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ASYMPTOTIC INEQUALITIES AND COMPARISON OF CLASSICAL MEANS

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Abstract. Inequalities between linear combinations of means are the subject of interest for decades. In this paper we propose a new approach to this subject, using the concept of asymptotical expansion of means. This enables us to find necessary conditions and the optimal values for coefficients in order that the inequality between linear combination of three or four means will be valid. We restrict ourselfs to the study of the most common classical mean, leaving detailed study of parametric means to the future works.

1. Introduction

Everyone has already seen examples of inequalities between various means, like

$$G + Q \leqslant 2A$$

or

$$2Q + H \leqslant 3A \tag{1.1}$$

where G, Q, A, H denote the usual bivariate means: geometric, quadratic, arithmetic and harmonic.

The main question here should not be why such inequalities hold, but, instead, why one should take coefficients 2, 1 and 3 in (1.1)? Are maybe better choices of these coefficients? Does inequality sign remains valid if we change some of them?

To be more precise, let us denote by M_k, $k = 1,2,3$, some of classical means from the list below. Then, the question which we will answer in this paper is: determine the constants a, b and c such that inequality

$$aM_1 + bM_2 + cM_3 \geqslant 0 \tag{1.2}$$

would be possible.

Of course, such problem is meaningful only if a, b and c are optimal: for any other choice a^*, b^*, c^* such that (1.2) remains valid it should be

$$a^*M_1 + b^*M_2 + c^*M_3 \geqslant aM_1 + bM_2 + cM_3.$$

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We will answer to one more question connected with this problem. Relation (1.1) can be interpreted as: $2Q+H$ is an approximation for $3A$. How good is this approximation?

In order to answer to this question, we must agree about measure for approximations. The natural one is the relative error. The good approximation for this type of error is to measure difference between mean with respect to average value of one of them. Such concept is closely related to the concept of moving averages and can be interpreted easily whenever means take his values into bounded subset away from origin.

In our previous papers [7] we discussed the concept of asymptotic expansion of means, when one consider the mean in the form

$$M(x+s,x+t) = x + c_1(s,t) + \frac{c_2(s,t)}{x} + \frac{c_3(s,t)}{x^2} + \dots.$$

Each mean is identified with its series of coefficients. The linear combination of means has also such form. Therefore, the quality of approximation will be denoted by the degree of the first nonvanishing coefficient.

In this paper we are dealing with standard means. Our goal is to present a complete and detailed analysis of all possible combinations of standard means which leads to optimal inequalities. Some new relations between these means will be derived. However, we do not intend to give analytic proof here, because one may wonder if this is even possible in some cases. The technique developed here are applicable to all other bivariate means which will not be considered in this paper. Some of them, like parameter means (power means, generalized logarithmic mean, Stolarsky mean and Gini mean), Neuman-Sandor or Seiffert mean will be discused separately in a forthcoming papers [8], [10], [18]. Se also [13]–[17] for similar expansions of means and related inequalities.

2. Asymptotic inequalities between mean

Let us mention briefly the means which we are studing and its notation. We are choosen the following ones:

$$N(s,t) = \frac{s^2+t^2}{s+t}, \qquad Q(s,t) = \sqrt{\frac{s^2+t^2}{2}}$$

$$A(s,t) = \frac{s+t}{2}, \qquad L(s,t) = \frac{t-s}{\log t - \log s},$$

$$I(s,t) = \frac{1}{e}\left(\frac{t^t}{s^s}\right)^{\frac{1}{t-s}}, \qquad H(s,t) = \frac{2st}{s+t},$$

$$G(s,t) = \sqrt{st}.$$

Together with already mentioned ones, one can find here contraharmonic mean N, identric mean I and logaritmic mean L. See [5], [6] and [12] for definition and elementary properties of these means.

DEFINITION 2.1. Let M be bivariate function, and

$$M(x+s,x+t) = c_k(s,t)x^{-k+1} + \mathscr{O}(x^{-k}). \tag{2.1}$$

If $c_k(s,t) > 0$ for all s and t, then we say that M is *asymptotically* greater than zero, and write $M \succ 0$. Of course, this is equivalent to $0 \prec M$.

THEOREM 2.2. *If $M(x+s,x+t) \geqslant 0$ for all values $x,s,t > 0$, then $M \succ 0$.*

Proof. For x large enough, the sign of $M(x+s,x+t)$ is the same as the sign of the first term in its asymptotic expansion. □

Therefore, one may consider asymptotic inequalities as a necessary condition for the inequality between comparable means. See [1] and [4] for the analysis of the asymptotic behavior of bivariate means.

The asymptotic comparison of linear combination of means is easy, it is sufficient to know the asymptotic expansion of these means. For example, in [7] the following expansions are derived

$$Q(x+s,x+t) = x+\alpha+\frac{\beta^2}{2x}-\frac{\alpha\beta^2}{2x^2}+\frac{\beta^2(4\alpha^2-\beta^2)}{8x^3}+\frac{\alpha\beta^2(4\alpha^2+3\beta^2)}{8x^4}+\ldots,$$
$$G(x+s,x+t) = x+\alpha-\frac{\beta^2}{2x}+\frac{\alpha\beta^2}{2x^2}-\frac{\beta^2(4\alpha^2+\beta^2)}{8x^3}+\frac{\alpha\beta^2(4\alpha^2+3\beta^2)}{8x^4}+\ldots$$

where s and t are substituted by more appropriate variables, $t=\alpha+\beta$, $s=\alpha-\beta$. Since it holds $A(x+s,x+t)=x+\alpha$, adding the previous two expansions one obtain

$$2A-G-Q \sim \frac{\beta^4}{4x^3}.$$

Hence,

$$2A-G-Q \succ 0.$$

This is a strong suggestion that the inequality

$$2A-G-Q \geqslant 0$$

may be valid, and if it is satisfied for all values of arguments, then the choice of the coefficients are optimal. The optimality will be proved in Theorem 3.1.

The analysis of such relations can be made under additional assumption $\alpha=0$ which simplify these coefficients. Let x, s and t, $(s<t)$ be given. Denote $x'=x+\alpha$, $s'=s-\alpha$, $t'=t-\alpha$. Then we have $x' \geqslant x^*$ and

$$M(x+s,x+t) = M(x'+s',x'+t')$$

where $s'+t'=0$, so for the asymptotic comparison of two means it is sufficient to consider the case $\alpha=0$.

We shall ilustrate the theorem above by taking some concrete means. In order to do this, the following list of coefficients will be useful. It is obtained in the paper

[7]. This time, the means are written in the falling order, which is clearly indicated by the values of the second coefficients. We are dealing here only with bivariate non-parametric mean. The asymptotic expansion of parameter means, which cover more general means like power means, Stolarsky, generalized logarithmic or Gini means, are treated in [8].

	x	β^2/x	β^4/x^3	β^6/x^5	β^8/x^7
N	1	1	0	0	0
Q	1	$\frac{1}{2}$	$-\frac{1}{8}$	$\frac{1}{16}$	$-\frac{5}{128}$
A	1	0	0	0	0
I	1	$-\frac{1}{6}$	$-\frac{13}{360}$	$-\frac{737}{45360}$	$-\frac{50801}{5443200}$
L	1	$-\frac{1}{3}$	$-\frac{4}{45}$	$-\frac{44}{945}$	$-\frac{428}{14175}$
G	1	$-\frac{1}{2}$	$-\frac{1}{8}$	$-\frac{1}{16}$	$-\frac{5}{128}$
H	1	-1	0	0	0

Let us consider the linear combination of three means (1.2). Since the first coefficients are equal, we should suppose that

$$a+b+c=0,$$

so, $b=-a-c$. Then, we may choose c such that x^{-1} term of the linear combination vanishes. This will eliminate x^{-2} term too, and the approximation will be of order $\mathscr{O}(x^{-3})$. For example, for the Q, A, H combination, we have

$$aQ-(a+c)A+cH=(\frac{a}{2}-c)\frac{\beta^2}{2x}+\mathscr{O}(x^{-3})$$

therefore, the critical value is $c=\frac{1}{2}a$. So we may take $a=2$, $b=-3$, $c=1$ to obtain

$$2Q-3A+H\sim-\frac{\beta^4}{4x^3}.$$

Therefore, the conclusion is the following:

1) If $a+b+c>0$ then $aQ+bA+cH\succ 0$, with order $\mathscr{O}(x)$.

2) Suppose $a+b+c=0$. If $a-2c>0$, then $aQ-(a+c)A+cH\succ 0$, with order $\mathscr{O}(x^{-1})$.

3) Suppose additionally that $a=2c$. Then $2Q-3A+H\prec 0$, with order $\mathscr{O}(x^{-3})$.

As one can see, the choice for the coefficients are made such that every new restriction increase the asymptotic order. We shall show that such choice of the coefficients is optimal. It is completely trivial to make similar analysis for any combinations of three means. Obtained relations are strong indicator for the inequalities between means of the same form, but as we shall see, this is not always true.

In the table below a complete analysis for the linear combination of three mean are given. In order to keep tables to decent size, in the first table we consider combinations which includes arithmetic mean, and other combinations will be presented in the second one.

	N	Q	A	I	L	G	H	$\times\beta^4/x^3$	
1	1	−2	1					1/4	$\geqslant 0$
2	1		−7	6				−13/60	$\leqslant 0$
3	1		−4		3			−4/15	$\leqslant 0$
4	1		−3			2		−1/4	$\leqslant 0$
5	1		−2				1	0	$= 0$
6		1	−4	3				−7/30	$\leqslant 0$
7		2	−5		3			−31/60	$\leqslant 0$
8		1	−2			1		−1/4	$\leqslant 0$
9		2	−3				1	−1/4	$\leqslant 0$
10			1	−2	1			−1/60	$\leqslant 0$
11			2	−3		1		−1/60	$\leqslant 0$
12			5	−6			1	13/60	$\geqslant 0$
13			1		−3	2		1/60	$\geqslant 0$
14			2		−3		1	4/15	$\geqslant 0$
15			1			−2	1	1/4	$\geqslant 0$

Table 1. Optimal coefficients derived from asymptotical analysis. Inequalities are the true ones.

This table should be read as follows. The optimal coefficient are given and the sign of the first asymptotic coefficient determines the sign of inequality. For example, the seventh row reads as:

$$5A - 2Q - 3L \sim \frac{31\beta^4}{60x^3} \succ 0$$

and the sign $\geqslant 0$ or $\leqslant 0$ in the last column means that this asymptotic inequality is true one, which can be (easily) verified by CAS, we used Mathematica for this purpose. Of course, this do not means that these inequalities are proved in a traditional way, using calculus and verification of such type can be a tedious job, at least for some of them.

The proofs for majority of inequalities stated here are known and easy. However, to the best of our knowledge, we cannot find the analytic proof in the bibliography for some of them. So, we stated here the first conjecture, for the inequalities connected with identric mean (which does not mean that all others are already proved, but inequalities with identric mean are harder to prove).

CONJECTURE 2.3. *The following inequalities are valid, and they are of maximal order:*

$$N+6I \leqslant 7A, \tag{2.2}$$

$$Q+3I \leqslant 4A, \tag{2.3}$$

$$A+L \leqslant 2I \tag{2.4}$$

$$2A+G \leqslant 3I \tag{2.5}$$

$$5A+H \geqslant 6I. \tag{2.6}$$

The second table is:

	N	Q	I	L	G	H	$\times\beta^4/x^3$	
16	4	-7	3				$23/30$	$\geqslant 0$
17	5	-8		3			$11/15$	$-$
18	2	-3			1		$1/4$	$-$
19	3	-4				1	$1/2$	$\geqslant 0$
20	1		-8	7			$-1/3$	$\leqslant 0$
21	2		-9		7		$-11/20$	$\leqslant 0$
22	5		-12			7	$13/30$	$\geqslant 0$
23	1			-9	8		$-1/5$	$-$
24	1			-3		2	$4/15$	$\geqslant 0$
25	1				-4	3	$1/2$	$\geqslant 0$
26		1	-5	4			$-3/10$	$\leqslant 0$
27		1	-3		2		$-4/15$	$\leqslant 0$
28		5	-9			4	$-3/10$	$-$
29		1		-6	5		$-13/60$	$-$
30		4		-9		5	$3/10$	$\geqslant 0$
31		1			-3	2	$1/4$	$\geqslant 0$
32			1	-2	1		$1/60$	$\geqslant 0$
33			4	-5		1	$3/10$	$\geqslant 0$
34			3		-5	2	$31/60$	$\geqslant 0$
35				3	-4	1	$7/30$	$\geqslant 0$

Table 2. Optimal coefficients derived from asymptotical analysis. Inequalities marked with $-$ are only asymptotical and does not hold in general.

The analytic proof for the majority of inequalities stated here are not known. The inequalities marked with $-$ will be analysed in the sequell.

3. From asymptotic to the true inequalities

We shall prove that coefficients calculated this way are optimal.

THEOREM 3.1. *Let M_1, M_2, M_3 be any three different means mentioned above and a, b, c coefficients given in the tables such that*

$$aM_1 + bM_2 + cM_3 \succ 0 \tag{3.1}$$

is satisfied. If a^*, b^*, c^* are arbitrary choice of coefficients for which the inequality

$$a^*M_1(s,t) + b^*M_2(s,t) + c^*M_3(s,t) \geqslant 0 \tag{3.2}$$

is valid for all $s,t > 0$, and the approximation is of the same order, then it holds

$$a^*M_1 + b^*M_2 + c^*M_3 \succ aM_1 + bM_2 + cM_3. \tag{3.3}$$

Proof. Denote

$$F = aM_1 + bM_2 + cM_3 = c_0 + c_2(s,t)\beta^2 + c_4\beta^4 + \dots,$$
$$F^* = a^*M_1 + b^*M_2 + c^*M_3 = c_0^* + c_2^*(s,t)\beta^2 + c_4^*\beta^4 + \dots,$$

The choosen coefficients a, b, c are uniquely determined such that $c_2(s,t) \equiv 0$ is satisfied. If a^*, b^*, c^* is another choice such that (3.2) is satisfied, then we must have $c_0^* \geqslant 0$. In the case $c_0^* > 0$, (3.3) is obviously fulfilled, so we shall assume $c_0^* = 0$. In this case, for the same reason, $c_2^*(s,t) > 0$ must be satisfied for all s,t. Therefore, (3.3) must be true.

COROLLARY 3.2. *Let a, b, c be any choice of coefficients from the tables above. If the inequality*

$$aM_1(s,t) + bM_2(s,t) + cM_3(s,t) \geqslant 0$$

is satisfied for all t, $s > 0$, then this choice of coefficients is optimal.

COROLLARY 3.3. *All choices of coefficients for the true inequalities in Tables 1 and 2 are optimal, for inequalities of order $\mathscr{O}(x^{-3})$.*

Because of the homogeneity property, if one want to compare two means M_1 and M_2, it is sufficient to do this for all s and t from the curve $\varphi(s,t) = 0$ which connect two axes. Namely, let (x,y) be arbitrary point in the first quadrant. Take $x' = kx$, $y' = ky$ and choose k such that it holds $\varphi(x',y') = 0$. Then $M_1(x,y)/M_2(x,y) = M_1(x',y')/M_2(x',y')$.

It is usual to choose the curve to be $s + t = 1$ or $s^2 + t^2 = 1$, where $0 \leqslant s \leqslant 1$. For example, in the Figure 1 the comparison of the means Q, A and G was made in both cases, $s + t = 1$ on the left and $s^2 + t^2 = 1$ on the right.

Here, on the figure on the left the graphs of the following functions are plotted: $Q(s, 1-s) = \sqrt{(2s^2 - 2s + 1)/2}$, $A(s, 1-s) = 1/2$ and $G(s, 1-s) = \sqrt{(1-s)s}$, similarly on the right side.

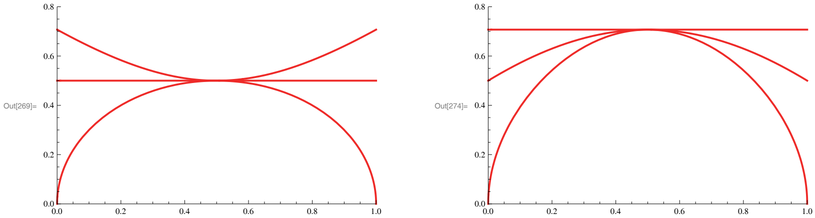

Figure 1.

Because of the symmetry, it is enough to consider the case $0 \leqslant s \leqslant \frac{1}{2}$, but we prefere symmetric larger picture.

In some cases the choice of the curve $\varphi(s,t) = t - \dfrac{1}{s} = 0$, which approaches asymptotically to the axes, is appropriate.

In the sequel we will use the curve $s+t-1=0$ for the comparison purpose.

The asymptotic inequality compares two mean in a cone close to the line $t=s$, i.e., for s, t near to $\frac{1}{2}$, if one looks the values on the curve $s+t-1=0$.

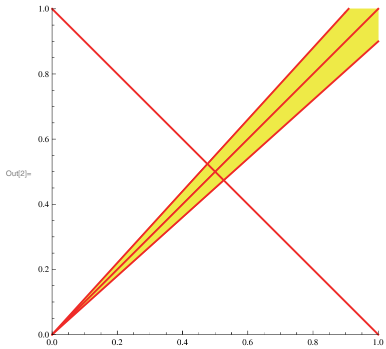

Figure 2.

In order to obtain true inequality, the whole range $0 \leqslant s \leqslant \frac{1}{2}$ should be considered. But, in the majority of the cases, it is sufficient to check the boundary $s=0$ only.

4. The inequalities of opposite sign

Let M_1, M_2, M_3 be three means under consideratrion. We may suppose that $M_1 \leqslant M_2 \leqslant M_3$. Then, the main inequality

$$aM_1 + bM_2 + cM_3 \geqslant 0$$

because of assumed connection $a+b+c=0$ can be also written in the following form

$$(1-\mu)M_1 + \mu M_3 \lesseqgtr M_2.$$

In this section we will consider the double inequality of the type

$$(1-\mu)M_1+\mu M_3 \leqslant M_2 \leqslant (1-\nu)M_1+\nu M_3. \tag{4.1}$$

One side of these inequalities is solved in the previous section, which is connected with asymptotic inequalities of order $\mathscr{O}(x^{-3})$. Now we shall consider the other side of the same inequalities which will be of order $\mathscr{O}(x^{-1})$.

For the proof of inequalities (4.1) it is sufficient to consider values of means on the line $t+s=1$. Therefore, (4.1) can be written in the form

$$\mu \leqslant \frac{M_2(s,1-s)-M_1(s,1-s)}{M_3(s,1-s)-M_1(s,1-s)} \leqslant \nu. \tag{4.2}$$

Hence, the choice of the optimal coefficients μ and ν is connected with the values of infimum and supremum of the function in the middle. For the most of considered means, this function will be monotone, and these extrema will be obtained at the edges $s=0$ and $s=1/2$. Of course, for the value $s=1/2$ this function is undefined, so the calculation should be made using asymptotic expansion. These coefficients are given in Table 1 and Table 2. The value at the other side $s=0$ can be easily obtained in some cases, in other it can be calculated using Taylor expansion. We shall present the result in Table 3 and Table 4 below.

It should be noticed that there will be few cases when the fraction will not be monotonic function: they are marked with $-$ in the first two, and also in the next two tables. These cases will be analysed in the subsequent sections.

Let $M_1 \leqslant M_2 \leqslant M_3$ be tree given means and

$$F(s,t)=M_2(s,t)-rM_3(s,t)-(1-r)M_1(s,t). \tag{4.3}$$

Then $F(s,1-s) \lesseqgtr 0$ is equivalent to

$$r \gtreqless \frac{M_2(s,1-s)-M_1(s,1-s)}{M_3(s,1-s)-M_1(s,1-s)}.$$

If the value 0 of F in an optimal inequality is obtained in a point s different from 0 or $1/2$, then it should be its extremal value.

THEOREM 4.1. *The critical value r^* of the coefficients in the inequality*

$$M_2(s,t)-rM_3(s,t)-(1-r)M_1 \lesseqgtr 0 \tag{4.4}$$

will be obtained in a point $0<s^*<1/2$ if the following is satisfied:

$$\begin{aligned} &M_2(s^*,1-s^*)-r^*M_3(s^*,1-s^*)-(1-r^*)M_1(s^*,1-s^*)=0,\\ &\frac{\mathrm{d}M_2}{\mathrm{d}s}(s^*,1-s^*)-r^*\frac{\mathrm{d}M_3}{\mathrm{d}s}(s^*,1-s^*)-(1-r^*)\frac{\mathrm{d}M_1}{\mathrm{d}s}(s^*,1-s^*)=0. \end{aligned} \tag{4.5}$$

The following table is supplement to the Table 1. All obtained inequalities are of the order $\mathscr{O}(x^{-1})$

	N	Q	A	I	L	G	H	
1	$\sqrt{2}-1$	-1	$2-\sqrt{2}$					$\leqslant 0$
2	$\frac{e}{2(e-1)}$		-1	$\frac{e-2}{2(e-1)}$				$\geqslant 0$
3	$\frac{1}{2}$		-1		$\frac{1}{2}$			$\geqslant 0$
4	$\frac{1}{2}$		-1			$\frac{1}{2}$		$\geqslant 0$
5	$\frac{1}{2}$		-1				$\frac{1}{2}$	$= 0$
6		$\frac{e-2}{e\sqrt{2}-2}$	-1	$\frac{(\sqrt{2}-1)e}{e\sqrt{2}-2}$				$\geqslant 0$
7		$\frac{1}{\sqrt{2}}$	-1		$1-\frac{1}{\sqrt{2}}$			$\geqslant 0$
8		$\frac{1}{\sqrt{2}}$	-1			$1-\frac{1}{\sqrt{2}}$		$\geqslant 0$
9		$\frac{1}{\sqrt{2}}$	-1				$1-\frac{1}{\sqrt{2}}$	$\geqslant 0$
10			$\frac{2}{e}$	-1	$1-\frac{2}{e}$			$\geqslant 0$
11			$\frac{2}{e}$	-1		$1-\frac{2}{e}$		$\geqslant 0$
12			$\frac{2}{e}$	-1			$1-\frac{2}{e}$	$\leqslant 0$
13			0		-1	1		$\leqslant 0$
14			0		-1		1	$\leqslant 0$
15			0			-1	1	$\leqslant 0$

Table 3. Optimal coefficients derived from the values on the edges. All inequalities are the true ones.

All inequalities from this and the following Table 4 have additional property that the sign of equality holds for $(s,t)=(0,1)$ as well as for $s=t$.

One can see that some items have degenerated coefficients, for which $\mu=0$. For example, from the row 15 from both tables, the following double inequality can be written

$$0\cdot A+H\leqslant G\leqslant \tfrac{1}{2}A+\tfrac{1}{2}H.$$

As before, we will extract rows with identric means in the form of conjecture, but some others also are waiting for the proof.

CONJECTURE 4.2. *The following inequalities are valid, and they are of order* $\mathscr{O}(x^{-1})$*:*

$$e\cdot N+(e-2)I\geqslant 2(e-1)A \tag{4.6}$$

$$(e-2)Q+(\sqrt{2}-1)eI\geqslant (e\sqrt{2}-2)A \tag{4.7}$$

$$2A+(e-2)L \geqslant e\cdot I \tag{4.8}$$

$$2A+(e-2)G \geqslant e\cdot I \tag{4.9}$$

$$2A+(e-2)H \leqslant e\cdot I \tag{4.10}$$

Note that the inequality sign is changed in the last inequality, despite the same coefficients as in previous two. These inequalities have opposite sign form ones given in Conjecture 2.3. Note also that (4.9) is proved in [3].

In the following table the values of coefficients are calculated for other combination of means. All inequalities is the true ones, except this from the first row 16.

	N	Q	I	L	G	H	
16	$\frac{e\sqrt{2}-2}{2(e-1)}$	-1	$\frac{(2-\sqrt{2})e}{2(e-1)}$				$-$
17	$\frac{1}{\sqrt{2}}$	-1		$1-\frac{1}{\sqrt{2}}$			$\geqslant 0$
18	$\frac{1}{\sqrt{2}}$	-1			$1-\frac{1}{\sqrt{2}}$		$\geqslant 0$
19	$\frac{1}{\sqrt{2}}$	-1				$1-\frac{1}{\sqrt{2}}$	$\leqslant 0$
20	$\frac{1}{e}$		-1	$1-\frac{1}{e}$			$\geqslant 0$
21	$\frac{1}{e}$		-1		$1-\frac{1}{e}$		$\geqslant 0$
22	$\frac{1}{e}$		-1	$1-\frac{1}{e}$			$\leqslant 0$
23	0			-1	1		$\leqslant 0$
24	0			-1		1	$\leqslant 0$
25	0				-1	1	$\leqslant 0$
26		$\frac{\sqrt{2}}{e}$	-1	$1-\frac{\sqrt{2}}{e}$			$\geqslant 0$
27		$\frac{\sqrt{2}}{e}$	-1		$1-\frac{\sqrt{2}}{e}$		$\geqslant 0$
28		$\frac{\sqrt{2}}{e}$	-1			$1-\frac{\sqrt{2}}{e}$	$\leqslant 0$
29		0		-1	1		$\leqslant 0$
30		0		-1		1	$\leqslant 0$
31		0			-1	1	$\leqslant 0$
32			0	-1	1		$\leqslant 0$
33			0	-1		1	$\leqslant 0$
34			0		-1	1	$\leqslant 0$
35				0	-1	1	$\leqslant 0$

Table 4. Optimal coefficients derived from the values on the edges. All inequalities are the true ones exept in the first row.

CONJECTURE 4.3. *The following inequalities are valid, and they are of order*

$\mathscr{O}(x^{-1})$:

$$e\cdot N+(e-1)L\geqslant e\cdot I, \tag{4.11}$$

$$e\cdot N+(e-1)G\geqslant e\cdot I, \tag{4.12}$$

$$e\cdot N+(e-1)H\leqslant e\cdot I, \tag{4.13}$$

$$\sqrt{2}\cdot Q+(e-\sqrt{2})L\geqslant e\cdot I, \tag{4.14}$$

$$\sqrt{2}\cdot Q+(e-\sqrt{2})G\geqslant e\cdot I, \tag{4.15}$$

$$\sqrt{2}\cdot Q+(e-\sqrt{2})H\leqslant e\cdot I. \tag{4.16}$$

5. The false asymptotic inequalities

Let us discuss the inequalities in rows 17–18, 23 and 28–29 of Table 2, which are marked with $-$ sign. This means that they do not hold for all $s,t>0$. We shall discuss also inequality from the row 16 of Table 3.

We shall show that they can be converted to the true inequalities on the cost of lovering the order of approximations. Let us discuss each inequality separately.

5.1. The I-Q-H case

We read in the row 28 of the Table 2 that

$$9I-5Q-4H\succ 0.$$

and the sign $-$ indicates that the inequality

$$I(s,t)-\tfrac{5}{9}Q(s,t)-\tfrac{4}{9}H(s,t)\geqslant 0 \tag{5.1}$$

is not true for all choices of arguments t, s. Because of the homogeneity of the means we can always suppose that t and s are normed in some way. Take $t=1-s$. In the limit case when $s\to 0$, we have $Q(0,1)=1/\sqrt{2}$, $I(0,1)=1/e$, and $H(0,1)=0$. Therefore, the value of this combination for $s=0$ is $-5/9\sqrt{2}+1/e\approx -0.025$. The relation $\frac{5}{9}Q+\frac{4}{9}H-I\geqslant 0$ holds if $s>0.09972\ldots$.

Let us denote

$$F(s,t)=I(s,t)-\nu Q(s,t)-(1-\nu)H(s,t).$$

Because of the symmetry and homogeneity property of observed means, in order to prove inequality $F(s,t)\geqslant 0$ for all $s,t>0$, it is enough to prove $F(s,1-s)\geqslant 0$ for $0<s\leqslant\frac{1}{2}$. The Figure 3 left shows the graph of this function for the initial choice of coefficients $\nu=5/9$. One solution is to take coefficients such that equality holds at $s=0$. This leads to inequality (4.16), the graph of the function

$$F(s,1-s)=I(s,1-s)-\frac{\sqrt{2}}{e}\cdot Q(s,1-s)-(1-\frac{\sqrt{2}}{e})H(s,1-s)$$

is plotted in the middle in Figure 3.

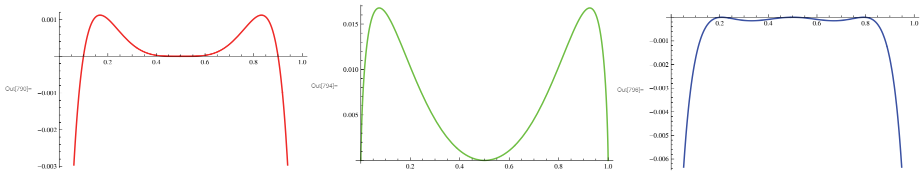

Figure 3. I-Q-H inequality.

If we wish to obtain inequality with opposite sign, then the order od approximation should be lowered. The optimal value of the coefficients can be determined numerically from (4.5). The exact value is not known to us. From Theorem 4.1 one can write:

CONJECTURE 5.1. *The critical value of the coefficients in the inequality*

$$I(s,t) - \nu Q(s,t) - (1-\nu)H(s,t) \leqslant 0 \tag{5.2}$$

will be obtained by conditions of Theorem 4.1. The approximative value is $s^* = 0.207884\cdots$ *and* $\nu^* = 0.559422\cdots$.

The graph with optimal values from Conjecture 5.1 is plotted on Figure 3 on the right.

5.2. The L-Q-G case

The asymptotic inequality

$$6L - Q - 5G \sim \frac{13\beta^4}{60x^3} \succ 0$$

do not implies true inequality of the same sign. The reason is the values of two means: $G(0,1) = 0$ and $L(0,1) = 0$, the inequality

$$L(s,1-s) - \tfrac{1}{6}Q(s,1-s) - \tfrac{5}{6}G(s,1-s) \geqslant 0$$

cannot holds for all $s \in [0,1/2]$. In fact this inequality is true for all $s > 0.00094\cdots$. The graph of the corresponding function $F(s,1-s)$ for the value $\nu = \frac{1}{6}$ is given in the Figure 4 on the left side, where

$$F(s,t) = L(s,t) - \nu Q(s,t) - (1-\nu)G(s,t).$$

But, despite the very small interval for which inequality is not satisfied, it cannot be improved in any way to nondegenerate inequality of the same sign and of the order $\mathscr{O}(x^{-1})$. From row 29 of Table 4 one see that this combination leeds to inequality $L - Q \geqslant 0$.

Let us find now the optimal coefficient of opposite inequality, of order $\mathscr{O}(x^{-1})$. Then

$$F \sim \Big(\frac{1}{6} - \nu\Big)\frac{\beta^2}{6x} + \frac{13\beta^4}{360x^3}$$

For inequality $F \leqslant 0$ to be true, it should be $\nu > \frac{1}{6}$.

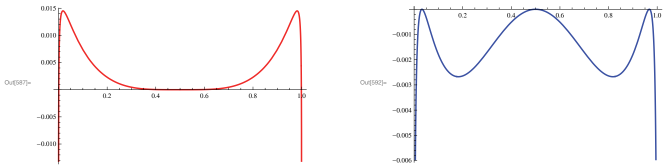

Figure 4. L-Q-G inequality.

CONJECTURE 5.2. *The critical value of the coefficients in the inequality*

$$L(s,t) - \nu Q(s,t) - (1-\nu)G(s,t) \leqslant 0 \tag{5.3}$$

will be obtained by conditions of Theorem 4.1. *The approximative value is* $s^* = 0.0322476\cdots$ *and* $\nu^* = 0.193626\cdots$.

For these values of ν^*, the graph of the inequality looks like on the Figure 4 on the right.

5.3. False inequalities with contraharmonic mean

The contraharmonic mean are included in this analysis because of its connection with other means, it can be expressed in a simple way as

$$N = 2A - H$$

and it leads to several cases of invalid asymptotic inequalities. The asymptotic expansion of this mean, in the case $\alpha = 0$, reduces to the two term:

$$N = x + \frac{\beta^2}{x}.$$

Further, for the value of mean in the boundary point we have $N(0,1) = 1$.

The following three asymptotic inequalities from the Table 2 are false:

$$F_1 := 2N - 3Q + G \succ 0, \tag{5.4}$$

$$F_2 := 5N - 8Q + 3L \succ 0, \tag{5.5}$$

$$F_3 := N - 9L + 8G \prec 0. \tag{5.6}$$

The reason for this is the same, the value in the boundary $s = 0$. We have

$$F_1(0,1) = -\frac{3}{\sqrt{2}} + 2 \approx -0.12132,$$

$$F_2(0,1) = -\frac{8}{\sqrt{2}} + 5 \approx -0.65685,$$

$$F_3(0,1) = 1.$$

Therefore, in order to obtain true inequalities we should change coefficients.

5.4. The Q-N-G case

Instead of coefficients from asymptotic inequality, one should take coefficients from Table 4 to obtain true inequality of the same sign:

THEOREM 5.3. *The inequality*

$$-\sqrt{2}Q(s,t)+(\sqrt{2}-1)G(s,t)+N(s,t)\geqslant 0 \tag{5.7}$$

is satisfied for all $s,t\geqslant 0$. *Inequality is of asymptotic order* $\mathscr{O}(x^{-1})$.

Proof. The following has to be proved:

$$-\sqrt{s^2+t^2}+(\sqrt{2}-1)\sqrt{st}+\frac{s^2+t^2}{s+t}\geqslant 0$$

for all $s,t>0$. Denote $u^2=st$. We can suppose $s+t=1$, and $s\leqslant t$ which leads to

$$(\sqrt{2}-1)u+1-2u^2\geqslant\sqrt{1-2u^2},\qquad 0<u\leqslant\tfrac{1}{2}.$$

The left side is positive, so after squaring this inequality is equivalent to

$$4u^4-4(\sqrt{2}-1)u^3+(1-2\sqrt{2})u^2+2(\sqrt{2}-1)u\geqslant 0,$$

i.e.

$$u(1-2u)(2\sqrt{2}-2-(3-2\sqrt{2})u-2u^2)\geqslant 0$$

which is evidently true for $0<u\leqslant\frac{1}{2}$.

The graphs of the initial inequality, and the one with new coefficients are plotted on the Figure 5, the picture on the left and in the middle.

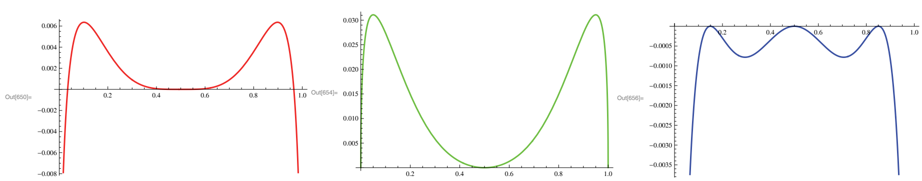

Figure 5. Q-N-G inequality

CONJECTURE 5.4. *The critical value of the coefficients in the inequality*

$$Q(s,t)-\nu N(s,t)-(1-\nu)G(s,t)\leqslant 0 \tag{5.8}$$

will be obtained by conditions of Theorem 4.1. *The approximative value is* $s^*=0.149410\cdots$ *and* $\nu^*=0.652843\cdots$.

The graph with optimal values from Conjecture 5.4 is plotted on Figure 5 on the right.

5.5. The Q-N-L case

This inequality is similar to the previous one, and we can summarise results in the following conjectures

CONJECTURE 5.5. *The inequality*

$$-\sqrt{2}Q(s,t)+(\sqrt{2}-1)L(s,t)+N(s,t)\geqslant 0 \tag{5.9}$$

is sastisfied for all $s,t\geqslant 0$. *The inequality is of asymptotic order* $\mathscr{O}(x^{-1})$.

The graph of initial asymptotic inequality and inequality from Conjecture 5.5 are plotted on the Figure 6, on the left and in the middle.

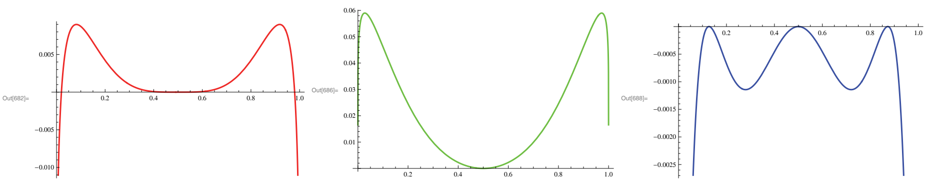

Figure 6. Q-N-L inequality

It should be noticed that the graph in the middle is misleading, the function $F=a^{*}Q+b^{*}L+c^{*}N$ has a value $F(0,1)=F(1,0)=0$, but the approach to zero is so slow that it cannot be plotted with computer program.

CONJECTURE 5.6. *The critical value of the coefficients in the inequality*

$$Q(s,t)-\nu N(s,t)-(1-\nu)L(s,t)\leqslant 0 \tag{5.10}$$

will be obtained by conditions of Theorem 4.1. The approximative value is $s^{*}=0.127164\cdots$ *and* $\nu^{*}=0.605253\cdots$.

The graph with optimal values from Conjecture 5.6 is plotted on Figure 6 on the right.

5.6. The L-N-G case

From the Table 2, row 23 we can write:

$$9L-N-8G\succ 0.$$

The corresponding true inequality is those from Table 4, and it is degenerate: $L-G\geqslant 0$. Therefore, the only interesting case will follow from the similar procedure as before:

CONJECTURE 5.7. *The critical value of the coefficients in the inequality*

$$L(s,t)-\nu N(s,t)-(1-\nu)G(s,t)\leqslant 0 \tag{5.11}$$

will be obtained by conditions of Theorem 4.1. The approximative value is $s^{*}=0.0185560\cdots$ *and* $\nu^{*}=0.129972$.

The graph of the initial asymptotic inequality and those from Conjecture 5.7 is plotted on the Figure 7.

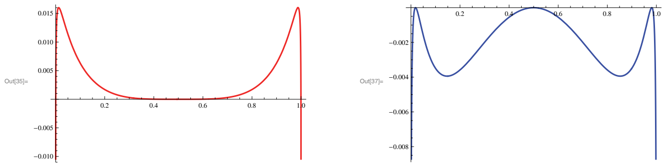

Figure 7. L-N-G inequality.

5.7. Q-N-I case

The asymptotical inequality leads to the true inequality:

$$Q(s,t) - \tfrac{4}{7}N(s,t) - \tfrac{3}{7}I(s,t) \leqslant 0.$$

However the inequality of opposite direction is obtained not in the endpoint, but in an internal point of interval.

CONJECTURE 5.8. *The critical value of the coefficients in the inequality*

$$Q(s,t) - \nu N(s,t) - (1-\nu)I(s,t) \geqslant 0 \tag{5.12}$$

will be obtained by conditions of Theorem 4.1. The approximative value is $s^* = 0.0431259\cdots$ *and* $\nu^* = 0.529081\cdots$.

The graph of the initial asymptotic inequality and those from Conjecture **??** is plotted on the Figure 8.

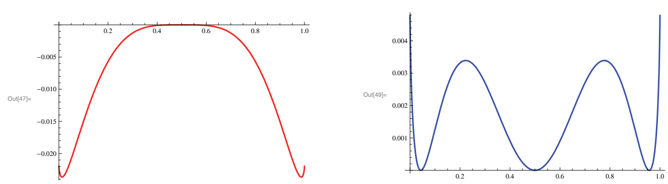

Figure 8. Q-N-I inequality.

6. Four means

By the same technique, taking combinations of four means, one can obtain asymptotic inequalities of order $\mathscr{O}(x^{-5})$. The contraharmonic mean is excluded and complete relations for other six means are given.

	A	Q	G	H	I	L	$\times\beta^6/x^5$	$\geqslant 0$
1	-1	1	-1	1			$1/8$	+
2	-32	1	-14		45		$13/63$	+
3	13	1	31			-45	$37/168$	+
4	18	13		14	-45		$389/252$	+
5	-18	32		31		-45	$86/21$	+
6	-18	1			31	-14	$2389/11340$	+
7	14		32	-1		-45	$2/21$	+
8	-31		-13	-1	45		$41/504$	+
9	1		1		-1	-1	$1/3240$	+
10	-18			-1	32	-13	$242/2835$	+
11		31	-18	32	-45		$239/63$	+
12		14	18	13		-45	$155/84$	+
13		1	18		13	-32	$613/2835$	+
14		1		1	-1	-1	$203/1620$	+
15			18	-1	14	-31	$1031/11340$	+

The last column suggest that all of these asymptotic inequalities are in fact the true ones. The opposite inequalities, which would include behaviour at the edge of interval, cannot be obtained easily, since connections depends on two parameters.

From the table we see three remarkable relations:

$$Q+H \succ A+G,$$
$$Q+H \succ I+L,$$
$$A+G \succ I+L,$$

each of them of order $\mathscr{O}(x^{-5})$. The last is true inequality:

$$A+G \geqslant I+L,$$

which is proved by H. Alzer in [2]. It is interesting that $A+G$ combinations is on different sides of the first and third one, so we have in fact a chain

$$Q+H \geqslant A+G \geqslant I+L. \tag{6.1}$$

THEOREM 6.1. *The following inequality between means is valid:*

$$Q+H \geqslant A+G. \tag{6.2}$$

Proof. We have to prove

$$\sqrt{\frac{s^2+t^2}{2}}+\frac{2st}{s+t}\geqslant\frac{s+t}{2}+\sqrt{st},\qquad\forall s,t>0.$$

By supstitution $s+t=2u$, $st=v^2$, this is equivalent to

$$\sqrt{2u^2-v^2}\geqslant u+v-\frac{v^2}{u}.$$

The right side is always positive, so this is equivalent to

$$u^4-2u^3v+2uv^3-v^4\geqslant 0,$$
$$(u^2-v^2)(u-v)^2\geqslant 0$$

which is true because of $u\geqslant v$.

Added in proofs. I would to thank the anonimous referee for his useful remarks which leads to the improvement of this paper. Among others, he/she pointing out that this result is well known *in the folklore*, with the following interesting proof. First, let us note that $H=G^2/A$ and $2A\geqslant Q+G$ — this one is mentioned in the introduction. Hence $A^2\geqslant(Q+G)^2/4\geqslant QG$ and we have from $Q^2-A^2=A^2-G^2$ and the observation above, the following:

$$Q-A=\frac{(A-G)(A+G)}{Q+A}\geqslant\frac{(A-G)G}{A}=G-H.$$

There is no nice combination of five means, to obtain an asymptotic inequality of order $\mathscr{O}(x^{-7})$. The coefficients are awfull. Here is the one which do not contain identric mean:

$$-16Q+310A-945L+688G-37H\sim\frac{137\beta^8}{60x^7}.$$

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