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GENERALIZATIONS OF HÖLDER'S INEQUALITIES ON TIME SCALES

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Abstract. Hölder's inequalities and their extensions have received considerable attention in the theory of differential and difference equations. In this paper, we establish some new generalizations and refinements of Hölder's inequality and some related inequalities on time scales. We also show that many existing inequalities related to the Hölder's inequality are special cases of the inequalities presented on time scales.

1. Introduction

Both the Hölder inequality and the Cauchy inequality play an important role in many areas of mathematics. Many of authors studied and obtained the generalizations, refinements, variations and applications of these inequalities in the literature (see [5], [9], [11], [12] and references therein).

In the book by Beckenbach and Bellman [3], the following well known Hölder inequality was studied. Let m,n be positive integers, and let $a_{ij} > 0$ $(1 \leqslant i \leqslant n)$, $(1 \leqslant j \leqslant m)$, $p_j > 0$ and $\sum\limits_{j=1}^{m} \frac{1}{p_j} = 1$. Then the inequality

$$\sum_{i=1}^{n} \prod_{j=1}^{m} a_{ij} \leqslant \prod_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ij}^{p_j} \right)^{\frac{1}{p_j}} \tag{1}$$

holds. Similarly, the integral form of the Hölder inequality is

$$\int_a^b \left(\prod_{j=1}^{m} f_j(x) \right) dx \leqslant \prod_{j=1}^{m} \left(\int_a^b f_j^{p_j}(x)\, dx \right)^{\frac{1}{p_j}} \tag{2}$$

where $f_j(x) > 0$ $(j = 1,2,\ldots,m)$, $x \in [a,b]$, $-\infty < a < b < +\infty$, $p_j > 0$, $\sum\limits_{j=1}^{m} \frac{1}{p_j} = 1$ and $f_j \in L^{p_j}[a,b]$.

If $m = 2$ and $p_1 = p_2 = 2$, then the inequalities (1) and (2) reduce to the well known Cauchy inequalities [8] of the discrete and the continuous versions, respectively.

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In the paper by Agarwall, Bohner and Peterson [2], a time scale version of the Cauchy and Hölder inequalities was derived. Let $a,\ b\in\mathbb{T}$. For rd-continuous functions $f,\ g:[a,b]\to\mathbb{R}$, we have

$$\int_a^b |f(t)\,g(t)|\,\Delta t \leqslant \left\{\int_a^b |f(t)|^p\,\Delta t\right\}^{\frac{1}{p}} \left\{\int_a^b |g(t)|^q\,\Delta t\right\}^{\frac{1}{q}}, \tag{3}$$

where $p>1$ and $q=\frac{p}{p-1}$.

In this paper we obtain some new generalizations and refinements of the Hölder inequality and some related inequalities. We conclude by presenting a generalization of the Hölder inequality for negative exponents.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, and they give rise to many applications. The cases when a time scale $\mathbb{T}=\left\{1,q,q^2,\ldots\right\}$ represents q-difference equations which have important applications in quantum theory.

On a time scale, the forward jump operator, the backward jump operator and the graininess function are defined by

$$\sigma(t):=\inf\{s\in\mathbb{T}:s>t\},\quad \rho(t):=\sup\{s\in\mathbb{T}:s<t\} \text{ and } \mu(t):=\sigma(t)-t$$

respectively. If $\sigma(t)>t$, we say that t is *right-scattered*, while if $\rho(t)<t$ we say that t is *left-scattered*. Points that are right-scattered and left-scattered at the same time are called isolated. Also, if $t<\sup\mathbb{T}$ and $\sigma(t)=t$, then t is called right-dense, and if $t>\inf\mathbb{T}$ and $\rho(t)=t$, then t is called left-dense. Points that are right-dense and left-dense at the same time are called dense.

A function $f:\ \mathbb{T}\to\mathbb{R}$ is called *rd-continuous* provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$.

Let $a,\ b\in\mathbb{T}$ and $f\in C_{rd}$.

i) If $\mathbb{T}=\mathbb{R}$, then

$$\int_a^b f(t)\,\Delta t=\int_a^b f(t)\,dt,$$

where the integral on the right is the usual Riemann integral from calculus.

ii) If $[a,b]$ consist of only isolated points, then

$$\int_a^b f(t)\,\Delta t=\begin{cases} \sum\limits_{t\in[a,b]}\mu(t)\,f(t) &, \quad a<b \\ 0 &, \quad a=b. \\ -\sum\limits_{t\in[a,b]}\mu(t)\,f(t) &, \quad a>b \end{cases}$$

We refer the reader to [4] for further results on time scale calculus.

The intervals with a $\mathbb{T}$ subscript are used to denote the intersection of the usual interval with $\mathbb{T}$; i.e., $[t_0,\infty)_{\mathbb{T}}:=[t_0,\infty)\cap\mathbb{T}$ for convenience.

2. Main results

We start with the following theorem:

THEOREM 1. *Let* $f_j > 0$ $(j = 1,2,...,m)$, $x \in [a,b]_{\mathbb{T}}$, *and* $f_j \in C_{rd}[a,b]$, $p_k > 0$, $\alpha_{kj} \in \mathbb{R}$ $(k = 1,2,...,s)$, $\sum\limits_{k=1}^{s} \frac{1}{p_k} = 1$ *and* $\sum\limits_{k=1}^{s} \alpha_{kj} = 0$, *then*

$$\int_a^b \left(\prod_{j=1}^{m} f_j(x) \right) \Delta x \leqslant \prod_{k=1}^{s} \left(\int_a^b \prod_{j=1}^{m} f_j^{1+p_k\alpha_{kj}}(x)\, \Delta x \right)^{\frac{1}{p_k}}. \tag{4}$$

Proof. For the sake of convenience, we denote positive functions by

$$g(x) = \left(\prod_{j=1}^{m} f_j^{1+p_k\alpha_{kj}}(x) \right)^{\frac{1}{p_k}}.$$

By using the assumptions of the theorem and computation, we have

$$\int_a^b \left(\prod_{j=1}^{m} f_j(x) \right) \Delta x = \int_a^b \left(\prod_{k=1}^{s} (g_k(x)) \right) \Delta x. \tag{5}$$

Then by using the Hölder's inequality (3), we obtain

$$\int_a^b \left(\prod_{k=1}^{s} (g_k(x)) \right) \Delta x \leqslant \prod_{k=1}^{s} \left(\int_a^b \left(g_k^{p_k}(x) \right) \Delta x \right)^{\frac{1}{p_k}}. \tag{6}$$

Substituting $g_k(x)$ and using (5) in Eq. (6) leads to the inequality (4) without any difficulty. $\square$

REMARK 1. If we choose $s = m$, $\alpha_{kj} = -\frac{1}{p_k}$ for $j \neq k$ and $\alpha_{kk} = 1 - \frac{1}{p_k}$, then the inequality (4) reduces to the Hölder inequality (1) and (2) for the case $\mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = \mathbb{R}$, respectively. By using the inequality (4), we have

$$\begin{aligned}
\int_a^b \left(\prod_{j=1}^{m} f_j(x) \right) \Delta x &\leqslant \prod_{k=1}^{m} \left(\int_a^b \prod_{j=1}^{m} f_j^{1+p_k\alpha_{kj}}(x)\, \Delta x \right)^{\frac{1}{p_k}} \\
&\leqslant \prod_{k=1}^{m} \left(\int_a^b \prod_{j=1}^{m} f_j^{1+p_k\left(-\frac{1}{p_k}\right)}(x)\, f_k^{1+p_k\left(1-\frac{1}{p_k}\right)}(x)\, \Delta x \right)^{\frac{1}{p_k}} \\
&\leqslant \prod_{k=1}^{m} \left(\int_a^b f_k^{p_k}(x)\, \Delta x \right)^{\frac{1}{p_k}}.
\end{aligned}$$

COROLLARY 1. *Under the assumptions of Theorem 1. If we choose* $s=m$, $\alpha_{kj}=-\frac{1}{p_k}$ *for* $j\neq k$ *and* $\alpha_{kk}=t\left(1-\frac{1}{p_k}\right)$ *with* $t\in\mathbb{T}$, *then we have*

$$\int_a^b\left(\prod_{j=1}^m f_j(x)\right)\Delta x\leqslant\prod_{k=1}^m\left(\int_a^b\prod_{j=1}^m f_j^{1+p_k\left(-\frac{t}{p_k}\right)}(x)\,f_k^{1+p_kt\left(1-\frac{1}{p_k}\right)}(x)\,\Delta x\right)^{\frac{1}{p_k}}$$
$$\leqslant\prod_{k=1}^m\left(\int_a^b\prod_{j=1}^m f_j^{1-t}(x)\,f_k^{p_kt}(x)\,\Delta x\right)^{\frac{1}{p_k}}. \tag{7}$$

The inequality (7) is one of the main results of [13] *for the case* $\mathbb{T}=\mathbb{Z}$.

THEOREM 2. *Let* $r\in\mathbb{R}$, $a,\ b\in\mathbb{T}$, $f_j>0\ (j=1,2,...,m)$, *and* $f_j\in C_{rd}[a,b]$, $rp_k>0\ (k=1,2,...,s)$, $\alpha_{kj}\in\mathbb{R}$, $\sum_{k=1}^{s}\frac{1}{p_k}=r$ *and* $\sum_{k=1}^{s}\alpha_{kj}=0$. *Then*

$$\int_a^b\left(\prod_{j=1}^m f_j(x)\right)\Delta x\leqslant\prod_{k=1}^s\left(\int_a^b\prod_{j=1}^m f_j^{1+rp_k\alpha_{kj}}(x)\,\Delta x\right)^{\frac{1}{rp_k}}. \tag{8}$$

Proof. By taking rp_k instead of p_k in the inequality (4) and using the assumptions of Theorem 2 one can easily obtain (8). □

COROLLARY 2. *Letting* $s=2$, $p_1=p$, $q_1=q$, $\alpha_{1j}=-\alpha_{2j}=\alpha_j$ *and under the assumptions of Theorem 2, we have*

$$\int_a^b\left(\prod_{j=1}^m f_j(x)\right)\Delta x\leqslant\left(\int_a^b\prod_{j=1}^m f_j^{1+rp\alpha_j}(x)\,\Delta x\right)^{\frac{1}{rp}}\left(\int_a^b\prod_{j=1}^m f_j^{1-rq\alpha_j}(x)\,\Delta x\right)^{\frac{1}{rq}}. \tag{9}$$

For the case $\mathbb{T}=\mathbb{Z}$ and $r=1$ in (9), we obtain a Hölder type generalization of the Callebaut inequality established by Masjed-Jamei [7]

$$\left(\sum_{i=1}^n a_{i1}a_{i2}\ldots a_{im}\right)^2\leqslant\sum_{i=1}^n a_{i1}^{(1+\alpha_1)}a_{i2}^{(1+\alpha_2)}\ldots a_{im}^{(1+\alpha_m)}\sum_{i=1}^n a_{i1}^{(1-\alpha_1)}a_{i2}^{(1-\alpha_2)}\ldots a_{im}^{(1-\alpha_m)}.$$

For the case $\mathbb{T}=\mathbb{R}$ and $r=1$, we have the continuous version of the inequality (9) takes the form

$$\int_a^b\left(\prod_{j=1}^m f_j(x)\right)dx\leqslant\left(\int_a^b\prod_{j=1}^m f_j^{1+\alpha_j}(x)\,dx\right)^{\frac{1}{2}}\left(\int_a^b\prod_{j=1}^m f_j^{1-\alpha_j}(x)\,dx\right)^{\frac{1}{2}}$$

which has interesting applications in statistics (see [7] for details).

THEOREM 3. *Assume that all the conditions of Theorem 1 hold. Then we have*

$$\int_a^b \left(\prod_{j=1}^m f_j(x)\right)\Delta x \leqslant \phi(c) \leqslant \prod_{k=1}^s \left(\int_a^b \prod_{j=1}^m f_j^{1+p_k\alpha_{kj}}(x)\Delta x\right)^{\frac{1}{p_k}}$$

where

$$\phi(c) \equiv \int_a^c \left(\prod_{j=1}^m f_j(x)\right)\Delta x + \prod_{k=1}^s \left(\int_c^b \prod_{j=1}^m f_j^{1+p_k\alpha_{kj}}(x)\Delta x\right)^{\frac{1}{p_k}}$$

is a nonincreasing function with $a \leqslant c \leqslant b$.

Proof. From the proof of the Theorem 1, we have

$$\int_a^b \left(\prod_{j=1}^m f_j(x)\right)\Delta x = \int_a^b \left(\prod_{k=1}^s (g_k(x))\right)\Delta x$$

where

$$g(x) = \left(\prod_{j=1}^m f_j^{1+p_k\alpha_{kj}}(x)\right)^{\frac{1}{p_k}}.$$

Using the Hölder inequality, we obtain

$$\begin{aligned}
\phi(\sigma(c)) &= \int_a^{\sigma(c)} \left(\prod_{j=1}^m f_j(x)\right)\Delta x + \prod_{k=1}^s \left(\int_{\sigma(c)}^b \prod_{j=1}^m f_j^{1+p_k\alpha_{kj}}(x)\Delta x\right)^{\frac{1}{p_k}} \\
&= \int_a^c \left(\prod_{k=1}^s (g_k(x))\right)\Delta x + \int_c^{\sigma(c)} \left(\prod_{k=1}^s (g_k(x))\right)\Delta x + \prod_{k=1}^s \left(\int_{\sigma(c)}^b g_k^{p_k}(x)\Delta x\right)^{\frac{1}{p_k}} \\
&\leqslant \int_a^c \left(\prod_{k=1}^s (g_k(x))\right)\Delta x + \prod_{k=1}^s \left(\int_c^{\sigma(c)} g_k^{p_k}(x)\Delta x\right)^{\frac{1}{p_k}} + \prod_{k=1}^s \left(\int_{\sigma(c)}^b g_k^{p_k}(x)\Delta x\right)^{\frac{1}{p_k}} \\
&= \int_a^c \left(\prod_{k=1}^s (g_k(x))\right)\Delta x + \prod_{k=1}^s \left(\int_c^{\sigma(c)} g_k^{p_k}(x)\Delta x + \int_{\sigma(c)}^b g_k^{p_k}(x)\Delta x\right)^{\frac{1}{p_k}} \\
&= \int_a^c \left(\prod_{k=1}^s (g_k(x))\right)\Delta x + \prod_{k=1}^s \left(\int_c^b g_k^{p_k}(x)\Delta x\right)^{\frac{1}{p_k}} \\
&= \phi(c),
\end{aligned}$$

i.e. $\phi(c)$ is a nonincreasing function with $a \leqslant c \leqslant b$. This completes the proof. □

THEOREM 4. *Assume that all the conditions of Theorem 2 are hold. Then we have*

$$\int_a^b \left(\prod_{j=1}^m f_j(x)\right)\Delta x \leqslant \phi(c) \leqslant \prod_{k=1}^s \left(\int_a^b \prod_{j=1}^m f_j^{1+rp_k\alpha_{kj}}(x)\,\Delta x\right)^{\frac{1}{rp_k}} \tag{10}$$

where

$$\phi(c) \equiv \int_a^c \left(\prod_{j=1}^m f_j(x)\right)\Delta x + \prod_{k=1}^s \left(\int_c^b \prod_{j=1}^m f_j^{1+rp_k\alpha_{kj}}(x)\,\Delta x\right)^{\frac{1}{rp_k}}$$

is a nonincreasing function with $a \leqslant c \leqslant b$.

Proof. Since $rp_k > 0$ and $\sum\limits_{k=1}^{s} \frac{1}{p_k} = r$, by taking rp_k instead of p_k in the inequality in Theorem 3, we obtain (10). The proof of the theorem is similar to the previous theorem. Therefore we omit this proof. $\square$

REMARK 2. If we choose $s = 2$, Theorem 4 presents refinement of (9). Moreover, letting $s = m = 2$, $r = 1$, $\alpha_{kj} = 1 - \frac{1}{p_k}$ for $j \neq k$ and $\alpha_{kk} = -\frac{1}{p_k}$, then Theorem 4 leads to the main results of [1] for the case $\mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = \mathbb{R}$.

In the paper by Wing-Sum [10], Hölder's inequality for negative exponents was studied. Finally, we derive a time scale version of Hölder's inequality for negative exponents.

THEOREM 5. *Let* $a, b \in \mathbb{T}$, $0 < p < 1$ *and* $q \in \mathbb{R}$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$. *For rd-continuous functions* $f, g : [a,b] \to \mathbb{R}$, *we have*

$$\int_a^b |f(t)g(t)|\,\Delta t \geqslant \left\{\int_a^b |f(t)|^p\,\Delta t\right\}^{\frac{1}{p}} \left\{\int_a^b |g(t)|^q\,\Delta t\right\}^{\frac{1}{q}} \tag{11}$$

unless $\int_a^b |g(t)|^q\,\Delta t = 0$, *in which case the right-hand side of (11) does not make sense.*

Proof. In the case we are concerned with, we have $0 < \int_a^b |g(t)|^q\,\Delta t < \infty$. Since $q < 0$, this implies that $g(t) > 0$ for almost all $t \in \mathbb{T}$. Let $r = \frac{1}{p}$, and define $\varphi(t) = g^{-\frac{1}{r}}(t)$ and $\psi(t) = g^{\frac{1}{r}}(t)f^{\frac{1}{r}}(t)$. It is easy to see that $\varphi^s(t) = g^u$. Using the Hölder's inequality (9), we have

$$\int_a^b |f(t)|^p \Delta t = \int_a^b |\varphi(t)\psi(t)| \Delta t \leqslant \left\{ \int_a^b |\psi(t)|^r \Delta t \right\}^{\frac{1}{r}} \left\{ \int_a^b |\varphi(t)|^s \Delta t \right\}^{\frac{1}{s}}$$

$$= \left(\int_a^b |f(t) g(t)| \Delta t \right)^{\frac{1}{p}} \left(\int_a^b |g(t)|^u \Delta t \right)^{\frac{1}{s}}.$$

Hence we obtain

$$\int_a^b |f(t) g(t)| \Delta t \geqslant \left(\int_a^b |f(t)|^p \Delta t \right)^{\frac{1}{p}} \left(\int_a^b |g(t)|^u \Delta t \right)^{-\frac{1}{ps}}$$

and since - $\frac{1}{ps} = \frac{1}{u}$, the proof is completed. □

THEOREM 6. *Let $p_1, \ldots, p_{m-1} < 0$, and $p_m \in \mathbb{R}$ such that $\sum_{j=1}^{m} \frac{1}{p_j} = 1$. Let $f_j(x) > 0$, $j = 1, 2, \ldots, m$, $x \in [a,b]_{\mathbb{T}}$, and $f_j \in C_{rd}[a,b]$, then*

$$\int_a^b \left(\prod_{j=1}^{m} |f_j(x)| \right) \Delta x \geqslant \prod_{j=1}^{m} \left(\int_a^b |f_j(x)|^{p_j} \Delta x \right)^{\frac{1}{p_j}} \tag{12}$$

unless $\int_a^b |f_j(x)|^{p_j} \Delta x = 0$ for some $j = 1, 2, \ldots, m-1$, in which case the right-hand side of (12) does not make sense.

Proof. We use mathematical induction on m. It is obvious that when $m = 2$, the inequality (12) reduces to the classical Hölder's inequality (11) for negative numbers. We assume that (12) holds for some integer $m \geqslant 2$. We claim that it also holds for $m+1$. Since $p_1, \ldots, p_m < 0$ and $p_{m+1} \in \mathbb{R}$ such that $\sum_{j=1}^{m+1} \frac{1}{p_j} = 1$. Note that $0 < p_{m+1} < 1$. Since

$$p_1 < 0, \quad 0 < \frac{p_1}{p_1 - 1} < 1, \quad \frac{1}{p_1} + \frac{1}{\frac{p_1}{p_1 - 1}} = 1,$$

by the Hölder's inequality (11), we have

$$\int_a^b \left(\prod_{j=1}^{m+1} |f_j(x)| \right) \Delta x = \int_a^b \left(|f_1(x)| \prod_{j=2}^{m+1} |f_j(x)| \right) \Delta x$$

$$\geqslant \left(\int_a^b |f_1(x)|^{p_1} \Delta x \right)^{\frac{1}{p_1}} \left[\int_a^b \left(\prod_{j=2}^{m+1} |f_j(x)| \right)^{\frac{p_1}{p_1 - 1}} \Delta x \right]^{\frac{p_1 - 1}{p_1}}$$

unless $\int_a^b |f_1(x)|^{p_1} \Delta x = 0$. Now since

$$\frac{p_j(p_1-1)}{p_1} < 0 \quad \text{for some} \quad j = 2,\dots,m, \qquad \frac{p_{m+1}(p_1-1)}{p_1} > 0,$$

we have

$$\sum_{j=2}^{m+1} \frac{1}{p_j \frac{p_1-1}{p_1}} = \frac{p_1}{p_1-1} \sum_{j=2}^{m+1} \frac{1}{p_j} = \frac{p_1}{p_1-1}\left(1 - \frac{1}{p_1}\right) = 1.$$

By the induction hypothesis and (11), we obtain

$$\begin{aligned}
\int_a^b \left(\prod_{j=1}^{m+1} |f_j(x)| \right) \Delta x &\geqslant \left(\int_a^b |f_1(x)|^{p_1} \Delta x \right)^{\frac{1}{p_1}} \\
&\quad \times \left[\prod_{j=2}^{m+1} \left(\int_a^b \left(|f_j(x)|\right)^{\frac{p_1}{p_1-1} \frac{p_j(p_1-1)}{p_1}} \Delta x \right)^{\frac{p_1}{p_j(p_1-1)}} \right]^{\frac{p_1-1}{p_1}} \\
&= \left(\int_a^b |f_1(x)|^{p_1} \Delta x \right)^{\frac{1}{p_1}} \prod_{j=2}^{m+1} \left(\int_a^b |f_j(x)|^{p_j} \Delta x \right)^{\frac{1}{p_j}} \\
&= \prod_{j=1}^{m+1} \left(\int_a^b |f_j(x)|^{p_j} \Delta x \right)^{\frac{1}{p_j}}
\end{aligned}$$

unless $\int_a^b |f_j(x)|^{p_j} \Delta x = 0$ for some $j = 1,2,\dots,m$. This completes the proof. □

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