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STEFFENSEN'S GENERALIZATION OF ČEBYŠEV INEQUALITY

K. M. AWAN, J. PEČARIĆ AND ATIQ UR REHMAN

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Abstract. In this paper, we obtain Ostrowski-type bounds for the weighted Čebyšev functional. Also we give bounds of weighted Čebyšev functional in the case of Steffesen's generalization of Čebyšev inequality.

1. Introduction and preliminaries

Let $f,g:[a,b]\to\mathbb{R}$ and $p:[a,b]\to\mathbb{R}^+$ be Lebesgue integrable functions. Then we consider the weighted Čebyšev functional:

$$T(f,g;p):=\frac{1}{P(b)}\int_a^b p(t)f(t)g(t)dt-\frac{1}{P(b)}\int_a^b p(t)f(t)dt\cdot\frac{1}{P(b)}\int_a^b p(t)g(t)dt, \tag{1}$$

where $P(x)=\int_a^x p(t)dt$.

If $p(t)=1$ for all $t\in[a,b]$, we define Čebyšev functional $T(f,g)=T(f,g;1)$.

It is known that if f and g are monotonic in the same direction on interval $[a,b]$, then

$$T(f,g;p)\geqslant 0 \tag{2}$$

If f and g are monotonic in opposite directions on interval $[a,b]$, then the reverse of the inequality in (2) is valid. In both cases, equality in (2) holds if and only if either f or g is constant almost everywhere.

Steffensen [6] (see also [5, page 199]) noted that inequality (2) is also valid when f is an increasing function on $[a,b]$ and g satisfies the condition

$$\frac{1}{P(x)}\int_a^x p(t)g(t)dt\leqslant\frac{1}{P(b)}\int_a^b p(t)g(t)dt,\quad\text{where}\quad P(x)=\int_a^x p(t)dt, \tag{3}$$

for $x\in(a,b)$.

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The condition $p(t) > 0$ for $t \in [a,b]$ for the inequality (2) can be replaced by

$$0 \leqslant P(x) \leqslant P(b) \text{ for } a \leqslant x \leqslant b. \tag{4}$$

In 1970, A. M. Ostrowski [3] proved that if g is absolutely continuous on $[a,b]$ and $g' \in L_\infty[a,b]$, then

$$|T(f,g)| \leqslant \frac{1}{8}(b-a)(M-m)\|g'\|_\infty, \tag{5}$$

provided m and M are real numbers with property

$$-\infty < m \leqslant f \leqslant M < \infty \text{ and } \|g'\|_\infty = \sup_{t\in[a,b]} |g'(t)|.$$

The constant $\frac{1}{8}$ in (5) cannot be improved in the general case.

In [1], P. Cerone and S. Dragomir gave the bounds of the Čebyšev functional $T(f,g)$. Namely, they proved that if g is non-decreasing on $[a,b]$ and f is absolutely continuous on $[a,b]$ with $f' \in L_\infty[a,b]$, then

$$|T(f,g)| \leqslant \frac{1}{2(b-a)}\|f'\|_\infty \int_a^b (x-a)(b-x)dg(x) \tag{6}$$

holds and the constant $\frac{1}{2}$ is best possible. They deduce bound of $T(f,g)$ given by Čebyšev in [2] i.e. if g is absolutely continuous on $[a,b]$ and $g' \in L_\infty[a,b]$, then

$$|T(f,g)| \leqslant \frac{1}{12}\|f'\|_\infty\|g'\|_\infty(b-a)^2, \tag{7}$$

holds and the constant $\frac{1}{12}$ is sharp.

In this paper, we find Ostrowski-type bounds for weighted Čebyšev functional and deduce the results of [1] in the case of non-weighted Čebyšev functional. Also we give some bounds in the case of Steffesen generalization of weighted Čebyšev inequality.

2. Main results

Let f,g be integrable functions on $[a,b]$ and p be positive integrable function on $[a,b]$ with $P(b) := \int_a^b p(t)dt$. Then the weighted version of Korkin's identity is represented by

$$T(f,g;p) = \frac{1}{2P^2(b)} \int_a^b \int_a^b p(x)p(y)\left(f(x)-f(y)\right)\left(g(x)-g(y)\right)dxdy. \tag{8}$$

We use identity (8) to prove the following lemma, which later leads to the bounds of Čebyšev functional.

LEMMA 2.1. *Let $\phi : [a,b] \to \mathbb{R}$ be an absolutely continuous function, and $p : [a,b] \to \mathbb{R}^+$ an integrable function and $(\varphi')^2 \in L[a,b]$*

$$P(x) = \int_a^x p(t)dt \;\; and \;\; \tilde{P}(x) = P(x)\int_a^b tp(t)dt - P(b)\int_a^x tp(t)dt. \tag{9}$$

Then we have the inequality

$$T(\phi,\phi;p) \leqslant \frac{1}{P^2(b)}\int_a^b \tilde{P}(x)\left[\phi'(x)\right]^2 dx, \tag{10}$$

provided that the integral on right hand side of above inequality exists. Also the inequality in (10) is sharp.

Proof. We have (see [4])

$$T(f,g;p) = \frac{1}{P^2(b)}\int_a^b \left\{\int_a^x p(t)h(t)dt\right\} g'(x)dx,$$

where

$$h(t) = \int_a^b p(s)\left(f(s)-f(t)\right)ds.$$

If we take $l(x) = x$, then

$$T(l,g;p) = \frac{1}{P^2(b)}\int_a^b \int_a^x p(t)\int_a^b p(s)(s-t)dsdtg'(x)dx.$$

A simple computation yields that

$$T(l,g;p) = \frac{1}{P^2(b)}\int_a^b \tilde{P}(x)g'(x)dx. \tag{11}$$

Korkin's identity (8) gives us

$$\begin{aligned} T(\phi,\phi;p) &= \frac{1}{2P^2(b)}\int_a^b\int_a^b p(x)p(y)\left(\phi(x)-\phi(y)\right)^2 dxdy,\\ &= \frac{1}{2P^2(b)}\int_a^b\int_a^b p(x)p(y)(x-y)^2\left(\frac{\phi(x)-\phi(y)}{x-y}\right)^2 dxdy. \end{aligned}$$

Since ϕ is absolutely continuous, $\phi(t)-\phi(s) = \int_s^t \phi'(u)du$, and by using Cauchy-Schwarz inequality, we have

$$\begin{aligned} T(\phi,\phi;p) &= \frac{1}{2P^2(b)}\int_a^b\int_a^b p(x)p(y)(x-y)^2\left(\frac{\int_x^y \phi'(s)ds}{x-y}\right)^2 dxdy,\\ &\leqslant \frac{1}{2P^2(b)}\int_a^b\int_a^b p(x)p(y)(x-y)^2\left(\frac{1}{x-y}\int_x^y [\phi'(s)]^2 ds\right)dxdy,\\ &= \frac{1}{2P^2(b)}\int_a^b\int_a^b p(x)p(y)(x-y)\left(\int_x^y [\phi'(s)]^2 ds\right)dxdy.\\ &= T(l,\psi;p), \text{ by applying Korkin's identity with } \psi(x) = \int_a^x \left[\phi'(s)\right]^2 ds. \end{aligned}$$

Now using (11), we get inequality (10).

To prove the sharpness of (10), we assume this inequality is valid with a constant $C > 0$, that is,

$$T(\phi,\phi;p) \leqslant \frac{C}{P^2(b)} \int_a^b \tilde{P}(x) \left[\phi'(x)\right]^2 dx, \tag{12}$$

If we consider $\phi(x) = x$, then we observe that the left hand side of (12) is equal to $\frac{1}{P^2(b)} \int_a^b \tilde{P}(x)dx$ and the right hand side of (12) is equal to $\frac{C}{P^2(b)} \int_a^b \tilde{P}(x)dx$. Thus we deduce that $C \geqslant 1$. □

A non-weighted case of the above theorem is given in the following corollary. It is also proved in [1].

COROLLARY 2.2. *If $\phi : [a,b] \to \mathbb{R}$ is an absolutely continuous function with $(\varphi')^2 \in L[a,b]$, then we have the inequality*

$$T(\phi,\phi) \leqslant \frac{1}{2(b-a)} \int_a^b (x-a)(b-x)[\phi'(x)]^2 dx. \tag{13}$$

The constant $\frac{1}{2}$ is the best possible.

Proof. Using $p(t) = 1$ in (9), we get

$$\begin{aligned}
\tilde{P}(x) &= (x-a)\int_a^b t\,dt - (b-a)\int_a^x t\,dt, \\
&= \frac{1}{2}(x-a)(b^2-a^2) - \frac{1}{2}(b-a)(x^2-a^2), \\
&= \frac{1}{2}(x-a)(b-a)(b-x).
\end{aligned}$$

Now using this result in (10), along with the fact that $T(\phi,\phi,1) = T(\phi,\phi)$, we get the result.

The inequality in (10) is sharp and we get constant $\frac{1}{2}$ in the simplification of (9), hence it is best possible. □

Throughout the paper we keep the notations $P(x)$ and $\tilde{P}(x)$ used in Lemma 2.1.

THEOREM 2.3. *Let $f,g : [a,b] \to \mathbb{R}$ be two absolutely continuous functions with $(f')^2$, $(g')^2 \in L[a,b]$ and $p : [a,b] \to \mathbb{R}^+$ be an integrable function, then we have the inequalities*

$$\begin{aligned}
|T(f,g;p)| &\leqslant \frac{1}{P(b)} T^{\frac{1}{2}}(f,f;p) \left(\int_a^b \tilde{P}(x) \left[g'(x)\right]^2 dx \right)^{\frac{1}{2}}, \\
&\leqslant \frac{1}{P^2(b)} \left(\int_a^b \tilde{P}(x) \left[f'(x)\right]^2 dx \right)^{\frac{1}{2}} \left(\int_a^b \tilde{P}(x) \left[g'(x)]\right]^2 dx \right)^{\frac{1}{2}}.
\end{aligned}$$

The above inequalities are sharp.

Proof. By Cauchy-Schwartz inequality for double integrals, we have

$$|T(f,g;p)| \leqslant T^{\frac{1}{2}}(f,f;p)T^{\frac{1}{2}}(g,g;p). \tag{14}$$

Now using (8) and Lemma 2.1 in above inequality, we get the required result. □

If we consider $p(t)=1$, then we get Theorem 1 of [1], which is stated in the following corollary.

COROLLARY 2.4. *Let $f,g:[a,b]\to\mathbb{R}$ be two absolutely continuous functions on $[a,b]$ with $(f')^2$, $(g')^2\in L[a,b]$. Then we have the inequality*

$$\begin{aligned} T(f,g) &\leqslant \frac{1}{\sqrt{2}}\left[T(f,f)\right]^{\frac{1}{2}}\frac{1}{\sqrt{b-a}}\left(\int_a^b (x-a)(b-x)\left[g'(x)\right]^2dx\right)^{\frac{1}{2}}, \\ &\leqslant \frac{1}{2(b-a)}\left(\int_a^b (x-a)(b-x)\left[f'(x)\right]^2dx\right)^{\frac{1}{2}} \\ &\quad\times\left(\int_a^b (x-a)(b-x)\left[g'(x)\right]^2dx\right)^{\frac{1}{2}}. \end{aligned} \tag{15}$$

The constants $\frac{1}{\sqrt{2}}$ and $\frac{1}{2}$ in (15) are best possibles.

THEOREM 2.5. *Assume that $g:[a,b]\to\mathbb{R}$ is monotonic nondecreasing, $p:[a,b]\to\mathbb{R}^+$ be integrable function and $f:[a,b]\to\mathbb{R}$ is absolutely continuous with $f'\in L_\infty[a,b]$. Then we have the inequality*

$$|T(f,g;p)| \leqslant \frac{\|f'\|_\infty}{P^2(b)}\int_a^b \tilde{P}(x)dg(x), \tag{16}$$

The inequality (16) is sharp.

Proof. We have, by Korkin's identity, that

$$\begin{aligned} |T(f,g;p)| &= \frac{1}{2P^2(b)}\left|\int_a^b\int_a^b p(x)p(y)\left(f(x)-f(y)\right)\left(g(x)-g(y)\right)dxdy\right|, \\ &\leqslant \frac{1}{2P^2(b)}\int_a^b\int_a^b p(x)p(y)\left|\frac{f(x)-f(y)}{x-y}\right|\left|(x-y)(g(x)-g(y))\right|dxdy, \\ &\leqslant \frac{||f'||_\infty}{2P^2(b)}\int_a^b\int_a^b p(x)p(y)\left|(x-y)(g(x)-g(y))\right|dxdy, \\ &= \frac{||f'||_\infty}{2P^2(b)}\int_a^b\int_a^b p(x)p(y)(x-y)(g(x)-g(y))dxdy, \\ &= ||f'||_\infty T(l,g;p), \text{ where } l(x)=x \text{ for } x\in[a,b]. \end{aligned}$$

Now we have

$$T(l,g;p) = \frac{1}{P^2(b)}\int_a^b \tilde{P}(x)dg(x).$$

This leads us to (16). Now to prove sharpness of the inequality, we consider that there exists constant $D > 0$ such that

$$|T(f,g;p)| \leqslant \frac{D}{P^2(b)} \|f'\|_\infty \int_a^b \tilde{P}(x)dg(x). \tag{17}$$

If we choose $f(x) = g(x) = x$, $x \in [a,b]$, then we observe that the left hand side of (17) is equal to $\frac{1}{P^2(b)} \int_a^b \tilde{P}(x)dx$ and the right hand side of (17) is equal to $\frac{D}{P^2(b)} \int_a^b \tilde{P}(x)dx$. Thus we deduce that $D \geqslant 1$. □

The following result is a non-weighted case of the above result and has been proved in [1].

COROLLARY 2.6. *Assume that $g : [a,b] \to \mathbb{R}$ is monotonic nondecreasing on $[a,b]$ and $f : [a,b] \to \mathbb{R}$ is absolutely continuous with $f' \in L_\infty[a,b]$. Then we have the inequality*

$$|T(f,g)| \leqslant \frac{1}{2(b-a)} ||f||_\infty \int_a^b (x-a)(b-x)dg(x). \tag{18}$$

The constant $\frac{1}{2}$ is best possible.

Proof. Putting $p(t) = 1$ for $t \in [a,b]$ in (16) gives the required result. □

THEOREM 2.7. *Assume that $f,g : [a,b] \to \mathbb{R}$ are absolutely continuous functions with $f', g' \in L_\infty[a,b]$ and g is non-decreasing on $[a,b]$. Also assume that $p : [a,b] \to \mathbb{R}^+$ is integrable function. Then we have an inequality*

$$\begin{aligned} |T(f,g;p)| &\leqslant \frac{||f'|||_\infty}{P^2(b)} \int_a^b \tilde{P}(x)dg(x), \\ &\leqslant \frac{1}{P^2(b)} \|f'\|_\infty \|g'\|_\infty \int_a^b \tilde{P}(x)dx. \end{aligned} \tag{19}$$

The above inequalities are sharp.

Proof. Since g is absolutely continuous and $g \in L_\infty[a,b]$, therefore

$$\begin{aligned} \int_a^b \tilde{P}(x)dg(x) &= \int_a^b \tilde{P}(x)g'(x)dx \\ &\leqslant \|g'\|_\infty \int_a^b \tilde{P}(x)dx. \end{aligned} \tag{20}$$

Using the above result in Theorem 2.5, we get the required result. Sharpness of the inequalities is obvious from the sharpness of inequality in Theorem 2.5. □

If we consider $p(t) = 1$ for all $t \in [a,b]$ in Theorem 2.7, then we have

$$|T(f,g)| \leqslant \frac{1}{2(b-a)} \|f'\|_\infty \int_a^b (x-a)(b-x)dg(x), \qquad (21)$$
$$\leqslant \frac{1}{12} \|f'\|_\infty \|g'\|_\infty (b-a).$$

This gives us refinement of inequality (7), it has been proved in [1]. Also note that Theorem 2.7 provides us weighted version of inequalities (6) and (7).

In Theorem 2.5 and 2.7, the weight function p is positive on $[a,b]$. This condition can be weaken if we use Steffensen's generalization of Čebyšev inequality.

THEOREM 2.8. *Assume that $g:[a,b] \to \mathbb{R}$ is monotonic nondecreasing, $p:[a,b] \to \mathbb{R}$ be integrable function such that (3) is valid and $f:[a,b] \to \mathbb{R}$ is absolutely continuous with $f' \in L_\infty[a,b]$. Then inequality (16) is valid.*

Proof. As it is given in [4]

$$T(f,g;p) = \frac{1}{P^2(b)} \left(\int_a^b \bar{P}(x) \int_a^x P(t)dg(t)df(x) + \int_a^b P(x) \int_x^b \bar{P}(t)dg(t)df(x) \right),$$

where $\bar{P}(x) = P(b) - P(x)$.
This gives us

$$|T(f,g;p)| \leqslant \frac{||f'||_\infty}{P^2(b)} \left\{ \int_a^b \bar{P}(x)dx \int_a^x P(t)dg(t) + \int_a^b P(x)dx \int_x^b \bar{P}(t)dg(t) \right\}$$
$$= ||f'||_\infty T(l,g;p), \text{ where } l(x) = x \text{ for } x \in [a,b].$$

Combining the above expression with (11) gives us the required result. □

THEOREM 2.9. *Assume that $g:[a,b] \to \mathbb{R}$ is monotonic nondecreasing, $p:[a,b] \to \mathbb{R}$ be integrable function such that (3) is valid and $f,g:[a,b] \to \mathbb{R}$ is absolutely continuous with $f',g' \in L_\infty[a,b]$. Then inequality (19) is valid.*

Proof. Using inequality (20) and Theorem 2.8, we get the required result. □

THEOREM 2.10. *Assume that $f,g:[a,b] \to \mathbb{R}$ are continuous on $[a,b]$ and differentiable on $[a,b]$ with $g'(t) \neq 0$ for each $t \in (a,b)$. Also assume that $p:[a,b] \in \mathbb{R}^+$ be integrable function. Then we have the inequalites*

$$T(f,g;p) \leqslant \left\| \frac{f'}{g'} \right\|_\infty T(g,g;p)$$
$$\leqslant \frac{1}{P^2(b)} \left\| \frac{f'}{g'} \right\|_\infty \int_a^b \tilde{P}(x) \left[g'(x)\right]^2 dx. \qquad (22)$$

The above inequalities are sharp.

Proof. Let $t,s \in (a,b)$, with $t \neq s$. By Cauchy mean value theorem there is $\xi \in (t,s)$ such that

$$\frac{f(t)-f(s)}{g(t)-g(s)} = \frac{f'(\xi)}{g'(\xi)},$$

where $g'(\xi) \neq 0$.

Thus, for any $t,s \in (a,b)$ with $t \neq s$ and $g'(t) \neq 0$ for each $t \in [a,b]$, we have

$$\left|\frac{f(t)-f(s)}{g(t)-g(s)}\right| \leqslant \left\|\frac{f'}{g'}\right\|_{\infty}.$$

Using the Korkin's identity (8), we deduce

$$\begin{aligned}
T(f,g;p) &= \frac{1}{2P^2(b)}\int_a^b\int_a^b p(s)p(t)\left(\frac{f(s)-f(t)}{g(s)-g(t)}\right)(g(s)-g(t))^2 dsdt,\\
&\leqslant \frac{1}{2P^2(b)}\int_a^b\int_a^b p(s)p(t)\left|\frac{f(s)-f(t)}{g(s)-g(t)}\right|(g(s)-g(t))^2 dsdt,\\
&\leqslant \frac{1}{2P^2(b)}\left\|\frac{f'}{g'}\right\|_{\infty}\int_a^b\int_a^b p(s)p(t)(g(s)-g(t))^2 dsdt\\
&= \left\|\frac{f'}{g'}\right\|_{\infty} T(g,g;p).
\end{aligned}$$

This gives us the first inequality in (22). The second inequality follows by applying Lemma 2.1 to first inequality.

The sharpness of the inequalities can be proved in a way similar as in Theorem 2.5. □

COROLLARY 2.11. *Assume that* $f,g:[a,b]\to\mathbb{R}$ *are continuous on* $[a,b]$ *and differentiable on* $[a,b]$ *with* $g'(t)\neq 0$ *for each* $t\in(a,b)$, *then the inequalities are valid:*

$$\begin{aligned}
T(f,g) &\leqslant \left\|\frac{f'}{g'}\right\|_{\infty} T(g,g),\\
&\leqslant \frac{1}{2(b-a))}\left\|\frac{f'}{g'}\right\|_{\infty}\int_a^b (x-a)(b-x)\left[g'(x)\right]^2 dx.
\end{aligned} \tag{23}$$

The first inequality in (23) *and the constant* $\frac{1}{2}$ *in second inequality are sharp.*

Proof. Considering $p(t)=1$ for $t\in[a,b]$ in Theorem 2.10, we get the required result. □

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K. M. Awan
Department of Mathematics, University of Sargodha
Sargodha, Pakistan
e-mail: khalid819@uos.edu.pk

J. Pečarić
Abdus Salam School of Mathematical Sciences, GC University
68-B, New Muslim Town
Lahore 54600, Pakistan
and
Faculty of Textile Technology, University of Zagreb
Pierottijeva 6
10000 Zagreb, Croatia
e-mail: pecaric@mahazu.hazu.hr

Atiq ur Rehman
Department of Mathematics
COMSATS Institute of Information Technology
Attock, Pakistan
e-mail: atiq@mathcity.org