STEFFENSEN’S GENERALIZATION OF ČEBYŠEV INEQUALITY

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Abstract. In this paper, we obtain Ostrowski-type bounds for the weighted Čebyšev functional. Also we give bounds of weighted Čebyšev functional in the case of Steffensen’s generalization of Čebyšev inequality.

1. Introduction and preliminaries

Let $f, g : [a, b] \to \mathbb{R}$ and $p : [a, b] \to \mathbb{R}^+$ be Lebesgue integrable functions. Then we consider the weighted Čebyšev functional:

$$T(f, g; p) := \frac{1}{P(b)} \int_a^b p(t)f(t)g(t)dt - \frac{1}{P(b)} \int_a^b p(t)f(t)dt \cdot \frac{1}{P(b)} \int_a^b p(t)g(t)dt,$$

where $P(x) = \int_a^x p(t)dt$.

If $p(t) = 1$ for all $t \in [a, b]$, we define Čebyšev functional $T(f, g) = T(f, g; 1)$.

It is known that if $f$ and $g$ are monotonic in the same direction on interval $[a, b]$, then

$$T(f, g; p) \geq 0$$

(2)

If $f$ and $g$ are monotonic in opposite directions on interval $[a, b]$, then the reverse of the inequality in (2) is valid. In both cases, equality in (2) holds if and only if either $f$ or $g$ is constant almost everywhere.

Steffensen [6] (see also [5, page 199]) noted that inequality (2) is also valid when $f$ is an increasing function on $[a, b]$ and $g$ satisfies the condition

$$\frac{1}{P(x)} \int_a^x p(t)g(t)dt \leq \frac{1}{P(b)} \int_a^b p(t)g(t)dt,$$

where $P(x) = \int_a^x p(t)dt$,

(3)

for $x \in (a, b)$.


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The condition $p(t) > 0$ for $t \in [a, b]$ for the inequality (2) can be replaced by

$$0 \leq P(x) \leq P(b) \quad \text{for} \quad a \leq x \leq b.$$  \hspace{1cm} (4)

In 1970, A. M. Ostrowski [3] proved that if $g$ is absolutely continuous on $[a, b]$ and $g' \in L_\infty[a, b]$, then

$$|T(f, g)| \leq \frac{1}{8}(b-a)(M-m)\|g'\|_\infty,$$  \hspace{1cm} (5)

provided $m$ and $M$ are real numbers with property

$$-\infty < m \leq f \leq M < \infty \quad \text{and} \quad \|g'\|_\infty = \sup_{t \in [a, b]} |g'(t)|.$$

The constant $\frac{1}{8}$ in (5) cannot be improved in the general case.

In [1], P. Cerone and S. Dragomir gave the bounds of the Čebyšev functional $T(f, g)$. Namely, they proved that if $g$ is non-decreasing on $[a, b]$ and $f$ is absolutely continuous on $[a, b]$ with $f' \in L_\infty[a, b]$, then

$$|T(f, g)| \leq \frac{1}{2(b-a)}\|f'\|_\infty \int_a^b (x-a)(b-x) dg(x)$$  \hspace{1cm} (6)

holds and the constant $\frac{1}{2}$ is best possible. They deduce bound of $T(f, g)$ given by Čebyšev in [2] i.e. if $g$ is absolutely continuous on $[a, b]$ and $g' \in L_\infty[a, b]$, then

$$|T(f, g)| \leq \frac{1}{12}\|f'\|_\infty \|g'\|_\infty (b-a)^2,$$  \hspace{1cm} (7)

holds and the constant $\frac{1}{12}$ is sharp.

In this paper, we find Ostrowski-type bounds for weighted Čebyšev functional and deduce the results of [1] in the case of non-weighted Čebyšev functional. Also we give some bounds in the case of Steffensen generalization of weighted Čebyšev inequality.

2. Main results

Let $f, g$ be integrable functions on $[a, b]$ and $p$ be positive integrable function on $[a, b]$ with $P(b) := \int_a^b p(t)dt$. Then the weighted version of Korkin’s identity is represented by

$$T(f, g; p) = \frac{1}{2P^2(b)} \int_a^b \int_a^b p(x)p(y)(f(x) - f(y))(g(x) - g(y))dxdy.$$  \hspace{1cm} (8)

We use identity (8) to prove the following lemma, which later leads to the bounds of Čebyšev functional.
Lemma 2.1. Let $\phi : [a, b] \to \mathbb{R}$ be an absolutely continuous function, and $p : [a, b] \to \mathbb{R}^+$ an integrable function and $(\phi')^2 \in L[a, b]\] 

\begin{equation}
P(x) = \int_a^x p(t)dt \quad \text{and} \quad \tilde{P}(x) = P(x)\int_a^b tp(t)dt - P(b)\int_a^x tp(t)dt. \tag{9}\end{equation}

Then we have the inequality

\begin{equation}
T(\phi, \phi; p) \leq \frac{1}{P^2(b)} \int_a^b \tilde{P}(x) [\phi'(x)]^2 dx, \tag{10}\end{equation}

provided that the integral on right hand side of above inequality exists. Also the inequality in (10) is sharp.

Proof. We have (see [4])

\begin{equation}
T(f, g; p) = \frac{1}{P^2(b)} \int_a^b \left\{ \int_a^x p(t)h(t)dt \right\} g'(x)dx, \end{equation}

where

\begin{equation}
h(t) = \int_a^b p(s) (f(s) - f(t)) ds. \end{equation}

If we take $l(x) = x$, then

\begin{equation}
T(l, g; p) = \frac{1}{P^2(b)} \int_a^b \int_a^x p(t) \int_a^b p(s)(s-t)ds dt g'(x)dx. \end{equation}

A simple computation yields that

\begin{equation}
T(l, g; p) = \frac{1}{P^2(b)} \int_a^b \tilde{P}(x)g'(x)dx. \tag{11}\end{equation}

Korkin’s identity (8) gives us

\begin{equation}
T(\phi, \phi; p) = \frac{1}{2P^2(b)} \int_a^b \int_a^b p(x)p(y) (\phi(x) - \phi(y))^2 dxdy, \end{equation}

\begin{equation}
= \frac{1}{2P^2(b)} \int_a^b \int_a^b p(x)p(y)(x-y)^2 \left( \frac{\phi(x) - \phi(y)}{x-y} \right)^2 dxdy. \end{equation}

Since $\phi$ is absolutely continuous, $\phi(t) - \phi(s) = \int_s^t \phi'(u)du$, and by using Cauchy-Schwarz inequality, we have

\begin{equation}
T(\phi, \phi; p) = \frac{1}{2P^2(b)} \int_a^b \int_a^b p(x)p(y)(x-y)^2 \left( \frac{\int_s^x \phi'(s)ds}{x-y} \right)^2 dxdy, \end{equation}

\begin{equation}
\leq \frac{1}{2P^2(b)} \int_a^b \int_a^b p(x)p(y)(x-y)^2 \left( \frac{1}{x-y} \int_x^y [\phi'(s)]^2 ds \right) dxdy, \end{equation}

\begin{equation}
= \frac{1}{2P^2(b)} \int_a^b \int_a^b p(x)p(y)(x-y) \left( \int_x^y [\phi'(s)]^2 ds \right) dxdy.
\end{equation}

\begin{equation}
= T(l, \psi; p), \text{ by applying Korkin’s identity with } \psi(x) = \int_a^x [\phi'(s)]^2 ds. \end{equation}
Now using (11), we get inequality (10).

To prove the sharpness of (10), we assume this inequality is valid with a constant $C > 0$, that is,

$$T(\phi, \phi; p) \leq \frac{C}{P^2(b)} \int_a^b \tilde{P}(x) \left[ \phi'(x) \right]^2 dx, \quad (12)$$

If we consider $\phi(x) = x$, then we observe that the left hand side of (12) is equal to $\frac{1}{P^2(b)} \int_a^b \tilde{P}(x)dx$ and the right hand side of (12) is equal to $\frac{C}{P^2(b)} \int_a^b \tilde{P}(x)dx$. Thus we deduce that $C \geq 1$. □

A non-weighted case of the above theorem is given in the following corollary. It is also proved in [1].

**Corollary 2.2.** If $\phi : [a,b] \to \mathbb{R}$ is an absolutely continuous function with $(\phi')^2 \in L[a,b]$, then we have the inequality

$$T(\phi, \phi) \leq \frac{1}{2(b-a)} \int_a^b (x-a)(b-x)|\phi'(x)|^2 dx. \quad (13)$$

The constant $\frac{1}{2}$ is the best possible.

**Proof.** Using $p(t) = 1$ in (9), we get

$$\tilde{P}(x) = (x-a) \int_a^b t dt - (b-a) \int_a^x t dt,$$

$$= \frac{1}{2}(x-a)(b^2-a^2) - \frac{1}{2}(b-a)(x^2-a^2),$$

$$= \frac{1}{2}(x-a)(b-a)(b-x).$$

Now using this result in (10), along with the fact that $T(\phi, \phi, 1) = T(\phi, \phi)$, we get the result.

The inequality in (10) is sharp and we get constant $\frac{1}{2}$ in the simplification of (9), hence it is best possible. □

Throughout the paper we keep the notations $P(x)$ and $\tilde{P}(x)$ used in Lemma 2.1.

**Theorem 2.3.** Let $f, g : [a,b] \to \mathbb{R}$ be two absolutely continuous functions with $(f')^2$, $(g')^2 \in L[a,b]$ and $p : [a,b] \to \mathbb{R}^+$ be an integrable function, then we have the inequalities

$$|T(f, g; p)| \leq \frac{1}{P(b)} T^{\frac{1}{2}}(f, f; p) \left( \int_a^b \tilde{P}(x) \left[ g'(x) \right]^2 dx \right) \frac{1}{2},$$

$$\leq \frac{1}{P^2(b)} \left( \int_a^b \tilde{P}(x) \left[ f'(x) \right]^2 dx \right) \frac{1}{2} \left( \int_a^b \tilde{P}(x) \left[ g'(x) \right]^2 dx \right) \frac{1}{2}.$$

The above inequalities are sharp.
Proof. By Cauchy-Schwartz inequality for double integrals, we have

$$|T(f, g; p)| \leq T^{\frac{1}{2}}(f, f; p)T^{\frac{1}{2}}(g, g; p).$$

(14)

Now using (8) and Lemma 2.1 in above inequality, we get the required result. □

If we consider $p(t) = 1$, then we get Theorem 1 of [1], which is stated in the following corollary.

**Corollary 2.4.** Let $f, g : [a, b] \rightarrow \mathbb{R}$ be two absolutely continuous functions on $[a, b]$ with $(f')^2, (g')^2 \in L[a, b]$. Then we have the inequality

$$T(f, g) \leq \frac{1}{\sqrt{2}}|T(f, f)| \frac{1}{\sqrt{b-a}} \left( \int_{a}^{b} (x-a)(b-x) \left[ g'(x) \right]^2 dx \right)^{\frac{1}{2}},$$

$$\leq \frac{1}{2(b-a)} \left( \int_{a}^{b} (x-a)(b-x) \left[ f'(x) \right]^2 dx \right)^{\frac{1}{2}} \times \left( \int_{a}^{b} (x-a)(b-x) \left[ g'(x) \right]^2 dx \right)^{\frac{1}{2}}.$$

(15)

The constants $\frac{1}{\sqrt{2}}$ and $\frac{1}{2}$ in (15) are best possibles.

**Theorem 2.5.** Assume that $g : [a, b] \rightarrow \mathbb{R}$ is monotonic nondecreasing, $p : [a, b] \rightarrow \mathbb{R}^+$ be integrable function and $f : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous with $f' \in L_{\infty}[a, b]$. Then we have the inequality

$$|T(f, g; p)| \leq \frac{\|f'\|_{\infty}}{p^2(b)} \int_{a}^{b} \tilde{P}(x) dg(x),$$

(16)

The inequality (16) is sharp.

Proof. We have, by Korkin’s identity, that

$$|T(f, g; p)| = \frac{1}{2P^2(b)} \left| \int_{a}^{b} \int_{a}^{b} p(x)p(y) (f(x) - f(y)) (g(x) - g(y)) dx dy \right|,$$

$$\leq \frac{1}{2P^2(b)} \int_{a}^{b} \int_{a}^{b} p(x)p(y) \left| \frac{f(x) - f(y)}{x-y} \right| |(x-y)(g(x) - g(y))| dx dy,$$

$$\leq \frac{\|f'\|_{\infty}}{2P^2(b)} \int_{a}^{b} \int_{a}^{b} p(x)p(y) |(x-y)(g(x) - g(y))| dx dy,$$

$$= \frac{\|f'\|_{\infty}}{2P^2(b)} \int_{a}^{b} \int_{a}^{b} p(x)p(y)(x-y)(g(x) - g(y)) dx dy,$$

$$= \|f'\|_{\infty} T(l, g; p), \text{ where } l(x) = x \text{ for } x \in [a, b].$$

Now we have

$$T(l, g; p) = \frac{1}{P^2(b)} \int_{a}^{b} \tilde{P}(x) dg(x).$$
This leads us to (16). Now to prove sharpness of the inequality, we consider that there exists constant $D > 0$ such that

$$|T(f,g;p)| \leq \frac{D}{P^2(b)} \|f'\|_\infty \int_a^b \tilde{P}(x)dg(x).$$

(17)

If we choose $f(x) = g(x) = x, x \in [a,b]$, then we observe that the left hand side of (17) is equal to $\frac{1}{P^2(b)} \int_a^b \tilde{P}(x)dx$ and the right hand side of (17) is equal to $\frac{D}{P^2(b)} \int_a^b \tilde{P}(x)dx$. Thus we deduce that $D \geq 1$. □

The following result is a non-weighted case of the above result and has been proved in [1].

**Corollary 2.6.** Assume that $g : [a,b] \to \mathbb{R}$ is monotonic nondecreasing on $[a,b]$ and $f : [a,b] \to \mathbb{R}$ is absolutely continuous with $f' \in L_\infty[a,b]$. Then we have the inequality

$$|T(f,g)| \leq \frac{1}{2(b-a)} \|f\|_\infty \int_a^b (x-a)(b-x)dg(x).$$

(18)

The constant $\frac{1}{2}$ is best possible.

*Proof.* Putting $p(t) = 1$ for $t \in [a,b]$ in (16) gives the required result. □

**Theorem 2.7.** Assume that $f,g : [a,b] \to \mathbb{R}$ are absolutely continuous functions with $f',g' \in L_\infty[a,b]$ and $g$ is non-decreasing on $[a,b]$. Also assume that $p : [a,b] \to \mathbb{R}^+$ is integrable function. Then we have an inequality

$$|T(f,g;p)| \leq \frac{\|f'\|_\infty}{P^2(b)} \int_a^b \tilde{P}(x)dg(x),$$

$$\leq \frac{1}{P^2(b)} \|f'\|_\infty \|g'\|_\infty \int_a^b \tilde{P}(x)dx.$$  

(19)

The above inequalities are sharp.

*Proof.* Since $g$ is absolutely continuous and $g \in L_\infty[a,b]$, therefore

$$\int_a^b \tilde{P}(x)dg(x) = \int_a^b \tilde{P}(x)g'(x)dx \leq \|g'\|_\infty \int_a^b \tilde{P}(x)dx.$$  

(20)

Using the above result in Theorem 2.5, we get the required result. Sharpness of the inequalities is obvious from the sharpness of inequality in Theorem 2.5. □
If we consider \( p(t) = 1 \) for all \( t \in [a,b] \) in Theorem 2.7, then we have

\[
|T(f,g)| \leq \frac{1}{2(b-a)} \|f'\|_{\infty} \int_a^b (x-a)(b-x) dg(x),
\]

\[
\leq \frac{1}{12} \|f'\|_{\infty} \|g'\|_{\infty} (b-a).
\]  

This gives us refinement of inequality (7), it has been proved in [1]. Also note that Theorem 2.7 provides us weighted version of inequalities (6) and (7).

In Theorem 2.5 and 2.7, the weight function \( p \) is positive on \([a,b]\). This condition can be weaken if we use Steffensen’s generalization of Čebyšev inequality.

**Theorem 2.8.** Assume that \( g : [a,b] \to \mathbb{R} \) is monotonic nondecreasing, \( p : [a,b] \to \mathbb{R} \) be integrable function such that (3) is valid and \( f : [a,b] \to \mathbb{R} \) is absolutely continuous with \( f' \in L_{\infty}[a,b] \). Then inequality (16) is valid.

**Proof.** As it is given in [4]

\[
T(f,g;p) = \frac{1}{P^2(b)} \left( \int_a^b P(x) dx \int_a^x P(t) dt \right) \left( \int_a^b f'(t) dt + \int_a^b f(t) df(t) \right),
\]

where \( P(x) = P(b) - P(x) \).

This gives us

\[
|T(f,g;p)| \leq \frac{\|f'\|_{\infty}}{P^2(b)} \left\{ \int_a^b \tilde{P}(x) dx \int_a^x P(t) dt \right\} \int_a^b \tilde{P}(t) dt + \int_a^b P(x) dx \int_a^b P(t) dt
\]

= \|f'\|_{\infty} T(l,g;p), \text{ where } l(x) = x \text{ for } x \in [a,b].

Combining the above expression with (11) gives us the required result. \( \square \)

**Theorem 2.9.** Assume that \( g : [a,b] \to \mathbb{R} \) is monotonic nondecreasing, \( p : [a,b] \to \mathbb{R} \) be integrable function such that (3) is valid and \( f, g : [a,b] \to \mathbb{R} \) is absolutely continuous with \( f', g' \in L_{\infty}[a,b] \). Then inequality (19) is valid.

**Proof.** Using inequality (20) and Theorem 2.8, we get the required result. \( \square \)

**Theorem 2.10.** Assume that \( f, g : [a,b] \to \mathbb{R} \) are continuous on \([a,b]\) and differentiable on \([a,b]\) with \( g'(t) \neq 0 \) for each \( t \in (a,b) \). Also assume that \( p : [a,b] \in \mathbb{R}^+ \) be integrable function. Then we have the inequalities

\[
T(f,g;p) \leq \frac{f'}{g'} \|T(g,g;p)\|
\]

\[
\leq \frac{1}{P^2(b)} \left\| \frac{f'}{g'} \right\|_{\infty} \int_a^b \tilde{P}(x) \left[ g'(x) \right]^2 dx.
\]  

The above inequalities are sharp.
Proof. Let \( t, s \in (a, b) \), with \( t \neq s \). By Cauchy mean value theorem there is \( \xi \in (t, s) \) such that
\[
\frac{f(t) - f(s)}{g(t) - g(s)} = \frac{f'(\xi)}{g'(\xi)},
\]
where \( g'(\xi) \neq 0 \).

Thus, for any \( t, s \in (a, b) \) with \( t \neq s \) and \( g'(t) \neq 0 \) for each \( t \in [a, b] \), we have
\[
\left| \frac{f(t) - f(s)}{g(t) - g(s)} \right| \leq \left\| \frac{f'}{g'} \right\|_{\infty}.
\]

Using the Korkin’s identity (8), we deduce
\[
T(f, g; p) = \frac{1}{2P^2(b)} \int_a^b \int_a^b p(s)p(t) \left( \frac{f(s) - f(t)}{g(s) - g(t)} \right) (g(s) - g(t))^2 ds dt,
\]
\[
\leq \frac{1}{2P^2(b)} \int_a^b \int_a^b p(s)p(t) \left( \frac{f(s) - f(t)}{g(s) - g(t)} \right) (g(s) - g(t))^2 ds dt,
\]
\[
\leq \frac{1}{2P^2(b)} \left\| \frac{f'}{g'} \right\|_{\infty} \int_a^b \int_a^b p(s)p(t)(g(s) - g(t))^2 ds dt
\]
\[
= \left\| \frac{f'}{g'} \right\|_{\infty} T(g, g; p).
\]

This gives us the first inequality in (22). The second inequality follows by applying Lemma 2.1 to first inequality.

The sharpness of the inequalities can be proved in a way similar as in Theorem 2.5. \( \Box \)

Corollary 2.11. Assume that \( f, g : [a, b] \to \mathbb{R} \) are continuous on \([a, b]\) and differentiable on \([a, b]\) with \( g'(t) \neq 0 \) for each \( t \in (a, b) \), then the inequalities are valid:

\[
T(f, g) \leq \left\| \frac{f'}{g'} \right\|_{\infty} T(\frac{g}{g'}, g).
\]

(23)

\[
\leq \frac{1}{2(b-a)} \left\| \frac{f'}{g'} \right\|_{\infty} \int_a^b (x-a)(b-x) \left[ g'(x) \right]^2 dx.
\]

The first inequality in (23) and the constant \( \frac{1}{2} \) in second inequality are sharp.

Proof. Considering \( p(t) = 1 \) for \( t \in [a, b] \) in Theorem 2.10, we get the required result. \( \Box \)
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