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NEW GENERALIZED 2D NONLINEAR INEQUALITIES AND APPLICATIONS IN FRACTIONAL DIFFERENTIAL–INTEGRAL EQUATIONS

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Abstract. In this paper, we *study* some new generalized 2D nonlinear Gronwall-Bellman type inequalities, which provide explicit bounds for unknown functions concerned, and are useful in the analysis of qualitative and quantitative properties for solutions to fractional differential and differential-integral equations. The presented inequalities are of new forms compared with the existing results so far in the literature. For illustrating the validity of the results presented, we present one application for them, in which the boundedness, uniqueness, and continuous dependence on the initial value and parameter for the solution to a certain fractional differential-integral equation are investigated.

1. Introduction

As is known, various inequalities play important roles in the research of differential equations, integral equations as well as difference equations. Among these inequalities, the Gronwall-Bellman inequality [1,2] and its various generalizations are of particular importance as these inequalities provide explicit bounds for the unknown functions concerned. During the past decades, much effort has been done for developing such inequalities (for example, see [3–22] and the references therein). These generalizations of the Gronwall-Bellman inequality can be used as a handy tool in the analysis of qualitative and quantitative properties such as boundedness, uniqueness, and continuous dependence on initial data for solutions to certain differential equations, integral equations as well as difference equations. But we notice that most of the Gronwall-Bellman type inequalities established so far can only be used in the research of differential equations of integer order, while in order to fulfill qualitative and quantitative analysis for solutions to some certain differential equations of fractional order, the earlier inequalities established are inadequate. So it is necessary to establish new inequalities so as to obtain the desired analysis.

In [23], Ye et al. presented a new Gronwall-Bellman type inequality in the following theorem:

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THEOREM A. *Suppose $\beta > 0$, $a(t)$ is a nonnegative function locally integrable on $0 \leqslant t < T$ (some $T \leqslant \infty$) and $g(t)$ is a nonnegative, nondecreasing continuous function defined on $0 \leqslant t < T$, $g(t) \leqslant M$ (constant), and suppose $u(t)$ is nonnegative and locally integrable on $0 \leqslant t < T$ with*

$$u(t) \leqslant a(t) + g(t) \int_0^t (t-s)^{\beta-1} u(s) ds$$

on this interval. Then

$$u(t) \leqslant a(t) + \int_0^t \sum_{n=1}^{\infty} \Big[\frac{(g(t)\Gamma(\beta))^n}{\Gamma(n\beta)} (t-s)^{n\beta-1} a(s) \Big] ds, \quad 0 \leqslant t < T.$$

Theorem A has proved to be useful in the research of boundedness and continuous dependence on the order α and the initial condition for solutions to certain fractional differential equations with the fractional derivative defined in the sense of Riemann-Liouville fractional derivative.

The aim of this paper is to establish some new generalized nonlinear 2D Gronwall-Bellman type inequalities, which is the 2D extension of the inequality in Theorem A, and is of more general forms than the inequality above. Based on these inequalities, new explicit bounds for unknown functions concerned are obtained. The presented inequalities can be used as a handy tool in the qualitative as well as quantitative analysis of fractional differential and differential-integral equations. For illustrating the validity of the established results, we will present one application for them, in which the boundedness, uniqueness, and continuous dependence on initial data for the solution to a certain fractional differential-integral equation are investigated.

2. Main results

First we study the following inequality:

$$u^p(x,y) \leqslant a(x,y) + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} h(x,y) \int_0^y \int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1} L(s,t,u(s,t)) ds dt, \quad (x,y) \in D, \tag{1}$$

Similar inequality involving the form $L(s,t,\omega)$ can be found in [4]. Here α, $\beta > 0$, $p \geqslant 1$ is a constant, $D := \{(x,y) | 0 \leqslant x < X,\ 0 \leqslant y < Y\}$, $L \in C(D \times R_+, R_+)$ with $0 \leqslant L(s,t,u) - L(s,t,v) \leqslant T(u-v)$ for $u \geqslant v \geqslant 0$, where T is the Lipschitz constant, $u(x,y)$, $a(x,y)$, $h(x,y)$ are nonnegative functions locally integrable on D with $h(x,y)$ nondecreasing and bounded by M, where M is a positive constant.

Based on the inequality (1), we will *derive an explicit bound* for the function $u(x,y)$.

LEMMA 1. [24] *Assume that $a \geqslant 0$, $p \geqslant q \geqslant 0$, and $p \neq 0$, then for any $K > 0$*

$$a^{\frac{q}{p}} \leqslant \frac{q}{p} K^{\frac{q-p}{p}} a + \frac{p-q}{p} K^{\frac{q}{p}}.$$

THEOREM 2. *If the inequality denoted in (1) satisfies, then we have the following estimate for* $u(x,y)$ *:*

$$u(x,y) \leqslant \Big\{ \widetilde{a}(x,y) + \int_0^y \int_0^x \Big[\sum_{n=1}^{\infty} \Big(\frac{T}{p} K^{\frac{1-p}{p}} \Big)^n \frac{h^n(x,y)}{\Gamma(n\alpha)\Gamma(n\beta)} (x-s)^{n\alpha-1} \times (y-t)^{n\beta-1} \widetilde{a}(s,t) \Big] dsdt \Big\}^{\frac{1}{p}}, \ \ (x,y) \in D, \tag{2}$$

where

$$\widetilde{a}(x,y) = a(x,y) + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} h(x,y) \int_0^y \int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1} L\Big(s,t,\frac{p-1}{p}K^{\frac{1}{p}}\Big) dsdt,$$

and $K > 0$ *is a constant.*

Proof. Denote the right-hand side of (1) by $v(x,y)$. Then we have

$$u(x,y) \leqslant v^{\frac{1}{p}}(x,y), \ (x,y) \in D. \tag{3}$$

So it follows that

$$v(x,y) \leqslant a(x,y) + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} h(x,y) \int_0^y \int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1} L(s,t,v^{\frac{1}{p}}(s,t)) dsdt, \quad (x,y) \in D. \tag{4}$$

By use of Lemma 1 we obtain that

$$\begin{aligned}
v(x,y) &\leqslant a(x,y) + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} h(x,y) \int_0^y \int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1} \\
&\quad \times L\Big(s,t,\frac{1}{p}K^{\frac{1-p}{p}} v(s,t) + \frac{p-1}{p}K^{\frac{1}{p}}\Big) dsdt \\
&= a(x,y) + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} h(x,y) \int_0^y \int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1} \\
&\quad \times \Big[L\Big(s,t,\frac{1}{p}K^{\frac{1-p}{p}} v(s,t) + \frac{p-1}{p}K^{\frac{1}{p}}\Big) - L\Big(s,t,\frac{p-1}{p}K^{\frac{1}{p}}\Big) \\
&\quad + L\Big(s,t,\frac{p-1}{p}K^{\frac{1}{p}}\Big)\Big] dsdt \\
&\leqslant a(x,y) + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} h(x,y) \int_0^y \int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1} \Big[\frac{T}{p} K^{\frac{1-p}{p}} v(s,t) \\
&\quad + L\Big(s,t,\frac{p-1}{p}K^{\frac{1}{p}}\Big)\Big] dsdt \\
&= a(x,y) + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} h(x,y) \int_0^y \int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1} L\Big(s,t,\frac{p-1}{p}K^{\frac{1}{p}}\Big) dsdt \\
&\quad + \frac{T}{p}K^{\frac{1-p}{p}} \frac{1}{\Gamma(\alpha)\Gamma(\beta)} h(x,y) \int_0^y \int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1} v(s,t) dsdt \\
&= \widetilde{a}(x,y) + \frac{T}{p}K^{\frac{1-p}{p}} \frac{1}{\Gamma(\alpha)\Gamma(\beta)} h(x,y) \int_0^y \int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1} v(s,t) dsdt.
\end{aligned} \tag{5}$$

Define the operator G such that

$$Gz(x,y)=\frac{T}{p}K^{\frac{1-p}{p}}\frac{1}{\Gamma(\alpha)\Gamma(\beta)}h(x,y)\int_0^y\int_0^x(x-s)^{\alpha-1}(y-t)^{\beta-1}z(s,t)dsdt,$$

where z is locally integrable on D. Then from (5) we have

$$v(x,y)\leqslant\widetilde{a}(x,y)+Gv(x,y).$$

Furthermore,

$$v(x,y)\leqslant\sum_{k=0}^{n-1}G^k\widetilde{a}(x,y)+G^nv(x,y). \tag{6}$$

Next we will show that the following relation holds:

$$G^n\widetilde{a}(x,y)\leqslant\left(\frac{T}{p}K^{\frac{1-p}{p}}\right)^n\frac{1}{\Gamma(n\alpha)\Gamma(n\beta)}\int_0^y\int_0^x h^n(x,y)(x-s)^{n\alpha-1}(y-t)^{n\beta-1}\widetilde{a}(s,t)dsdt. \tag{7}$$

In order to prove the inequality above, we will use the mathematical induction method.

When $n=1$, (7) holds in equality. Assume that (7) holds for $n=k$. Then for $n=k+1$, we have

$$\begin{aligned}
&G^{k+1}\widetilde{a}(x,y)=G(G^k\widetilde{a}(x,y))\\
&\leqslant\left(\frac{T}{p}K^{\frac{1-p}{p}}\right)^{k+1}\frac{1}{\Gamma(k\alpha)\Gamma(k\beta)\Gamma(\alpha)\Gamma(\beta)}h(x,y)\int_0^y\int_0^x(x-s)^{\alpha-1}(y-t)^{\beta-1}\\
&\quad\left[\int_0^t\int_0^s(h(s,t))^k(s-\tau)^{k\alpha-1}(t-\xi)^{k\beta-1}\widetilde{a}(\tau,\xi)d\tau d\xi\right]dsdt\\
&\leqslant\left(\frac{T}{p}K^{\frac{1-p}{p}}\right)^{k+1}\frac{1}{\Gamma(k\alpha)\Gamma(k\beta)\Gamma(\alpha)\Gamma(\beta)}h^{k+1}(x,y)\int_0^y\int_0^x(x-s)^{\alpha-1}(y-t)^{\beta-1}\\
&\quad\left[\int_0^t\int_0^s(s-\tau)^{k\alpha-1}(t-\xi)^{k\beta-1}\widetilde{a}(\tau,\xi)d\tau d\xi\right]dsdt\\
&=\left(\frac{T}{p}K^{\frac{1-p}{p}}\right)^{k+1}\frac{1}{\Gamma(k\alpha)\Gamma(k\beta)\Gamma(\alpha)\Gamma(\beta)}h^{k+1}(x,y)\\
&\quad\int_0^y\int_0^x\int_0^t\int_0^s(x-s)^{\alpha-1}(y-t)^{\beta-1}(s-\tau)^{k\alpha-1}(t-\xi)^{k\beta-1}\widetilde{a}(\tau,\xi)d\tau d\xi dsdt\\
&=\left(\frac{T}{p}K^{\frac{1-p}{p}}\right)^{k+1}\frac{1}{\Gamma(k\alpha)\Gamma(k\beta)\Gamma(\alpha)\Gamma(\beta)}h^{k+1}(x,y)\\
&\quad\int_0^y\int_0^x\int_\xi^y\int_\tau^x(x-s)^{\alpha-1}(y-t)^{\beta-1}(s-\tau)^{k\alpha-1}(t-\xi)^{k\beta-1}\widetilde{a}(\tau,\xi)dsdtd\tau d\xi\\
&=\left(\frac{T}{p}K^{\frac{1-p}{p}}\right)^{k+1}\frac{1}{\Gamma(k\alpha)\Gamma(k\beta)\Gamma(\alpha)\Gamma(\beta)}h^{k+1}(x,y)\\
&\quad\int_0^y\int_0^x\left[\int_\xi^y(y-t)^{\beta-1}(t-\xi)^{k\beta-1}dt\right]\left[\int_\tau^x(x-s)^{\alpha-1}(s-\tau)^{k\alpha-1}ds\right]\widetilde{a}(\tau,\xi)d\tau d\xi.
\end{aligned} \tag{8}$$

On the other hand, letting $s=\tau+\rho(x-\tau)$, we obtain that

$$\begin{aligned}\int_{\tau}^{x}(x-s)^{\alpha-1}(s-\tau)^{k\alpha-1}ds &= (x-\tau)^{(k+1)\alpha-1}\int_0^1(1-\rho)^{\alpha-1}\rho^{k\alpha-1}d\rho\\ &= (x-\tau)^{(k+1)\alpha-1}B(k\alpha,\alpha)\\ &= (x-\tau)^{(k+1)\alpha-1}\frac{\Gamma(\alpha)\Gamma(k\alpha)}{\Gamma((k+1)\alpha)},\end{aligned} \tag{9}$$

where $B(k\alpha,\alpha)$ denotes the beta function. Similarly, for the integral $\int_{\xi}^{y}(y-t)^{\beta-1}(t-\xi)^{k\beta-1}dt$, if we let $t=\xi+\zeta(y-\xi)$, then we obtain that

$$\begin{aligned}\int_{\xi}^{y}(y-t)^{\beta-1}(t-\xi)^{k\beta-1}dt &= (y-\xi)^{(k+1)\beta-1}\int_0^1(1-\zeta)^{\beta-1}\zeta^{k\beta-1}d\zeta\\ &= (y-\xi)^{(k+1)\beta-1}B(k\beta,\beta)\\ &= (y-\xi)^{(k+1)\beta-1}\frac{\Gamma(\beta)\Gamma(k\beta)}{\Gamma((k+1)\beta)}.\end{aligned} \tag{10}$$

Combining (8)–(10) we deduce that (7) holds for $n=k+1$. So (7) is proved. Moreover, as $h(x,y)\leqslant M$, then

$$G^n v(x,y)\leqslant\Big(\frac{MT}{p}K^{\frac{1-p}{p}}\Big)^n\frac{1}{\Gamma(n\alpha)\Gamma(n\beta)}\int_0^y\int_0^x(x-s)^{n\alpha-1}(y-t)^{n\beta-1}v(s,t)dsdt.$$

Since when $n\to\infty$, $\Gamma(n\alpha)\Gamma(n\beta)$ tends to infinity faster than $(\frac{MT}{p}K^{\frac{1-p}{p}})^n(x-s)^{n\alpha-1}(y-t)^{n\beta-1}$, then one can see $\lim_{n\to\infty}G^n v(x,y)=0$. So combining with (6)–(7) we have proved the desired inequality (2). $\square$

COROLLARY 3. *Under the conditions of Theorem 2, furthermore, assume $a(x,y)$ is nondecreasing. Then we have the following estimate:*

$$u(x,y)\leqslant\{\widetilde{a}(x,y)\sum_{n=0}^{\infty}(\frac{T}{p}K^{\frac{1-p}{p}})^n\frac{(h(x,y)x^{\alpha}y^{\beta})^n}{\Gamma(n\alpha+1)\Gamma(n\beta+1)}\}^{\frac{1}{p}},\ (x,y)\in D.$$

Proof. From (2) we obtain

$$\begin{aligned}&u(x,y)\\ &\leqslant\Big\{\widetilde{a}(x,y)+\int_0^y\int_0^x\Big[\sum_{n=1}^{\infty}\Big(\frac{T}{p}K^{\frac{1-p}{p}}\Big)^n\frac{h^n(x,y)}{\Gamma(n\alpha)\Gamma(n\beta)}(x-s)^{n\alpha-1}(y-t)^{n\beta-1}\widetilde{a}(s,t)\Big]dsdt\Big\}^{\frac{1}{p}}\\ &\leqslant\widetilde{a}^{\frac{1}{p}}(x,y)\Big\{1+\int_0^y\int_0^x\Big[\sum_{n=1}^{\infty}\Big(\frac{T}{p}K^{\frac{1-p}{p}}\Big)^n\frac{h^n(x,y)}{\Gamma(n\alpha)\Gamma(n\beta)}(x-s)^{n\alpha-1}(y-t)^{n\beta-1}\Big]dsdt\Big\}^{\frac{1}{p}}\\ &=\Big\{\widetilde{a}(x,y)\sum_{n=0}^{\infty}\Big(\frac{T}{p}K^{\frac{1-p}{p}}\Big)^n\frac{(h(x,y)x^{\alpha}y^{\beta})^n}{\Gamma(n\alpha+1)\Gamma(n\beta+1)}\Big\}^{\frac{1}{p}}.\end{aligned}$$

So the proof is complete. $\square$

Now we study the following inequality with more general form than (1):

$$u^p(x,y) \leqslant a(x,y) + \int_0^y \int_0^x b(s,t)u^q(s,t)dsdt + \frac{1}{\Gamma(\alpha)\Gamma(\beta)}h(x,y)\int_0^y \int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1}L(s,t,u(s,t))dsdt, \quad (x,y) \in D, \tag{11}$$

where D, α, β, $u(x,y)$, $a(x,y)$, $h(x,y)$ are defined as in Theorem 2, $b(x,y)$ is a nonnegative function locally integrable on D, and p, q are constants with $p \geqslant q \geqslant 1$.

LEMMA 4. (see [25, Lemma 1] with $\mathbb{T} = \mathbb{R}$) *Suppose $u(x,y)$, $a(x,y)$, $b(x,y)$ are continuous functions with $b(x,y) \geqslant 0$. Then*

$$u(x,y) \leqslant a(x,y) + \int_{y_0}^y \int_{x_0}^x b(s,t)u(s,t)dsdt$$

implies

$$u(x,y) \leqslant a(x,y) + \int_{y_0}^y \int_{x_0}^x a(s,t)b(s,t)\exp\Big(\int_t^y \int_{x_0}^x b(\tau,\xi)d\tau d\xi\Big)dsdt.$$

Furthermore, if $a(x,y)$ is nondecreasing, then we have

$$u(x,y) \leqslant a(x,y)\exp\Big(\int_{y_0}^y \int_{x_0}^x b(s,t)dsdt\Big).$$

THEOREM 5. *If $a(x,y)$ is nondecreasing, and the inequality (11) satisfies, then we have*

$$u(x,y) \leqslant \exp\Big(\frac{q}{p^2}K^{\frac{q-p}{p}}\int_0^y \int_0^x b(s,t)dsdt\Big)\Big\{\widehat{a}(x,y) + \int_0^y \int_0^x \Big[\sum_{n=1}^{\infty}\Big(\frac{T}{p}K^{\frac{1-p}{p}}\Big)^n \times \frac{\widehat{h}^n(x,y)}{\Gamma(n\alpha)\Gamma(n\beta)}(x-s)^{n\alpha-1}(y-t)^{n\beta-1}\widehat{a}(s,t)\Big]dsdt\Big\}^{\frac{1}{p}}, \quad (x,y) \in D, \tag{12}$$

where

$$\widehat{a}(x,y) = a(x,y) + \frac{p-q}{p}K^{\frac{q}{p}}\int_0^y \int_0^x b(s,t)dsdt + \frac{1}{\Gamma(\alpha)\Gamma(\beta)}h(x,y)\int_0^y \int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1}L\Big(s,t,\frac{p-1}{p}K^{\frac{1}{p}}\Big)dsdt,$$

$$\widehat{h}(x,y) = \exp\Big(\frac{q}{p}K^{\frac{q-p}{p}}\int_0^y \int_0^x b(s,t)dsdt\Big)h(x,y).$$

Proof. Denote the right-hand side of (1) by $v(x,y)$. Then we have

$$u(x,y) \leqslant v^{\frac{1}{p}}(x,y), \ (x,y) \in D. \tag{13}$$

So by Lemma 1 it follows that

$$\begin{aligned}
v(x,y) \leqslant\, & a(x,y) + \int_0^y \int_0^x b(s,t) v^{\frac{q}{p}}(s,t) dsdt \\
& + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} h(x,y) \int_0^y \int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1} L\Big(s,t,v^{\frac{1}{p}}(s,t)\Big) dsdt \\
\leqslant\, & a(x,y) + \int_0^y \int_0^x b(s,t) \Big[\frac{q}{p} K^{\frac{q-p}{p}} v(s,t) + \frac{p-q}{p} K^{\frac{q}{p}}\Big] dsdt \\
& + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} h(x,y) \int_0^y \int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1} \\
& \times \Big[L\Big(s,t,\frac{1}{p} K^{\frac{1-p}{p}} v(s,t) + \frac{p-1}{p} K^{\frac{1}{p}}\Big) \\
& - L\Big(s,t,\frac{p-1}{p} K^{\frac{1}{p}}\Big) + L\Big(s,t,\frac{p-1}{p} K^{\frac{1}{p}}\Big)\Big] dsdt \\
\leqslant\, & a(x,y) + \int_0^y \int_0^x b(s,t) \Big[\frac{q}{p} K^{\frac{q-p}{p}} v(s,t) + \frac{p-q}{p} K^{\frac{q}{p}}\Big] dsdt \\
& + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} h(x,y) \int_0^y \int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1} \Big[\frac{T}{p} K^{\frac{1-p}{p}} v(s,t) \\
& + L\Big(s,t,\frac{p-1}{p} K^{\frac{1}{p}}\Big)\Big] dsdt, \ (x,y) \in D.
\end{aligned} \tag{14}$$

Let

$$\begin{aligned}
z(x,y) =\, & a(x,y) + \frac{p-q}{p} K^{\frac{q}{p}} \int_0^y \int_0^x b(s,t) dsdt \\
& + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} h(x,y) \int_0^y \int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1} L\Big(s,t,\frac{p-1}{p} K^{\frac{1}{p}}\Big) dsdt \\
& + \frac{T}{p} K^{\frac{1-p}{p}} \frac{1}{\Gamma(\alpha)\Gamma(\beta)} h(x,y) \int_0^y \int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1} v(s,t) dsdt.
\end{aligned}$$

Then we have

$$v(x,y) \leqslant z(x,y) + \frac{q}{p} K^{\frac{q-p}{p}} \int_0^y \int_0^x b(s,t) v(s,t) dsdt, \ (x,y) \in D. \tag{15}$$

By $z(x,y)$ is nondecreasing, applying Lemma 4 to (15) we get that

$$v(x,y) \leqslant z(x,y) \exp\Big(\frac{q}{p} K^{\frac{q-p}{p}} \int_0^y \int_0^x b(s,t) dsdt\Big), \ (x,y) \in D. \tag{16}$$

Moreover,

$$\begin{aligned}
z(x,y) \leqslant{}& a(x,y) + \frac{p-q}{p}K^{\frac{q}{p}}\int_0^y\int_0^x b(s,t)dsdt \\
&+ \frac{1}{\Gamma(\alpha)\Gamma(\beta)}h(x,y)\int_0^y\int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1}L\Big(s,t,\frac{p-1}{p}K^{\frac{1}{p}}\Big)dsdt \\
&+ \frac{T}{p}K^{\frac{1-p}{p}}\frac{1}{\Gamma(\alpha)\Gamma(\beta)}h(x,y)\int_0^y\int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1} \\
&\times\Big[z(s,t)\exp\Big(\frac{q}{p}K^{\frac{q-p}{p}}\int_0^t\int_0^s b(\tau,\xi)d\tau d\xi\Big)\Big]dsdt \\
\leqslant{}& a(x,y) + \frac{p-q}{p}K^{\frac{q}{p}}\int_0^y\int_0^x b(s,t)dsdt \\
&+ \frac{1}{\Gamma(\alpha)\Gamma(\beta)}h(x,y)\int_0^y\int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1}L\Big(s,t,\frac{p-1}{p}K^{\frac{1}{p}}\Big)dsdt \\
&+ \frac{T}{p}K^{\frac{1-p}{p}}\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\exp\Big(\frac{q}{p}K^{\frac{q-p}{p}}\int_0^y\int_0^x b(s,t)dsdt\Big) \\
&\times h(x,y)\int_0^y\int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1}z(s,t)dsdt \\
={}& \widehat{a}(x,y) + \frac{T}{p}K^{\frac{1-p}{p}}\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\widehat{h}(x,y)\int_0^y\int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1}z(s,t)dsdt, \\
&(x,y)\in D. \qquad (17)
\end{aligned}$$

We notice that the structure of (17) is the same as (5). So following in a similar manner to the proof in Theorem 2 we get that

$$z(x,y) \leqslant \widehat{a}(x,y)+\int_0^y\int_0^x \Big[\sum_{n=1}^{\infty}\Big(\frac{T}{p}K^{\frac{1-p}{p}}\Big)^n\frac{\widehat{h}^n(x,y)}{\Gamma(n\alpha)\Gamma(n\beta)}(x-s)^{n\alpha-1}(y-t)^{n\beta-1}\widehat{a}(s,t)\Big]dsdt, \quad (x,y)\in D. \qquad (18)$$

Combining (13), (16) and (18) we get the desired result. □

COROLLARY 6. *For Theorem 5, similar to the proof of Corollary 3, we can obtain the following estimate for* $u(x,y)$*:*

$$u(x,y)\leqslant \exp\Big(\frac{q}{p^2}K^{\frac{q-p}{p}}\int_0^y\int_0^x b(s,t)dsdt\Big)\Big\{\widehat{a}(x,y)\sum_{n=0}^{\infty}\Big(\frac{T}{p}K^{\frac{1-p}{p}}\Big)^n\frac{(\widehat{h}(x,y)x^{\alpha}y^{\beta})^n}{\Gamma(n\alpha+1)\Gamma(n\beta+1)}\Big\}^{\frac{1}{p}}, \quad (x,y)\in D.$$

3. Applications

In this section, we apply the inequalities established above to research boundedness, uniqueness, and continuous dependence on the initial value and parameter for the solution to a fractional differential-integral equation. Let us consider the following fractional differential-integral equation:

$$D_y^\beta u^3(x,y) = K_1 + J_x^\alpha f(x,y,u(x,y)),\ (x,y) \in D, \tag{19}$$

with the initial condition

$$D_y^{\beta-1} u^3(x,y)|_{y=0} = K_2, \tag{20}$$

where $0 < \alpha,\ \beta < 1,\ J_x^\alpha$ denotes the Riemann-Liouville fractional partial integral with respect to the variable x defined by $J_x^\alpha v(x,y) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} v(s,y)ds,\ D_y^\beta$ denotes the Riemann-Liouville fractional partial derivative with respect to the variable y defined by $D_y^\beta v(x,y) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dy} \int_0^y (y-t)^{-\beta} v(x,t)dt,\ f \in C(D\times R, R)$ with $|f(x,y,u)| \leqslant L(x,y,|u|)$, where $D,\ L$ are defined as in Theorem 2.

THEOREM 7. *For the IVP (19)–(20), we have the following estimate:*

$$u(x,y) \leqslant \sqrt[3]{W(x,y) + \int_0^y \int_0^x \Big[\sum_{n=1}^\infty \Big(\frac{T}{3} K^{-\frac{2}{3}}\Big)^n \frac{1}{\Gamma(n\alpha)\Gamma(n\beta)} (x-s)^{n\alpha-1}(y-t)^{n\beta-1} W(s,t)\Big] dsdt}, \tag{21}$$

where $W(x,y) = |\frac{K_2}{\Gamma(\beta)} y^{\beta-1} + \frac{K_1}{\Gamma(\beta+1)} y^\beta|$, $K > 0$ *is a constsnt, and* T *is defined as in Theorem 2.*

Proof. The equivalent integral form of the IVP (19)–(20) can be denoted as follows:

$$\begin{aligned} u^3(x,y) = {} & \frac{K_2}{\Gamma(\beta)} y^{\beta-1} + \frac{K_1}{\Gamma(\beta+1)} y^\beta \\ & + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^y \int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1} f(s,t,u(s,t)) dsdt. \end{aligned}$$

So

$$\begin{aligned} & |u(x,y)|^3 \\ & \leqslant \Big|\frac{K_2}{\Gamma(\beta)} y^{\beta-1} + \frac{K_1}{\Gamma(\beta+1)} y^\beta\Big| + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^y \int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1} |f(s,t,u(s,t))| dsdt \\ & \leqslant \Big|\frac{K_2}{\Gamma(\beta)} y^{\beta-1} + \frac{K_1}{\Gamma(\beta+1)} y^\beta\Big| + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^y \int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1} L(s,t,|u(s,t)|) dsdt \\ & = W(x,y) + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^y \int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1} L(s,t,|u(s,t)|) dsdt,\ (x,y) \in D. \end{aligned} \tag{22}$$

Then a suitable application of Theorem 2 (with $p = 3$) to (22) yields the desired result. $\square$

THEOREM 8. *If* $|f(x,y,u)-f(x,y,v)| \leqslant L(x,y,|u^3-v^3|)$*, where* L *is defined as in Theorem 2, and* $L(s,t,0)\equiv 0$*, then the IVP (19)–(20) has a unique solution.*

Proof. Suppose the IVP (19)–(20) has two solutions $u_1(x,y)$, $u_2(x,y)$. Then we have

$$\begin{aligned} u_1(x,y) &= \frac{K_2}{\Gamma(\beta)}y^{\beta-1}+\frac{K_1}{\Gamma(\beta+1)}y^{\beta} \\ &\quad +\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^y\int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1}f(s,t,u_1(s,t))dsdt, \end{aligned} \tag{23}$$

$$\begin{aligned} u_2(x,y) &= \frac{K_2}{\Gamma(\beta)}y^{\beta-1}+\frac{K_1}{\Gamma(\beta+1)}y^{\beta} \\ &\quad +\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^y\int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1}f(s,t,u_2(s,t))dsdt. \end{aligned} \tag{24}$$

Furthermore,

$$\begin{aligned} &u_1^3(x,y)-u_2^3(x,y) \\ &=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^y\int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1}[f(s,t,u_1(s,t))-f(s,t,u_2(s,t))]dsdt, \end{aligned} \tag{25}$$

which implies

$$\begin{aligned} &|u_1^3(x,y)-u_2^3(x,y)| \\ &\leqslant \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^y\int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1}|f(s,t,u_1(s,t))-f(s,t,u_2(s,t))|dsdt \\ &\leqslant \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^y\int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1}L(s,t,|u_1^3(s,t)-u_2^3(s,t)|)dsdt. \end{aligned} \tag{26}$$

Treating $|u_1^3(x,y)-u_2^3(x,y)|$ as one independent function, applying Theorem 2 to (26) we obtain $|u_1^3(x,y)-u_2^3(x,y)|\leqslant 0$, which implies $u_1(x,y)\equiv u_2(x,y)$. So the proof is complete. □

Now we research the continuous dependence on the initial value and parameter for the solution of the IVP (19)–(20).

THEOREM 9. *Let* $u(x,y)$ *be the solution of the IVP (19)–(20), and* $\overline{u}(x,y)$ *be the solution of the following IVP:*

$$\begin{cases} D_y^{\beta}\overline{u}^3(x,y)=\overline{K}_1+J_x^{\alpha}f(x,y,\overline{u}(x,y)), \\ D_y^{\beta-1}\overline{u}^3(x,y)|_{y=0}=\overline{K_2}. \end{cases} \tag{27}$$

If $|K_i-\overline{K_i}|<\varepsilon$, $i=1,2$*, where* ε *is arbitrarily small, and* $|f(x,y,u)-f(x,y,v)|\leqslant L(x,y,|u^3-v^3|)$*, where* L *is defined as in Theorem 2, and* $L(s,t,0)\equiv 0$*, then we have:*

$$\begin{aligned} |u^3(x,y)-\overline{u}^3(x,y)| \leqslant \varepsilon\Big\{\frac{y^{\beta-1}}{\Gamma(\beta)}+\frac{y^{\beta}}{\Gamma(\beta+1)}+\sum_{n=1}^{\infty}\Big\{\frac{1}{\Gamma(n\alpha)\Gamma(n\beta)}\frac{x^{n\alpha}}{n\alpha} \\ \times\Big[\frac{y^{(n+1)\beta}B(\beta+1,n\beta)}{\Gamma(\beta+1)}+\frac{y^{(n+1)\beta-1}B(\beta,n\beta)}{\Gamma(\beta)}\Big]\Big\}\Big\}, \end{aligned} \tag{28}$$

where $B(\alpha,\beta)$ denotes the beta function.

Proof. The equivalent integral form of the IVP (27) is denoted as follows:

$$\begin{aligned}\overline{u}^3(x,y) = &\frac{\overline{K}_2}{\Gamma(\beta)}y^{\beta-1} + \frac{\overline{K}_1}{\Gamma(\beta+1)}y^{\beta} \\ &+ \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^y\int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1}f(s,t,\overline{u}(s,t))dsdt. \qquad (29)\end{aligned}$$

So we have

$$\begin{aligned}u^3(x,y)-\overline{u}^3(x,y) = &\frac{K-\overline{K}}{\Gamma(\beta)}y^{\beta-1} + \frac{1}{\Gamma(\beta+1)}y^{\beta}(a(x)-\overline{a}(x)) \\ &+ \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^y\int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1} \\ &\times [f(s,t,u(s,t))-f(s,t,\overline{u}(s,t))]dsdt. \qquad (30)\end{aligned}$$

Furthermore,

$$\begin{aligned}|u^3(x,y)-\overline{u}^3(x,y)| \leqslant &\frac{|K_2-\overline{K}_2|}{\Gamma(\beta)}y^{\beta-1} + \frac{|K_1-\overline{K}_1|}{\Gamma(\beta+1)}y^{\beta} + \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^y\int_0^x (x-s)^{\alpha-1} \\ &\times (y-t)^{\beta-1}|f(s,t,u(s,t))-f(s,t,\overline{u}(s,t))|dsdt, \\ \leqslant &\,\varepsilon\Big(\frac{y^{\beta-1}}{\Gamma(\beta)} + \frac{y^{\beta}}{\Gamma(\beta+1)}\Big) + \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^y\int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1} \\ &\times L(s,t,|u_1^3(s,t)-u_2^3(s,t)|)dsdt. \qquad (31)\end{aligned}$$

Applying Theorem 2 to (31), after some basic computation we can get the desired result. □

4. Conclusions

In this paper, we have established some new generalized 2D nonlinear Gronwall-Bellman type inequalities. As one can see from the present example, the results established are useful in researching the qualitative as well as quantitative properties such as the boundedness, uniqueness, and continuous dependence on the initial value and parameter for solutions to certain fractional differential and differential-integral equations. Moreover, we note that in order to fulfill analysis for the solutions to fractional differential-integral equations with more complicated forms, it is necessary to establish corresponding Gronwall-Bellman type inequalities with more general forms.

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