(a,b) - \frac{M_{AG}(a,b)}{4}, \tag{1.20}$$
$$D_{SN_2-SA}(a,b) = \frac{4M_{SN_2}(a,b)}{5} - M_{SA}(a,b), \tag{1.21}$$
$$D_{AN_2-SN_2}(a,b) = 4M_{AN_2}(a,b) - \frac{4M_{SN_2}(a,b)}{5}, \tag{1.22}$$
$$D_{SN_1-SH}(a,b) = 2M_{SN_1}(a,b) - M_{SH}(a,b), \tag{1.23}$$
$$D_{SG-SN_1}(a,b) = \frac{3M_{SG}(a,b)}{2} - 2M_{SN_1}(a,b), \tag{1.24}$$

$$D_{SN_3-SA}(a,b)=\frac{3M_{SN_3}(a,b)}{4}-M_{SA}(a,b), \tag{1.25}$$

$$D_{SN_1-SN_3}(a,b)=\frac{2M_{SN_1}(a,b)}{3}-\frac{3M_{SN_3}(a,b)}{4}. \tag{1.26}$$

were introduced in [11, 12] and obtained the following results.

THEOREM 1.4. ([11, Theorem 3.1]) *The differences of means listed in equations* (1.2) *to* (1.12) *are Schur-harmonically convex on* $\mathbb{R}_+^2$.

THEOREM 1.5. ([12, Theorem 3.1]) *The differences given by* (1.13) *to* (1.26) *are Schur-harmonically convex functions in* $\mathbb{R}_+^2$.

Motivated by the above differences of means, we now introduce the following twenty one ratios of means:

$$Q_{SA}(a,b)=\frac{S(a,b)}{A(a,b)},\quad Q_{SN_2}(a,b)=\frac{S(a,b)}{N_2(a,b)},\quad Q_{SN_3}(a,b)=\frac{S(a,b)}{N_3(a,b)}, \tag{1.27}$$

$$Q_{SN_1}(a,b)=\frac{S(a,b)}{N_1(a,b)},\quad Q_{SG}(a,b)=\frac{S(a,b)}{G(a,b)},\quad Q_{SH}(a,b)=\frac{S(a,b)}{H(a,b)}, \tag{1.28}$$

$$Q_{AN_2}(a,b)=\frac{A(a,b)}{N_2(a,b)},\quad Q_{AN_3}(a,b)=\frac{A(a,b)}{N_3(a,b)},\quad Q_{AN_1}(a,b)=\frac{A(a,b)}{N_1(a,b)}, \tag{1.29}$$

$$Q_{AG}(a,b)=\frac{A(a,b)}{G(a,b)},\quad Q_{AH}(a,b)=\frac{A(a,b)}{H(a,b)},\quad Q_{N_2N_3}(a,b)=\frac{N_2(a,b)}{N_3(a,b)}, \tag{1.30}$$

$$Q_{N_2N_1}(a,b)=\frac{N_2(a,b)}{N_1(a,b)},\quad Q_{N_2G}(a,b)=\frac{N_2(a,b)}{G(a,b)},\quad Q_{N_2H}(a,b)=\frac{N_2(a,b)}{H(a,b)}, \tag{1.31}$$

$$Q_{N_3N_1}(a,b)=\frac{N_3(a,b)}{N_1(a,b)},\quad Q_{N_3G}(a,b)=\frac{N_3(a,b)}{G(a,b)},\quad Q_{N_3H}(a,b)=\frac{N_3(a,b)}{H(a,b)}, \tag{1.32}$$

$$Q_{N_1G}(a,b)=\frac{N_1(a,b)}{G(a,b)},\quad Q_{N_1H}(a,b)=\frac{N_1(a,b)}{H(a,b)},\quad Q_{GH}(a,b)=\frac{G(a,b)}{H(a,b)}. \tag{1.33}$$

In this paper, we will prove that the ratios (1.27) to (1.33) are Schur m-power convex and Schur-geometrically convex in $\mathbb{R}_+^2$ and establish some inequalities of the ratios of means.

2. Definitions and lemmas

In order to verify our main results, the following definitions and lemmas are necessary.

It is general knowledge that a set $\Omega\subseteq\mathbb{R}^n$ is said to be convex if

$$\lambda\boldsymbol{x}+(1-\lambda)\boldsymbol{y}=(\lambda x_1+(1-\lambda)y_1,\ldots,\lambda x_n+(1-\lambda)y_n)\in\Omega$$

for every $\boldsymbol{x},\boldsymbol{y}\in\Omega$ and $\lambda\in[0,1]$.

DEFINITION 2.1. ([2, pp. 8 and 80], [3, pp. 2313–2314], and [10]) Let

$$\boldsymbol{x}=(x_1,\ldots,x_n) \quad \text{and} \quad \boldsymbol{y}=(y_1,\ldots,y_n)\in\mathbb{R}^n.$$

1. The tuple $\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$, in symbols $\boldsymbol{x}\prec\boldsymbol{y}$, if $\sum_{i=1}^k x_{[i]}\leqslant\sum_{i=1}^k y_{[i]}$ for $k=1,2,\ldots,n-1$ and $\sum_{i=1}^n x_i=\sum_{i=1}^n y_i$, where $x_{[1]}\geqslant\cdots\geqslant x_{[n]}$ and $y_{[1]}\geqslant\cdots\geqslant y_{[n]}$ are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in a descending order.

2. A function $\varphi:\Omega\subseteq\mathbb{R}^n\to\mathbb{R}$ is said to be Schur-convex on Ω if $\boldsymbol{x}\prec\boldsymbol{y}$ on Ω implies $\varphi(\boldsymbol{x})\leqslant\varphi(\boldsymbol{y})$. A function φ is said to be Schur-concave on Ω if and only if $-\varphi$ is Schur-convex.

DEFINITION 2.2. ([1]) Let $\Omega\subset\mathbb{R}_+^n$.

1. The set Ω is said to be geometrically convex if $\left(x_1^{\lambda}y_1^{1-\lambda},\ldots,x_n^{\lambda}y_n^{1-\lambda}\right)\in\Omega$ for every $\boldsymbol{x},\boldsymbol{y}\in\Omega$ and $\lambda\in[0,1]$.

2. A function $\varphi:\Omega\to\mathbb{R}_+$ is said to be Schur-geometrically convex on Ω if $\ln\boldsymbol{x}=(\ln x_1,\cdots,\ln x_n)\prec\ln\boldsymbol{y}=(\ln y_1,\cdots,\ln y_n)$ implies $\varphi(\boldsymbol{x})\leqslant\varphi(\boldsymbol{y})$ for every $\boldsymbol{x},\boldsymbol{y}\in\Omega$.

DEFINITION 2.3. ([13, Definition 1.3]) Let $\Omega\subset\mathbb{R}_+^n$.

1. A set Ω is said to be harmonically convex if $\frac{\boldsymbol{xy}}{\lambda\boldsymbol{x}+(1-\lambda)\boldsymbol{y}}\in\Omega$ for every $\boldsymbol{x},\boldsymbol{y}\in\Omega$ and $\lambda\in[0,1]$, where $\boldsymbol{xy}=\sum_{i=1}^n x_iy_i$ and $\frac{1}{\boldsymbol{x}}=\left(\frac{1}{x_1},\cdots,\frac{1}{x_n}\right)$.

2. A function $\varphi:\Omega\to\mathbb{R}_+$ is said to be Schur-harmonically convex on Ω if $\frac{1}{\boldsymbol{x}}\prec\frac{1}{\boldsymbol{y}}$ implies $\varphi(\boldsymbol{x})\leqslant\varphi(\boldsymbol{y})$.

DEFINITION 2.4. ([14]) Let $f:\mathbb{R}_+\to\mathbb{R}$ be defined by

$$f(x)=\begin{cases}\dfrac{x^m-1}{m}, & m\neq 0;\\ \ln x, & m=0.\end{cases} \tag{2.1}$$

Then a function $\phi:\Omega\subset\mathbb{R}_+^n\to\mathbb{R}$ is said to be Schur m-power convex on Ω if

$$(f(x_1),f(x_2),\ldots,f(x_n))\prec(f(y_1),f(y_2),\ldots,f(y_n))$$

for all $(x_1,x_2,\ldots,x_n)\in\Omega$ and $(y_1,y_2,\ldots,y_n)\in\Omega$ implies $\phi(x)\leqslant\phi(y)$.

If $-\phi$ is Schur m-power convex, then we say that ϕ is Schur m-power concave.

If putting $f(x)=x,\ln x,\frac{1}{x}$ in Definition 2.4, then definitions of the Schur-convex, Schur-geometrically convex, and Schur-harmonically convex functions can be deduced respectively.

LEMMA 2.1. ([2, 10]) *Let $\Omega\subset\mathbb{R}_+^n$ be a symmetric and geometrically convex set with inner point and $\varphi:\Omega\to\mathbb{R}_+$ be a symmetric and differentiable function in Ω°. Then φ is a Schur-convex function on Ω if and only if*

$$(x_1-x_2)\left[\frac{\partial\varphi(\boldsymbol{x})}{\partial x_1}-\frac{\partial\varphi(\boldsymbol{x})}{\partial x_2}\right]\geqslant 0,\quad \boldsymbol{x}\in\Omega^\circ. \tag{2.2}$$

LEMMA 2.2. ([1]) *Let $\Omega \subset \mathbb{R}^n_+$ be a symmetric and geometrically convex set with inner point and $\varphi : \Omega \to \mathbb{R}_+$ be a symmetric and differentiable function in Ω°. Then φ is a Schur-geometrically convex function on Ω if and only if*

$$(\ln x_1 - \ln x_2)\left[x_1 \frac{\partial \varphi(\boldsymbol{x})}{\partial x_1} - x_2 \frac{\partial \varphi(\boldsymbol{x})}{\partial x_2}\right] \geqslant 0, \quad \boldsymbol{x} \in \Omega^\circ. \tag{2.3}$$

LEMMA 2.3. ([13, Lemma 2.4]) *Let $\Omega \subset \mathbb{R}^n_+$ be a symmetric and harmonically convex set with inner points and let $\varphi : \Omega \to \mathbb{R}_+$ be a continuously symmetric function which is differentiable on Ω°. Then φ is Schur-harmonically convex on Ω if and only if*

$$(x_1 - x_2)\left[x_1^2 \frac{\partial \varphi(\boldsymbol{x})}{\partial x_1} - x_2^2 \frac{\partial \varphi(\boldsymbol{x})}{\partial x_2}\right] \geqslant 0, \quad \boldsymbol{x} \in \Omega^\circ. \tag{2.4}$$

LEMMA 2.4. ([14]) *Let $\Omega \subset \mathbb{R}^n_+$ be a symmetric set with nonempty interior Ω° and $\varphi : \Omega \to \mathbb{R}_+$ be continuous on Ω and differentiable in Ω°. Then φ is Schur m-power convex on Ω if and only if φ is symmetric on Ω and*

$$\frac{x_1^m - x_2^m}{m}\left[x_1^{1-m} \frac{\partial \varphi(\boldsymbol{x})}{\partial x_1} - x_2^{1-m} \frac{\partial \varphi(\boldsymbol{x})}{\partial x_2}\right] \geqslant 0, \quad \textit{if } m \neq 0 \tag{2.5}$$

and

$$(\ln x_1 - \ln x_2)\left[x_1 \frac{\partial \varphi(\boldsymbol{x})}{\partial x_1} - x_2 \frac{\partial \varphi(\boldsymbol{x})}{\partial x_2}\right] \geqslant 0, \quad \textit{if } m = 0 \tag{2.6}$$

for all $\boldsymbol{x} \in \Omega^\circ$.

3. Main results

Now we set off to prove our main results.

THEOREM 3.1. *For $m \neq 0$, the ratios of means given in (1.27) to (1.33) are Schur m-power convex functions in $\mathbb{R}^2_+$.*

Proof. It is easy to show that

$$\begin{aligned}
&\frac{a^m - b^m}{m}\left[a^{1-m} \frac{\partial Q_{GH}(a,b)}{\partial a} - b^{1-m} \frac{\partial Q_{GH}(a,b)}{\partial b}\right] \\
&= \frac{(a-b)(a^m - b^m)}{4m\sqrt{ab}}\left(\frac{1}{a^m} + \frac{1}{b^m}\right) \geqslant 0, \\
&\frac{a^m - b^m}{m}\left[a^{1-m} \frac{\partial Q_{N_1G}(a,b)}{\partial a} - b^{1-m} \frac{\partial Q_{N_1G}(a,b)}{\partial b}\right] \\
&= \frac{(a-b)(a^m - b^m)}{4m\sqrt{ab}}\left(\frac{1}{a^m} + \frac{1}{b^m}\right) \geqslant 0,
\end{aligned}$$

$$\frac{a^m-b^m}{m}\left[a^{1-m}\frac{\partial Q_{N_3N_1}(a,b)}{\partial a}-b^{1-m}\frac{\partial Q_{N_3N_1}(a,b)}{\partial b}\right]$$
$$=\frac{2\sqrt{ab}(a-b)(a^m-b^m)}{3m\left(\sqrt{a}+\sqrt{b}\right)^4}\left(\frac{1}{a^m}+\frac{1}{b^m}\right)\geqslant 0.$$

Therefore, by Lemma 2.4, it follows that $Q_{GH}(a,b)$, $Q_{N_1G}(a,b)$, and $Q_{N_3N_1}(a,b)$ are Schur m-power convex functions on $\mathbb{R}_+^2$.

A direct calculation yields

$$\frac{a^m-b^m}{m}\left[a^{1-m}\frac{\partial Q_{N_2N_3}(a,b)}{\partial a}-b^{1-m}\frac{\partial Q_{N_2N_3}(a,b)}{\partial b}\right]$$
$$=\frac{a^m-b^m}{m}\left[\frac{3\sqrt{2}ab\left(\sqrt{a}-\sqrt{b}\right)a^{1-m}}{8\left(a+b+\sqrt{ab}\right)^2\sqrt{ab(a+b)}}-\frac{3\sqrt{2b}a\left(\sqrt{b}-\sqrt{a}\right)b^{1-m}}{8\left(a+b+\sqrt{ab}\right)^2\sqrt{ab(a+b)}}\right]$$
$$=\frac{3\sqrt{2ab}\left(\sqrt{a}-\sqrt{b}\right)(a^m-b^m)}{8m\left(a+b+\sqrt{ab}\right)^2\sqrt{a+b}}\left(\frac{1}{a^m}+\frac{1}{b^m}\right)\geqslant 0,$$
$$\frac{a^m-b^m}{m}\left[a^{1-m}\frac{\partial Q_{AN_2}(a,b)}{\partial a}-b^{1-m}\frac{\partial Q_{AN_2}(a,b)}{\partial b}\right]$$
$$=\frac{a^m-b^m}{m}\left[\frac{\sqrt{2}b\left(\sqrt{a}-\sqrt{b}\right)a^{1-m}}{2\left(\sqrt{a}+\sqrt{b}\right)^2\sqrt{ab(a+b)}}-\frac{\sqrt{2}a\left(\sqrt{b}-\sqrt{a}\right)b^{1-m}}{2\left(\sqrt{a}+\sqrt{b}\right)^2\sqrt{ab(a+b)}}\right]$$
$$=\frac{\sqrt{2ab}(a-b)(a^m-b^m)}{2m\left(\sqrt{a}+\sqrt{b}\right)^3\sqrt{a+b}}\left(\frac{1}{a^m}+\frac{1}{b^m}\right)\geqslant 0,$$

and

$$\frac{a^m-b^m}{m}\left[a^{1-m}\frac{\partial Q_{SA}(a,b)}{\partial a}-b^{1-m}\frac{\partial Q_{SA}(a,b)}{\partial b}\right]$$
$$=\frac{a^m-b^m}{m}\left[a^{1-m}\frac{\sqrt{2}b(a-b)}{(a+b)^2\sqrt{a^2+b^2}}-b^{1-m}\frac{\sqrt{2}a(b-a)}{(a+b)^2\sqrt{a^2+b^2}}\right]$$
$$=\frac{\sqrt{2}ab(a-b)(a^m-b^m)}{m(a+b)^2\sqrt{a^2+b^2}}\left(\frac{1}{a^m}+\frac{1}{b^m}\right)\geqslant 0.$$

From Lemma 2.4, it follows that the ratios $Q_{N_2N_3}(a,b)$, $Q_{AN_2}(a,b)$, and $Q_{SA}(a,b)$ are Schur m-power convex functions in $\mathbb{R}_+^2$.

Notice that

$$Q_{N_1H}(a,b)=Q_{N_1G}(a,b)Q_{GH}(a,b),\qquad Q_{N_3G}(a,b)=Q_{N_3N_1}(a,b)Q_{N_1G}(a,b), \tag{3.1}$$
$$Q_{N_3H}(a,b)=Q_{N_3N_1}(a,b)Q_{N_1H}(a,b),\qquad Q_{N_2N_1}(a,b)=Q_{N_2N_3}(a,b)Q_{N_3N_1}(a,b), \tag{3.2}$$
$$Q_{N_2G}(a,b)=Q_{N_2N_3}(a,b)Q_{N_3G}(a,b),\qquad Q_{N_2H}(a,b)=Q_{N_2N_3}(a,b)Q_{N_3H}(a,b), \tag{3.3}$$
$$Q_{AN_3}(a,b)=Q_{AN_2}(a,b)Q_{N_2N_3}(a,b),\qquad Q_{AN_1}(a,b)=Q_{AN_2}(a,b)Q_{N_2N_1}(a,b), \tag{3.4}$$
$$Q_{AG}(a,b)=Q_{AN_2}(a,b)Q_{N_2G}(a,b),\qquad Q_{AH}(a,b)=Q_{AN_2}(a,b)Q_{N_2H}(a,b), \tag{3.5}$$

$$Q_{SN_2}(a,b)=Q_{SA}(a,b)Q_{AN_2}(a,b), \qquad Q_{SN_3}(a,b)=Q_{SA}(a,b)Q_{AN_3}(a,b), \tag{3.6}$$
$$Q_{SN_1}(a,b)=Q_{SA}(a,b)Q_{AN_1}(a,b), \qquad Q_{SG}(a,b)=Q_{SA}(a,b)Q_{AG}(a,b), \tag{3.7}$$
$$Q_{SH}(a,b)=Q_{SA}(a,b)Q_{AH}(a,b) \tag{3.8}$$

and that a quotient of finitely many Schur m-power convex functions is also Schur m-power convex. Conclusively, the rest ratios listed in (1.27) to (1.33) are Schur m-power convex in $\mathbb{R}^2_+$. The proof of Theorem 3.1 is complete. □

COROLLARY 3.1. *Under the conditions of Theorem 3.1, if $m=\pm 1$, then the ratios of means given in (1.27) to (1.33) are Schur-convex and Schur-harmonically convex in $\mathbb{R}^2_+$.*

COROLLARY 3.2. *For $a,b\in\mathbb{R}_+$ and $m\in\mathbb{R}$ with $m\neq 0$, let $p_{a,b,m}(t)=(1-t)a^m+tb^m$ for $0\leqslant t\leqslant 1$. Then*

$$\frac{S\big(a^m,b^m\big)}{A\big(a^m,b^m\big)}\geqslant\frac{S(p_{a,b;m}(t),p_{a,b;m}(1-t))}{A(p_{a,b;m}(t),p_{a,b;m}(1-t))}\geqslant 0,$$
$$\frac{A\big(a^m,b^m\big)}{N_2\big(a^m,b^m\big)}\geqslant\frac{A(p_{a,b;m}(t),p_{a,b;m}(1-t))}{N_2(p_{a,b;m}(t),p_{a,b;m}(1-t))}\geqslant 0,$$
$$\frac{N_2\big(a^m,b^m\big)}{N_3\big(a^m,b^m\big)}\geqslant\frac{N_2(p_{a,b;m}(t),p_{a,b;m}(1-t))}{N_3(p_{a,b;m}(t),p_{a,b;m}(1-t))}\geqslant 0,$$
$$\frac{N_3\big(a^m,b^m\big)}{N_1\big(a^m,b^m\big)}\geqslant\frac{N_3(p_{a,b;m}(t),p_{a,b;m}(1-t))}{N_1(p_{a,b;m}(t),p_{a,b;m}(1-t))}\geqslant 0,$$
$$\frac{N_1\big(a^m,b^m\big)}{G\big(a^m,b^m\big)}\geqslant\frac{N_1(p_{a,b;m}(t),p_{a,b;m}(1-t))}{G(p_{a,b;m}(t),p_{a,b;m}(1-t))}\geqslant 0,$$
$$\frac{G\big(a^m,b^m\big)}{H\big(a^m,b^m\big)}\geqslant\frac{G(p_{a,b;m}(t),p_{a,b;m}(1-t))}{H(p_{a,b;m}(t),p_{a,b;m}(1-t))}\geqslant 0.$$

Proof. It is easy to see that

$$\left(\frac{a^m+b^m}{2},\frac{a^m+b^m}{2}\right)\prec\big((1-t)a^m+tb^m,ta^m+(1-t)b^m\big)\prec\big(a^m,b^m\big),\quad 0\leqslant t\leqslant 1.$$

Further by Theorem 3.1 and Definition 2.4, the proof of Corollary 3.2 is complete. □

THEOREM 3.2. *The ratios of means given in (1.27) to (1.33) are Schur-geometrically convex functions in $\mathbb{R}^2_+$.*

Proof. It is easy to see that

$$(\ln a-\ln b)\left[a\frac{\partial Q_{GH}(a,b)}{\partial a}-b\frac{\partial Q_{GH}(a,b)}{\partial b}\right]=\frac{(\ln a-\ln b)(a-b)}{2\sqrt{ab}}\geqslant 0,$$
$$(\ln a-\ln b)\left[a\frac{\partial Q_{N_1G}(a,b)}{\partial a}-b\frac{\partial Q_{N_1G}(a,b)}{\partial b}\right]=\frac{(\ln a-\ln b)(a-b)}{2\sqrt{ab}}\geqslant 0,$$

$$(\ln a-\ln b)\left[a\frac{\partial Q_{N_3N_1}(a,b)}{\partial a}-b\frac{\partial Q_{N_3N_1}(a,b)}{\partial b}\right]=\frac{4\sqrt{ab}(\ln a-\ln b)(a-b)}{3\left(\sqrt{a}+\sqrt{b}\right)^4}\geqslant 0,$$

$$(\ln a-\ln b)\left[a\frac{\partial Q_{AN_2}(a,b)}{\partial a}-b\frac{\partial Q_{AN_2}(a,b)}{\partial b}\right]=\frac{\sqrt{2ab}(\ln a-\ln b)(a-b)}{\left(\sqrt{a}+\sqrt{b}\right)^3\sqrt{a+b}}\geqslant 0,$$

$$(\ln a-\ln b)\left[a\frac{\partial Q_{SA}(a,b)}{\partial a}-b\frac{\partial Q_{SA}(a,b)}{\partial b}\right]=\frac{2\sqrt{2}ab(\ln a-\ln b)(a-b)}{(a+b)^2\sqrt{a^2+b^2}}\geqslant 0,$$

and

$$(\ln a-\ln b)\left[a\frac{\partial Q_{N_2N_3}(a,b)}{\partial a}-b\frac{\partial Q_{N_2N_3}(a,b)}{\partial b}\right]$$
$$=\frac{3\sqrt{2}ab(\ln a-\ln b)(a-b)}{4\left(\sqrt{a}+\sqrt{b}\right)\left(a+b+\sqrt{ab}\right)^2\sqrt{a+b}}\geqslant 0.$$

By the equalities (3.1) and (3.8), it follows that the ratios (1.27) to (1.33) are Schur-geometrically convex in $\mathbb{R}_+^2$. The proof of Theorem 3.2 is complete. □

COROLLARY 3.3. *For $a,b\in\mathbb{R}_+$ and $0\leqslant t\leqslant 1$, let $g_{a,b}(t)=a^{1-t}b^t$. Then*

$$\frac{S(a,b)}{A(a,b)}\geqslant\frac{S(g_{a,b}(t),g_{a,b}(1-t))}{A(g_{a,b}(t),g_{a,b}(1-t))}\geqslant 0,\quad \frac{A(a,b)}{N_2(a,b)}\geqslant\frac{A(g_{a,b}(t),g_{a,b}(1-t))}{N_2(g_{a,b}(t),g_{a,b}(1-t))}\geqslant 0,$$
$$\frac{N_2(a,b)}{N_3(a,b)}\geqslant\frac{N_2(g_{a,b}(t),g_{a,b}(1-t))}{N_3(g_{a,b}(t),g_{a,b}(1-t))}\geqslant 0,\quad \frac{N_3(a,b)}{N_1(a,b)}\geqslant\frac{N_3(g_{a,b}(t),g_{a,b}(1-t))}{N_1(g_{a,b}(t),g_{a,b}(1-t))}\geqslant 0,$$
$$\frac{N_1(a,b)}{G(a,b)}\geqslant\frac{N_1(g_{a,b}(t),g_{a,b}(1-t))}{G(g_{a,b}(t),g_{a,b}(1-t))}\geqslant 0,\quad \frac{G(a,b)}{H(a,b)}\geqslant\frac{G(g_{a,b}(t),g_{a,b}(1-t))}{H(g_{a,b}(t),g_{a,b}(1-t))}\geqslant 0.$$

Proof. This follows from

$$(\ln\sqrt{ab},\ln\sqrt{ab})\prec(\ln(a^{1-t}b^t),\ln(a^tb^{1-t}))\prec(\ln a,\ln b),\quad 0\leqslant t\leqslant 1$$

and making use of Theorem 3.2 and Definition 2.4. The proof of Corollary 3.3 is complete. □

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