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PARABOLIC FRACTIONAL MAXIMAL AND INTEGRAL OPERATORS WITH ROUGH KERNELS IN PARABOLIC GENERALIZED MORREY SPACES

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Abstract. Let P be a real $n\times n$ matrix, whose all the eigenvalues have positive real part, $A_t = t^P$, $t > 0$, $\gamma = \mathrm{tr}P$ is the homogeneous dimension on $\mathbb{R}^n$ and Ω is an A_t-homogeneous of degree zero function, integrable to a power $s > 1$ on the unit sphere generated by the corresponding parabolic metric. We study the parabolic fractional maximal and integral operators $M^P_{\Omega,\alpha}$ and $I^P_{\Omega,\alpha}$, $0 < \alpha < \gamma$ with rough kernels in the parabolic generalized Morrey space $\mathscr{M}_{p,\varphi,P}(\mathbb{R}^n)$. We find conditions on the pair (φ_1,φ_2) for the boundedness of the operators $M^P_{\Omega,\alpha}$ and $I^P_{\Omega,\alpha}$ from the space $\mathscr{M}_{p,\varphi_1,P}(\mathbb{R}^n)$ to another one $\mathscr{M}_{q,\varphi_2,P}(\mathbb{R}^n)$, $1 < p < q < \infty$, $1/p - 1/q = \alpha/\gamma$, and from the space $\mathscr{M}_{1,\varphi_1,P}(\mathbb{R}^n)$ to the weak space $W\mathscr{M}_{q,\varphi_2,P}(\mathbb{R}^n)$, $1 \leqslant q < \infty$, $1 - 1/q = \alpha/\gamma$. We also find conditions on φ for the validity of the Adams type theorems $M^P_{\Omega,\alpha}$, $I^P_{\Omega,\alpha}$: $\mathscr{M}_{p,\varphi^{\frac{1}{p}},P}(\mathbb{R}^n) \to \mathscr{M}_{q,\varphi^{\frac{1}{q}},P}(\mathbb{R}^n), 1 < p < q < \infty$.

1. Introduction

The boundedness of classical operators of the real analysis, such as the maximal operator, fractional maximal operators, fractional integral operators and singular integral operators etc, from one weighted Lebesgue space to another one is well studied by now, and there are well known various applications of such results in partial differential equations. Besides Lebesgue spaces, Morrey spaces, both the classical ones (the idea of their definition having appeared in [22]) and generalized ones, also play an important role in the theory of partial differential equations, see [12, 19, 20, 29, 30].

In this paper, we find conditions for the boundedness of the parabolic fractional maximal and integral operators with rough kernel from a parabolic generalized Morrey space to another one, including also the case of weak boundedness, and prove Adams type boundedness theorems for this operators. To precisely formulate the results of this paper, we need the notions given below.

Note that we deal not exactly with the parabolic metric, but with a general anisotropic metric ρ of generalized homogeneity, the parabolic metric being its particular

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case, but we keep the term parabolic in the title and text of the paper, following the existing tradition, see for instance [7].

Everywhere in the sequel $A \lesssim B$ means that $A \leqslant CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

1.1. Parabolic homogeneous space $\{\mathbb{R}^n, \rho, dx\}$

For $x \in \mathbb{R}^n$ and $r > 0$, we denote the open ball centered at x of radius r by $B(x,r)$, its complement by $^{\complement}B(x,r)$ and $|B(x,r)|$ will stand for the Lebesgue measure of $B(x,r)$.

Let P be a real $n \times n$ matrix, whose all the eigenvalues have positive real part. Let $A_t = t^P \ (t > 0)$, and set

$$\gamma = \mathrm{tr} P.$$

Then, there exists a quasi-distance ρ associated with P such that (see [8])

$$(a)\ \ \rho(A_t x) = t\rho(x), \ \ t > 0, \ \ \text{for every} \ \ x \in \mathbb{R}^n;$$

$$(b)\ \ \rho(0) = 0, \ \ \rho(x) = \rho(-x) \geqslant 0$$
$$\text{and} \ \ \rho(x-y) \leqslant k(\rho(x-z) + \rho(y-z));$$

$$(c)\ \ dx = \rho^{\gamma-1} d\sigma(w) d\rho, \ \ \text{where} \ \ \rho = \rho(x), w = A_{\rho^{-1}} x$$
$$\text{and } d\sigma(w) \text{ is a measure on the unit ellipsoid } \ S_\rho = \{w : \rho(w) = 1\}.$$

Then, $\{\mathbb{R}^n, \rho, dx\}$ becomes a space of homogeneous type in the sense of Coifman-Weiss (see [8]) and a homogeneous group in the sense of Folland-Stein (see [10]). Moreover, we always assume that there hold the following properties of the quasidistance ρ:

(d) For every x,

$$c_1 |x|^{\alpha_1} \leqslant \rho(x) \leqslant c_2 |x|^{\alpha_2}, \ \ \text{if} \ \ \rho(x) \geqslant 1;$$
$$c_3 |x|^{\alpha_3} \leqslant \rho(x) \leqslant c_4 |x|^{\alpha_4}, \ \ \text{if} \ \ \rho(x) \leqslant 1$$

and

$$\rho(\theta x) \leqslant \rho(x) \ \ \text{for} \ \ 0 < \theta < 1,$$

with some positive constants α_i and c_i $(i = 1, \dots, 4)$. Similar properties hold also for the quasimetric ρ^* associated with the adjoint matrix P^*.

The following are some important examples of the above defined matrices P and distances ρ.

1. Let $(Px, x) \geqslant (x, x)$, $x \in \mathbb{R}^n$. In this case, $\rho(x)$ is defined as the unique solution $\rho(x) = t$ of $|A_{t^{-1}} x| = 1$, and $k = 1$. This is the case studied by Calderon and Torchinsky in [7].

2. Let P be a diagonal matrix with positive diagonal entries, and $t = \rho(x)$, $x \in \mathbb{R}^n$ be the unique solution of $|A_{t^{-1}} x| = 1$.

2_a) When all the diagonal entries are greater than or equal to 1, O. V. Besov, V. P. Il'in, P. I. Lizorkin in [3] and E. B. Fabes and N. M. Rivière in [9] studied the weak $(1,1)$ and strong (p,p) estimates of singular integral operators.

2_b) If there are diagonal entries smaller than 1, then ρ satisfies the above (a)–(d) with $k > 1$.

3. In [28] Stein and Wainger defined and studied some problems in harmonic analysis on this kind of spaces. Consider a one parameter group of dilations on $\mathbb{R}^n$, $A_t : \mathbb{R}^n \to \mathbb{R}^n$ for each $t > 0$ with the following properties:

(i) $A_{st} = A_s A_t$ and A_1 is the identity;

(ii) $\lim_{t \to 0} A_t x = 0$ for every $x \in \mathbb{R}^n$;

(iii) $A_t x = t^P x = \exp\{P \log t\} x$.

Then all eigenvalues of P have positive real part. Then in this case there exist $0 < \beta_1 < \beta_2$ and $0 < c_1' < c_2'$ such that A_t has the following properties:

(iv) for every x

$$c_1' t^{\beta_1} |x| < |A_t x| < c_2' t^{\beta_2} |x|, \quad \text{for } t \geqslant 1$$

and

$$(c_2')^{-1} t^{\beta_2} |x| < |A_t x| < (c_1')^{-1} t^{\beta_1} |x|, \quad \text{for} \quad t \leqslant 1.$$

By [28], if $|A_{tx}|$ were strictly monotonic, then we might define the unique solutions of $|A_t x| = 1$ by $\rho(x)$. Otherwise, there is a positive definite symmetric matrix B such that

$$\langle A_t x \rangle = \langle A_t x \rangle_B = (B A_t x, A_t x)^{\frac{1}{2}}$$

is strictly increasing and thus ρ can be defined as follows: For $x \neq 0$, $\rho(x)$ is the unique positive t such that $\langle A_t^{-1}(x) \rangle = 1$. For $x = 0$, set $\rho(x) = 0$. Then for $x \neq 0$

$$x = A_{\rho(x)} w(x),$$

where $\langle w(x) \rangle = 1$ and $w(x)$ is unique. Let $\rho^*(\xi)$ be the quasi-distance function corresponding to the group $A_t^* = t^{P^*} = \exp(P^* \log t)$. Then $\xi = A^*_{\rho^*(\xi)}(w^*(\xi))$ where

$$\langle w^*(\xi) \rangle = (B_1 w^*(\xi), w^*(\xi))^{\frac{1}{2}}$$

for an appropriate positive definite symmetric matrix B.

It was pointed out in [28] that both ρ and ρ^* satisfy (1.1)–(1.4), and one can easily see that $\alpha_1 = \alpha_4 = \frac{1}{\beta_2}$, $\alpha_1 = \alpha_3 = \frac{1}{\beta_1}$, and c_i depends on on the matrices P and B. Moreover, in this case

$$dx = \rho^{\gamma - 1} d\sigma(w) d\rho,$$

where $d\sigma(w)$ is a C^∞ measure on the ellipsoid $\rho(w)^2 = (Bw, w) = 1$.

In the standard parabolic case $P_0 = \mathrm{diag}(1, \ldots, 1, 2)$ we have

$$\rho(x) = \sqrt{\frac{|x'|^2 + \sqrt{|x'|^4 + x_n^2}}{2}}, \qquad x = (x', x_n).$$

The balls $\mathscr{E}(x,r) = \{y \in \mathbb{R}^n : \rho(x-y) < r\}$ with respect to the quasidistance ρ are ellipsoids. For its Lebesgue measure one has

$$|\mathscr{E}(x,r)| = v_\rho r^\gamma,$$

where v_ρ is the volume of the unit ellipsoid. By ${}^{\complement}\mathscr{E}(x,r) = \mathbb{R}^n \setminus \mathscr{E}(x,r)$ we denote the complement of $\mathscr{E}(x,r)$.

1.2. Parabolic generalized Morrey spaces

We define the parabolic Morrey space $\mathscr{M}_{p,\lambda,P}(\mathbb{R}^n)$ via the norm

$$\|f\|_{\mathscr{M}_{p,\lambda,P}} = \sup_{x\in\mathbb{R}^n, t>0} \left(t^{-\lambda} \int_{\mathscr{E}(x,t)} |f(y)|^p dy\right)^{1/p} < \infty,$$

where $1 \leqslant p \leqslant \infty$ and $0 \leqslant \lambda \leqslant \gamma$.

If $\lambda = 0$, then $\mathscr{M}_{p,0,P}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$; if $\lambda = \gamma$, then $\mathscr{M}_{p,\gamma,P}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$; if $\lambda < 0$ or $\lambda > \gamma$, then $\mathscr{M}_{p,\lambda,P} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

We also denote by $W\mathscr{M}_{p,\lambda,P}(\mathbb{R}^n)$ the weak parabolic Morrey space of functions $f \in WL_p^{\text{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{W\mathscr{M}_{p,\lambda,P}} = \sup_{x\in\mathbb{R}^n, t>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(\mathscr{E}(x,r))} < \infty,$$

where $WL_p(\mathscr{E}(x,r))$ denotes the weak L_p-space of measurable functions f for which

$$\|f\|_{WL_p(\mathscr{E}(x,r))} = \sup_{t>0} t\, |\{y \in \mathscr{E}(x,r) : |f(y)| > t\}|^{1/p}.$$

Note that $WL_p(\mathbb{R}^n) = W\mathscr{M}_{p,0,P}(\mathbb{R}^n)$,

$$\mathscr{M}_{p,\lambda,P}(\mathbb{R}^n) \subset W\mathscr{M}_{p,\lambda,P}(\mathbb{R}^n) \text{ and } \|f\|_{W\mathscr{M}_{p,\lambda,P}} \leqslant \|f\|_{\mathscr{M}_{p,\lambda,P}}.$$

If $P = I$, then $\mathscr{M}_{p,\lambda}(\mathbb{R}^n) \equiv \mathscr{M}_{p,\lambda,I}(\mathbb{R}^n)$ is the classical Morrey space.

We introduce the parabolic generalized Morrey spaces following the known ideas of defining generalized Morrey spaces ([15, 24, 26] etc).

DEFINITION 1.1. Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \leqslant p < \infty$. The space $\mathscr{M}_{p,\varphi,P} \equiv \mathscr{M}_{p,\varphi,P}(\mathbb{R}^n)$, called the parabolic generalized Morrey space, is defined by the norm

$$\|f\|_{\mathscr{M}_{p,\varphi,P}} = \sup_{x\in\mathbb{R}^n, t>0} \varphi(x,t)^{-1} |\mathscr{E}(x,t)|^{-\frac{1}{p}} \|f\|_{L_p(\mathscr{E}(x,t))}.$$

DEFINITION 1.2. Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \leqslant p < \infty$. The space $W\mathcal{M}_{p,\varphi,P} \equiv W\mathcal{M}_{p,\varphi,P}(\mathbb{R}^n)$, called the weak parabolic generalized Morrey space, is defined by the norm

$$\|f\|_{W\mathcal{M}_{p,\varphi,P}} = \sup_{x\in\mathbb{R}^n, t>0} \varphi(x,t)^{-1} |\mathscr{E}(x,t)|^{-\frac{1}{p}} \|f\|_{WL_p(\mathscr{E}(x,t))}.$$

If $P = I$, then $\mathcal{M}_{p,\varphi}(\mathbb{R}^n) \equiv \mathcal{M}_{p,\varphi,I}(\mathbb{R}^n)$ and $W\mathcal{M}_{p,\varphi}(\mathbb{R}^n) \equiv W\mathcal{M}_{p,\varphi,I}(\mathbb{R}^n)$ are the generalized Morrey space and the weak generalized Morrey space, respectively.

According to this definition, we recover the space $\mathcal{M}_{p,\lambda,P}(\mathbb{R}^n)$ under the choice $\varphi(x,r) = r^{\frac{\lambda-\gamma}{p}}$:

$$\mathcal{M}_{p,\lambda,P}(\mathbb{R}^n) = \mathcal{M}_{p,\varphi,P}(\mathbb{R}^n)\Big|_{\varphi(x,r)=r^{\frac{\lambda-\gamma}{p}}}.$$

1.3. Operators under consideration

Let $S_\rho = \{w \in \mathbb{R}^n : \rho(w) = 1\}$ be the unit ρ-sphere (ellipsoid) in $\mathbb{R}^n$ $(n \geqslant 2)$ equipped with the normalized Lebesgue surface measure $d\sigma$ and Ω be A_t-homogeneous of degree zero, i.e. $\Omega(A_t x) \equiv \Omega(x)$, $x \in \mathbb{R}^n$, $t > 0$. The parabolic fractional maximal function $M^P_{\Omega,\alpha} f$ and the parabolic fractional integral $I^P_{\Omega,\alpha} f$ by with rough kernels, $0 < \alpha < \gamma$, of a function $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$ are defined by

$$M^P_{\Omega,\alpha} f(x) = \sup_{t>0} |\mathscr{E}(x,t)|^{-1+\frac{\alpha}{\gamma}} \int_{\mathscr{E}(x,t)} |\Omega(x-y)|\, |f(y)| dy,$$

$$I^P_{\Omega,\alpha} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)\, f(y)}{\rho(x-y)^{\gamma-\alpha}} dy.$$

If $\Omega \equiv 1$, then $M^P_\alpha \equiv M^P_{1,\alpha}$ and $I^P_\alpha \equiv I^P_{1,\alpha}$ are the parabolic fractional maximal operator and the parabolic fractional integral operator, respectively. If $\alpha = 0$, then $M^P_\Omega \equiv M^P_{\Omega,0}$ is the parabolic maximal operator with rough kernel. If $P = I$, then $M_{\Omega,\alpha} \equiv M^I_{\Omega,\alpha}$ is the fractional maximal operator with rough kernel, and $M \equiv M^I_{\Omega,0}$ is the Hardy-Littlewood maximal operator with rough kernel. It is well known that the parabolic fractional maximal operators play an important role in harmonic analysis (see [10, 27]).

We prove the boundedness of the parabolic fractional maximal and integral operators $M^P_{\Omega,\alpha}$, $I^P_{\Omega,\alpha}$ with rough kernel from one parabolic generalized Morrey space $\mathcal{M}_{p,\varphi_1,P}(\mathbb{R}^n)$ to another one $\mathcal{M}_{q,\varphi_2,P}(\mathbb{R}^n)$, $1 < p < q < \infty$, $1/p - 1/q = \alpha/\gamma$, and from the space $\mathcal{M}_{1,\varphi_1,P}(\mathbb{R}^n)$ to the weak space $W\mathcal{M}_{q,\varphi_2,P}(\mathbb{R}^n)$, $1 \leqslant q < \infty$, $1 - 1/q = \alpha/\gamma$. We also prove the Adams type boundedness of the operators $M^P_{\Omega,\alpha}$, $I^P_{\Omega,\alpha}$ from $\mathcal{M}_{p,\varphi^{\frac{1}{p}},P}(\mathbb{R}^n)$ to $\mathcal{M}_{q,\varphi^{\frac{1}{q}},P}(\mathbb{R}^n)$ for $1 < p < q < \infty$ and from $\mathcal{M}_{1,\varphi,P}(\mathbb{R}^n)$ to $W\mathcal{M}_{q,\varphi^{\frac{1}{q}},P}(\mathbb{R}^n)$ for $1 < q < \infty$.

2. Preliminaries

In the papers [24, 25], where the maximal and other operator were studied in generalized Morrey spaces, the following condition was imposed on $\varphi(x,r)$:

$$c^{-1}\varphi(x,r) \leqslant \varphi(x,t) \leqslant c\,\varphi(x,r), \tag{2.1}$$

whenever $r \leqslant t \leqslant 2r$, jointly with the condition:

$$\int_r^\infty \varphi(x,t)^p \frac{dt}{t} \leqslant C\,\varphi(x,r)^p$$

for the maximal or singular operators and the condition

$$\int_r^\infty t^{\alpha p}\varphi(x,t)^p \frac{dt}{t} \leqslant C\,r^{\alpha p}\varphi(x,r)^p \tag{2.2}$$

for potential and fractional maximal operators, where c and C do not depend on r and x.

The results of [24, 25] imply the following statement.

THEOREM 2.1. *Let* $1 \leqslant p < \infty$, $0 < \alpha < \frac{\gamma}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\gamma}$ *and* $\varphi(x,t)$ *satisfy the conditions* (2.1) *and* (2.2). *Then* M_α^P *and* I_α^P *are bounded from* $\mathscr{M}_{p,\varphi,P}(\mathbb{R}^n)$ *to* $\mathscr{M}_{q,\varphi,P}(\mathbb{R}^n)$ *for* $p > 1$ *and from* $\mathscr{M}_{1,\varphi,P}(\mathbb{R}^n)$ *to* $W\mathscr{M}_{q,\varphi,P}(\mathbb{R}^n)$ *for* $p = 1$.

In [11] the following statement was proved by fractional integral operator with rough kernels $I_{\Omega,\alpha}$, containing the result in [21, 24].

THEOREM 2.2. *Let* $\Omega \in L_s(S^{n-1})$, $1 < s \leqslant \infty$, *be homogeneous of degree zero. Let* $0 < \alpha < n$, $s' < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ *and* $\varphi(x,r)$ *satisfy the condition* (2.1) *and*

$$\int_r^\infty t^{\alpha p}\varphi(x,t)^p \frac{dt}{t} \leqslant C\,r^{\alpha p}\varphi(x,r)^p, \tag{2.3}$$

where C *does not depend on* x *and* r. *Then the operator* $I_{\Omega,\alpha}$ *is bounded from* $M_{p,\varphi}$ *to* $M_{q,\varphi}$.

The following statements, containing results obtained in [21], [24] was proved in [13, 15] (see also [4]–[6], [14]–[17]).

THEOREM 2.3. *Let* $0 < \alpha < \gamma$, $1 \leqslant p < \frac{\gamma}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\gamma}$ *and* (φ_1, φ_2) *satisfy the condition*

$$\int_r^\infty t^{\alpha-1}\varphi_1(x,t)dt \leqslant C\,\varphi_2(x,r), \tag{2.4}$$

where C *does not depend on* x *and* r. *Then the operator* I_α^P *is bounded from* $M_{p,\varphi_1,P}$ *to* $M_{q,\varphi_2,P}$ *for* $p > 1$ *and from* $M_{1,\varphi_1,P}$ *to* WLM_{q,φ_2} *for* $p = 1$.

Let v be a weight on $(0,\infty)$. We denote by $L_{\infty,v}(0,\infty)$ the space of all functions $g(t)$, $t>0$ with finite norm

$$\|g\|_{L_{\infty,v}(0,\infty)} = \operatorname*{ess\,sup}_{t>0} v(t)|g(t)|$$

and write $L_\infty(0,\infty) \equiv L_{\infty,1}(0,\infty)$. Let $\mathfrak{M}(0,\infty)$ be the set of all Lebesgue-measurable functions on $(0,\infty)$ and $\mathfrak{M}^+(0,\infty)$ its subset of all nonnegative functions. By $\mathfrak{M}^+(0,\infty;\uparrow)$ we denote the cone of all functions in $\mathfrak{M}^+(0,\infty)$ non-decreasing on $(0,\infty)$ and introduce also the set

$$\mathbb{A} = \left\{ \varphi \in \mathfrak{M}^+(0,\infty;\uparrow) : \lim_{t\to 0+} \varphi(t) = 0 \right\}.$$

Let u be a non-negative continuous function on $(0,\infty)$. We define the supremal operator $\overline{S}_u$ on $g \in \mathfrak{M}(0,\infty)$ by

$$(\overline{S}_u g)(t) := \|u\, g\|_{L_\infty(t,\infty)}, \ \ t \in (0,\infty).$$

The following theorem was proved in [5].

THEOREM 2.4. *Let v_1, v_2 be non-negative measurable functions satisfying $0 < \|v_1\|_{L_\infty(t,\infty)} < \infty$ for any $t>0$ and let u be a continuous non-negative function on $(0,\infty)$. Then the operator $\overline{S}_u$ is bounded from $L_{\infty,v_1}(0,\infty)$ to $L_{\infty,v_2}(0,\infty)$ on the cone $\mathbb{A}$ if and only if*

$$\left\| v_2 \overline{S}_u \left(\|v_1\|^{-1}_{L_\infty(\cdot,\infty)} \right) \right\|_{L_\infty(0,\infty)} < \infty. \tag{2.5}$$

We are going to use the following statement on the boundedness of the weighted Hardy operator

$$H_w^* g(t) := \int_t^\infty g(s) w(s) ds, \ 0<t<\infty,$$

where w is a fixed function non-negative and measurable on $(0,\infty)$.

The following theorem was proved in [18].

THEOREM 2.5. *Let v_1, v_2 and w be positive almost everywhere and measurable functions on $(0,\infty)$. The inequality*

$$\operatorname*{ess\,sup}_{t>0} v_2(t) H_w^* g(t) \leqslant C \operatorname*{ess\,sup}_{t>0} v_1(t) g(t) \tag{2.6}$$

holds for some $C>0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \operatorname*{ess\,sup}_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\operatorname*{ess\,sup}_{s<\tau<\infty} v_1(\tau)} < \infty. \tag{2.7}$$

Moreover, if C^* is the minimal value of C in (2.6), then $C^* = B$.

REMARK 2.1. In (2.6) and (2.7) it is assumed that $\frac{1}{\infty} = 0$ and $0 \cdot \infty = 0$.

3. Boundedness of the parabolic fractional operators in the spaces $L_p(\mathbb{R}^n)$

In this section we prove the (p,p)-boundedness of the operator M^P_Ω and the (p,q)-boundedness of the operators $I^P_{\Omega,\alpha}$ and $M^P_{\Omega,\alpha}$.

THEOREM 3.1. *Let $\Omega \in L_s(S_\rho)$, $1 < s \leqslant \infty$, be A_t-homogeneous of degree zero. Then the operator M^P_Ω is bounded in the space $L_p(\mathbb{R}^n)$, $p > s'$.*

Proof. In the case $s = \infty$ the statement of Theorem 3.1 is known and may be found in [8] and [27]. So we assume that $1 < s < \infty$.

Note that

$$\begin{aligned} \|\Omega(x-\cdot)\|_{L_s(\mathscr{E}(x,t))} &= \Big(\int_{\mathscr{E}(0,t)} |\Omega(y)|^s dy\Big)^{1/s} \\ &= \Big(\int_0^t r^{\gamma-1} dr \int_{S_\rho} |\Omega(\omega)|^s d\sigma(\omega)\Big)^{1/s} \\ &= c_0\, \|\Omega\|_{L_s(S_\rho)}\, |\mathscr{E}(x,t)|^{1/s}, \end{aligned} \tag{3.1}$$

where $c_0 = \big(\gamma v_\rho\big)^{-1/s}$ and $v_\rho = |\mathscr{E}(0,1)|$.

The case $p = \infty$ is easy. Indeed, making use of (3.1), we get

$$\|M^P_\Omega f\|_{L_\infty} \leqslant \|f\|_{L_\infty} \sup_{t>0} |\mathscr{E}(x,t)|^{-1+\frac{1}{s'}} \|\Omega(x-\cdot)\|_{L_s(\mathscr{E}(x,t))} \leqslant c_0\, \|\Omega\|_{L_s(S_\rho)}\, \|f\|_{L_\infty}.$$

So we assume that $s' < p < \infty$. Applying Hölder's inequality, we get

$$M^P_\Omega f(x) \leqslant \sup_{t>0} |\mathscr{E}(x,t)|^{-1} \|\Omega(x-\cdot)\|_{L_s(\mathscr{E}(x,t))}\, \|f\|_{L_{s'}(\mathscr{E}(x,t))}. \tag{3.2}$$

Then from (3.2) and (3.1) we have

$$\begin{aligned} M^P_\Omega f(x) &\leqslant c_0\, \|\Omega\|_{L_s(S_\rho)} \sup_{t>0} |\mathscr{E}(x,t)|^{-1+1/s}\, \|f\|_{L_{s'}(\mathscr{E}(x,t))} \\ &= c_0\, \|\Omega\|_{L_s(S_\rho)} \Big(\sup_{t>0} |\mathscr{E}(x,t)|^{-1} \||f|^{s'}\|_{L_1(\mathscr{E}(x,t))}\Big)^{1/s'} \\ &= c_0\, \|\Omega\|_{L_s(S_\rho)} \Big(M^P(|f|^{s'})(x)\Big)^{1/s'}. \end{aligned} \tag{3.3}$$

Therefore, from (3.3) for $1 \leqslant s' < p < \infty$ we get

$$\begin{aligned} \|M^P_\Omega f\|_{L_p} &\leqslant c_0\, \|\Omega\|_{L_s(S_\rho)} \Big\|\Big(M^P(|f|^{s'})(x)\Big)^{1/s'}\Big\|_{L_p} \\ &= c_0\, \|\Omega\|_{L_s(S_\rho)} \|M^P|f|^{s'}\|^{1/s'}_{L_{p/s'}} \lesssim \||f|^{s'}\|^{1/s'}_{L_{p/s'}} = \|f\|_{L_p}. \quad \square \end{aligned}$$

THEOREM 3.2. *Suppose that* $0 < \alpha < \gamma$ *and the function* $\Omega \in L_{\frac{\gamma}{\gamma-\alpha}}(S_\rho)$, *is* A_t*-homogeneous of degree zero. Let* $1 \leqslant p < \frac{\gamma}{\alpha}$ *and* $1/p - 1/q = \alpha/\gamma$. *Then the fractional integral operator* $I^P_{\Omega,\alpha}$ *is bounded from* $L_p(\mathbb{R}^n)$ *to* $L_q(\mathbb{R}^n)$ *for* $p > 1$ *and from* $L_1(\mathbb{R}^n)$ *to* $WL_q(\mathbb{R}^n)$ *for* $p = 1$.

Proof. We denote

$$K(x) := \frac{\Omega(x)}{\rho(x)^{\gamma-\alpha}}$$

for brevity, and may assume that $K(x) \geqslant 0$. We have

$$\left|\{x \in \mathbb{R}^n : I^P_\alpha f(x) > \lambda\}\right| \leqslant \left|\{x \in \mathbb{R}^n : I^P_\Omega f(x) > C^{-1}_{\gamma,\alpha}\lambda\}\right| \leqslant I_1 + I_2,$$

where

$$I_1 := \left|\left\{x \in \mathbb{R}^n : |K^1_\mu * f(x)| > \frac{\lambda}{2}\right\}\right|, \quad I_2 := \left|\left\{x \in \mathbb{R}^n : |K^2_\mu * f(x)| > \frac{\lambda}{2}\right\}\right|,$$

$$K^1_\mu(x) = (K(x) - \mu)\chi_{E(\mu)}(x) \text{ and } K^2_\mu(x) = K(x) - K^1_\mu(x),$$

$\mu > 0$ and $E(\mu) = \{x \in \mathbb{R}^n : K(x) > \mu\}$. Note that

$$|E(\mu)| \leqslant B\mu^{\frac{\gamma}{\gamma-\alpha}}. \tag{3.4}$$

where $B = \frac{1}{\alpha}\|\Omega\|^{\frac{\gamma}{\gamma-\alpha}}_{L_{\frac{\gamma}{\gamma-\alpha}}(S_\rho)}$ as seen from the following estimation:

$$\begin{aligned}|E(\mu)| &\leqslant \frac{1}{\mu}\int_{E(\mu)} \frac{|\Omega(x)|}{\rho(x)^{\gamma-\alpha}}dx \\ &= \frac{1}{\mu}\int_{S_\rho} \Omega(x')d\sigma(x') \int_0^{\left(\frac{|\Omega(x')|}{\mu}\right)^{\frac{1}{\gamma-\alpha}}} r^{\alpha-1}dr = B\mu^{\frac{\gamma}{\gamma-\alpha}}.\end{aligned}$$

By means of (3.4) we can prove the estimate

$$\|K^2_\mu\|_{L_{p'}} \leqslant \left(\frac{\gamma-\alpha}{\gamma}Bq\right)^{\frac{1}{p'}} \mu^{\frac{\gamma}{(\gamma-\alpha)q}}, \quad 1 \leqslant p < \frac{\gamma}{\alpha}.$$

For $p = 1$ it easily follows from (3.4), and for $p > 1$ we have

$$\begin{aligned}\int_{\mathbb{R}^n} |K^2_\mu(x)|^{p'}dx &= p'\int_0^\mu t^{p'-1}|E(t)|dt \\ &\leqslant p'B\int_0^\mu t^{p'-1-\frac{\gamma}{\gamma-\alpha}}dt \\ &= \frac{\gamma-\alpha}{\gamma}Bq\mu^{\frac{\gamma}{\gamma-\alpha}\frac{p'}{q}}.\end{aligned}$$

Then by the Young inequality we obtain

$$\|K_\mu^2 * f\|_{L_\infty} \leqslant \|K_\mu^2\|_{L_{p'}} \|f\|_{L_p} \leqslant \Big(\frac{\gamma-\alpha}{\gamma} Bq\Big)^{\frac{1}{p'}} \mu^{\frac{\gamma}{(\gamma-\alpha)q}} \; \|f\|_{L_p}.$$

Now for a $\lambda > 0$, we choose μ such that

$$\Big(\frac{\gamma-\alpha}{\gamma} Bq\Big)^{\frac{1}{p'}} \mu^{\frac{\gamma}{(\gamma-\alpha)q}} \; \|f\|_{L_p} = \frac{\lambda}{2},$$

then

$$\Big|\Big\{x \in \mathbb{R}^n : |K_\mu^2 * f(x)| > \frac{\lambda}{2}\Big\}\Big| = 0.$$

Thus

$$\begin{aligned}\Big|\{x \in \mathbb{R}^n : I_\alpha^P f(x) > \lambda\}\Big| &\leqslant \Big|\Big\{x \in \mathbb{R}^n : |K_\mu^1 * f(x)| > \frac{\lambda}{2}\Big\}\Big| \\ &\leqslant \Big(\frac{2}{\lambda} \; \|K_\mu^1 * f\|_{L_p}\Big)^p. \end{aligned} \tag{3.5}$$

The following estimations take (3.4) into account:

$$\begin{aligned}\int_{\mathbb{R}^n} |K_\mu^1(x)| dx &= \int_{E(\mu)} \big(|K(x)| - \mu\big) dx \\ &\leqslant \int_0^\infty |E(t+\mu)| dt \\ &\leqslant B \int_\mu^\infty t^{-\frac{\gamma}{\gamma-\alpha}} dt \\ &= \frac{\alpha B}{\gamma-\alpha} \; \mu^{-\frac{\alpha}{\gamma-\alpha}}. \end{aligned} \tag{3.6}$$

For all $f \in L_\infty(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, from (3.6) it follows that

$$|K_\mu^1 * f(x)| \leqslant \|f\|_{L_\infty} \int_{\mathbb{R}^n} |K_\mu^1(x)| dx \leqslant \frac{\alpha B}{\gamma-\alpha} \; \mu^{-\frac{\alpha}{\gamma-\alpha}} \; \|f\|_{L_\infty}. \tag{3.7}$$

For all $f \in L_1(\mathbb{R}^n)$, from (3.6) follows

$$\|K_\mu^1 * f\|_{L_1} \leqslant \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |K_\mu^1(x-y)||f(y)| dx dy \leqslant \frac{\alpha B}{\gamma-\alpha} \; \mu^{-\frac{\alpha}{\gamma-\alpha}} \; \|f\|_{L_1}. \tag{3.8}$$

Thus from (3.7) and (3.8) follows that the operator $T_1 : f \to K_\mu^1 * f$ is of (∞,∞) and $(1,1)$-type. Then by the Riesz-Thorin theorem the operator T_1 is also of (p,p)-type, $1 < p < \infty$, and

$$\|T_1 f\|_{L_p} \leqslant \frac{\alpha B}{\gamma-\alpha} \; \mu^{-\frac{\alpha}{\gamma-\alpha}} \; \|f\|_{L_p}. \tag{3.9}$$

From (3.5) and (3.9) we get

$$\Big|\{x\in\mathbb{R}^n : I^P_\alpha f(x)>\lambda\}\Big| \leqslant \Big(\frac{2}{\lambda}\,\|K^1_\mu * f\|_{L_p}\Big)^p$$
$$\leqslant C\Big(\frac{1}{\lambda}\,\|f\|_{L_p}\Big)^q, \tag{3.10}$$

where C is independent of λ and f.

To finish the proof, i.e. prove that the operator I^P_α is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ for $1<p<\frac{\gamma}{\alpha}$ and $1/p-1/q=\alpha/\gamma$, observe that the inequality (3.10) tells us that I^P_α is bounded from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ with $1-1/q=\alpha/\gamma$. We choose any p_0 such that $p<p_0<\frac{\gamma}{\alpha}$, and put $\frac{1}{q_0}=\frac{1}{p_0}-\frac{\alpha}{\gamma}$. By (3.10) the operator I^P_α is of weak (p_0,q_0)-type. Since it is also of weak $(1,q)$-type by the Marcinkiewicz interpolation theorem, we conclude that I^P_α is of (p,q)-type. □

COROLLARY 3.1. *Under the assumptions of Theorem 3.2, the fractional maximal operator $M^P_{\Omega,\alpha}$ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ for $p>1$ and from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ for $p=1$.*

Proof. It suffices to refer to the known fact that

$$M^P_{\Omega,\alpha}f(x)\leqslant C_{\gamma,\alpha}I^P_{\Omega,\alpha}f(x),\quad C_{\gamma,\alpha}=|\mathscr{E}(0,1)|^{\frac{\gamma-\alpha}{\gamma}},\qquad \square$$

Note that in the isotropic case $P=I$ Theorem 3.2 was proved in [23].

4. Parabolic fractional maximal operator with rough kernels in the spaces $\mathscr{M}_{p,\varphi,P}(\mathbb{R}^n)$

Note that in the next Section 5 we obtain boundedness results of Spanne and Adams type for the fractional integral operator $I^P_{\Omega,\alpha}$. Although $M^P_{\Omega,\alpha}f$ is dominated by $I^P_{\Omega,\alpha}f$ and consequently from the results of Section 5 there may be derived the corresponding results for $M^P_{\Omega,\alpha}f$, we obtain here a Spanne type statement for the operator $M^P_{\Omega,\alpha}f$ separately from Section 5, because for this operator we are able to obtain the boundedness results under weaker assumptions than in Section 5, see Remark 5.3.

Recall that in the classical isotropic case, i.e. in the case of of the operator M_α, $0\leqslant\alpha<n$ on $\mathbb{R}^n$ with Euclidean distance, sufficient conditions on for the boundedness of this operator in generalized Morrey spaces $\mathscr{M}_{p,\varphi}(\mathbb{R}^n)$ have been obtained in [2, 5, 15, 24].

LEMMA 4.1. *Suppose that $0<\alpha<\gamma$ and the function $\Omega\in L_{\frac{\gamma}{\gamma-\alpha}}(S_\rho)$ is A_t-homogeneous of degree zero. Let $1\leqslant p<\frac{\gamma}{\alpha}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{\gamma}$. Then for any ball $\mathscr{E}=\mathscr{E}(x,r)$ in $\mathbb{R}^n$ and $f\in L^{\rm loc}_p(\mathbb{R}^n)$ there hold the inequalities*

$$\|M^P_{\Omega,\alpha}f\|_{L_q(\mathscr{E}(x,r))}\lesssim \|f\|_{L_p(\mathscr{E}(x,2kr))}+r^{\frac{\gamma}{q}}\sup_{t>2kr}t^{-\gamma+\alpha}\|\Omega(x-\cdot)f(\cdot)\|_{L_1(\mathscr{E}(x,t))},\quad p>1,$$

$$\|M^P_{\Omega,\alpha}f\|_{WL_q(\mathscr{E}(x,r))} \lesssim \|f\|_{L_1(\mathscr{E}(x,2kr))} + r^{\frac{\gamma}{q}} \sup_{t>2kr} t^{-\gamma+\alpha}\|\Omega(x-\cdot)f(\cdot)\|_{L_1(\mathscr{E}(x,t))}, \quad p=1. \tag{4.1}$$

Proof. Given a ball $\mathscr{E}=\mathscr{E}(x,r)$, we split the function f as $f=f_1+f_2$, where $f_1=f\chi_{\mathscr{E}(x,2kr)}$ and $f_2=f\chi_{\complement_{(\mathscr{E}(x,2kr))}}$, and then

$$\|M^P_{\Omega,\alpha}f\|_{L_q(\mathscr{E})} \leqslant \|M^P_{\Omega,\alpha}f_1\|_{L_q(\mathscr{E})} + \|M^P_{\Omega,\alpha}f_2\|_{L_q(\mathscr{E})}.$$

Let $p>1$. By Corollary 3.1

$$\|M^P_{\Omega,\alpha}f_1\|_{L_q(\mathscr{E})} \lesssim \|f\|_{L_p(\mathscr{E}(x,2kr))}.$$

To estimate $M^P_{\Omega,\alpha}f_2(y)$, observe that if $\mathscr{E}(y,t)\cap{}^{\complement}(\mathscr{E}(x,2kr))\neq\emptyset$, where $y\in\mathscr{E}$, then $t>r$. Indeed, if $z\in\mathscr{E}(y,t)\cap{}^{\complement}(\mathscr{E}(x,2kr))$, then $t>\rho(y-z)\geqslant\frac{1}{k}\rho(x-z)-\rho(x-y)>2r-r=r$.

On the other hand, $\mathscr{E}(y,t)\cap{}^{\complement}(\mathscr{E}(x,2kr))\subset\mathscr{E}(x,2kt)$. Indeed, for $z\in\mathscr{E}(y,t)\cap{}^{\complement}(\mathscr{E}(x,2kr))$ we get $\rho(x-z)\leqslant k\rho(y-z)+k\rho(x-y)<k(t+r)<2kt$.

Hence

$$\begin{aligned}
M^P_{\Omega,\alpha}f_2(y) &= \sup_{t>0}\frac{1}{|\mathscr{E}(y,t)|^{1-\alpha/\gamma}}\int_{\mathscr{E}(y,t)\cap{}^{\complement}(\mathscr{E}(x,2kr))}|f(z)||\Omega(x-z)|dz\\
&\leqslant (2k)^{\gamma-\alpha}\sup_{t>r}\frac{1}{|\mathscr{E}(x,2kt)|^{1-\alpha/\gamma}}\int_{\mathscr{E}(x,2kt)}|f(z)||\Omega(x-z)|dz\\
&= (2k)^{\gamma-\alpha}\sup_{t>2kr}\frac{1}{|\mathscr{E}(x,t)|^{1-\alpha/\gamma}}\int_{\mathscr{E}(x,t)}|f(z)||\Omega(x-z)|dz.
\end{aligned}$$

Therefore, for all $y\in\mathscr{E}$ we have

$$M^P_{\Omega,\alpha}f_2(y) \leqslant (2k)^{\gamma-\alpha}\sup_{t>2kr}\frac{1}{|\mathscr{E}(x,t)|^{1-\alpha/\gamma}}\int_{\mathscr{E}(x,t)}|f(z)||\Omega(x-z)|dz. \tag{4.2}$$

Thus

$$\|M^P_{\Omega,\alpha}f\|_{L_q(\mathscr{E})} \lesssim \|f\|_{L_p(\mathscr{E}(x,2kr))} + |\mathscr{E}|^{\frac{1}{q}}\sup_{t>2kr}\frac{1}{|\mathscr{E}(x,t)|^{1-\alpha/\gamma}}\int_{\mathscr{E}(x,t)}|f(z)||\Omega(x-z)|dz.$$

Let $p=1$. We have

$$\|M^P_{\Omega,\alpha}f\|_{WL_q(\mathscr{E})} \leqslant \|M^P_{\Omega,\alpha}f_1\|_{WL_q(\mathscr{E})} + \|M^P_{\Omega,\alpha}f_2\|_{WL_q(\mathscr{E})}.$$

By Corollary 3.1 we get

$$\|M^P_{\Omega,\alpha}f_1\|_{WL_q(\mathscr{E})} \lesssim \|f\|_{L_1(\mathscr{E}(x,2kr))}.$$

Then by (4.2) we arrive at (4.1) and complete the proof. □

Similarly to Lemma 4.1 and Theorem 3.1 the following lemma may be proved.

LEMMA 4.2. *Let the function $\Omega \in L_s(S_\rho)$, $1 < s \leqslant \infty$, be A_t-homogeneous of degree zero. Then for $p > s'$ and any ball $\mathscr{E} = \mathscr{E}(x,r)$ the inequality*

$$\|M_\Omega^P f\|_{L_p(\mathscr{E}(x,r))} \lesssim \|f\|_{L_p(\mathscr{E}(x,2kr))} + r^{\frac{\gamma}{p}} \sup_{t>2kr} t^{-\gamma} \|\Omega(x-\cdot)f(\cdot)\|_{L_1(\mathscr{E}(x,t))}$$

holds for all $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$.

LEMMA 4.3. *Suppose that the function $\Omega \in L_{\frac{\gamma}{\gamma-\alpha}}(S_\rho)$ is A_t-homogeneous of degree zero. Let $0 < \alpha < \gamma$, $1 \leqslant p < \frac{\gamma}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\gamma}$. Then for $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ there hold the inequalities*

$$\|M_{\Omega,\alpha}^P f\|_{L_q(\mathscr{E}(x,r))} \lesssim r^{\frac{\gamma}{q}} \sup_{t>2kr} t^{-\frac{\gamma}{q}} \|f\|_{L_p(\mathscr{E}(x,t))}, \quad p > 1, \tag{4.3}$$

$$\|M_{\Omega,\alpha}^P f\|_{WL_q(\mathscr{E}(x,r))} \lesssim r^{\frac{\gamma}{q}} \sup_{t>2kr} t^{-\frac{\gamma}{q}} \|f\|_{L_1(\mathscr{E}(x,t))}, \quad p = 1. \tag{4.4}$$

Proof. Let $p > 1$ Denote

$$\mathscr{A}_1 := |\mathscr{E}|^{\frac{1}{q}} \left(\sup_{t>2kr} \frac{1}{|\mathscr{E}(x,t)|^{1-\alpha/\gamma}} \int_{\mathscr{E}(x,t)} |f(z)||\Omega(x-z)|dz \right),$$
$$\mathscr{A}_2 := \|f\|_{L_p(\mathscr{E}(x,2kr))}.$$

Applying Hölder's inequality, we get

$$\mathscr{A}_1 \lesssim |\mathscr{E}|^{\frac{1}{q}} \sup_{t>2kr} \|f\|_{L_p(\mathscr{E}(x,t))} \|\Omega(x-\cdot)\|_{L_{\frac{\gamma}{\gamma-\alpha}}(\mathscr{E}(x,t))} |\mathscr{E}(x,t)|^{\frac{\alpha}{\gamma}-\frac{1}{p}}$$
$$\lesssim |\mathscr{E}|^{\frac{1}{q}} \sup_{t>2kr} |\mathscr{E}(x,t)|^{-\frac{1}{q}} \|f\|_{L_p(\mathscr{E}(x,t))}.$$

On the other hand,

$$|\mathscr{E}|^{\frac{1}{q}} \sup_{t>2kr} |\mathscr{E}(x,t)|^{-\frac{1}{q}} \|f\|_{L_p(\mathscr{E}(x,t))}$$
$$\gtrsim |\mathscr{E}|^{\frac{1}{q}} \sup_{t>2kr} |\mathscr{E}(x,t)|^{-\frac{1}{q}} \|f\|_{L_p(\mathscr{E}(x,2kr))} \approx \mathscr{A}_2.$$

Since $\|M_{\Omega,\alpha}^P f\|_{L_q(\mathscr{E})} \leqslant \mathscr{A}_1 + \mathscr{A}_2$, by Lemma 4.1, we arrive at (4.3).

Let $p = 1$. The inequality (4.4) directly follows from (4.1). □

Similarly to Lemma 4.3 and Theorem 3.1 the following lemma is also proved.

LEMMA 4.4. *Suppose that the function $\Omega \in L_s(S_\rho)$, $1 < s \leqslant \infty$, is A_t-homogeneous of degree zero. Then for $p > s'$ and any ball $\mathscr{E} = \mathscr{E}(x,r)$, the inequality*

$$\|M_\Omega^P f\|_{L_p(\mathscr{E}(x,r))} \lesssim r^{\frac{\gamma}{q}} \sup_{t>2kr} t^{-\frac{\gamma}{p}} \|f\|_{L_p(\mathscr{E}(x,t))}$$

holds for $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$.

THEOREM 4.1. *Suppose that* $0<\alpha<\gamma$ *and the function* $\Omega \in L_{\frac{\gamma}{\gamma-\alpha}}(S_\rho)$ *is* A_t*-homogeneous of degree zero. Let* $1 \leqslant p < \frac{\gamma}{\alpha}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{\gamma}$, *and* (φ_1,φ_2) *satisfy the condition*

$$\sup_{r<t<\infty} t^{\alpha-\frac{\gamma}{p}} \operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau)\,\tau^{\frac{\gamma}{p}} \leqslant C\,\varphi_2(x,r), \tag{4.5}$$

where C *does not depend on* x *and* r. *Then the operator* $M^P_{\Omega,\alpha}$ *is bounded from* $\mathscr{M}_{p,\varphi_1,P}(\mathbb{R}^n)$ *to* $\mathscr{M}_{q,\varphi_2,P}(\mathbb{R}^n)$ *for* $p>1$ *and from* $\mathscr{M}_{1,\varphi_1,P}(\mathbb{R}^n)$ *to* $W\mathscr{M}_{q,\varphi_2,P}(\mathbb{R}^n)$ *for* $p=1$.

Proof. By Theorem 2.4 and Lemma 4.3 we get

$$\begin{aligned}\|M^P_{\Omega,\alpha}f\|_{\mathscr{M}_{q,\varphi_2,P}} &\lesssim \sup_{x\in\mathbb{R}^n,r>0} \varphi_2(x,r)^{-1} \sup_{t>r} t^{-\frac{\gamma}{q}}\,\|f\|_{L_p(\mathscr{E}(x,t))}\\ &\lesssim \sup_{x\in\mathbb{R}^n,r>0} \varphi_1(x,r)^{-1}\, r^{-\frac{\gamma}{p}}\,\|f\|_{L_p(\mathscr{E}(x,r))} = \|f\|_{\mathscr{M}_{p,\varphi_1,P}},\end{aligned}$$

if $p\in(1,\infty)$ and

$$\begin{aligned}\|M^P_{\Omega,\alpha}f\|_{W\mathscr{M}_{q,\varphi_2,P}} &\lesssim \sup_{x\in\mathbb{R}^n,r>0} \varphi_2(x,r)^{-1} \sup_{t>r} t^{-\frac{\gamma}{q}}\,\|f\|_{L_1(\mathscr{E}(x,t))}\\ &\lesssim \sup_{x\in\mathbb{R}^n,r>0} \varphi_1(x,r)^{-1}\, r^{-\gamma}\,\|f\|_{L_1(\mathscr{E}(x,r))} = \|f\|_{\mathscr{M}_{1,\varphi_1,P}},\end{aligned}$$

if $p=1$. □

In the same way, by means of Lemma 4.4 we can obtain the following theorem.

THEOREM 4.2. *Suppose that the function* $\Omega\in L_s(S_\rho)$, $1<s\leqslant\infty$ *is* A_t*-homogeneous of degree zero. Let* $p>s'$ *and* (φ_1,φ_2) *satisfy the condition*

$$\sup_{r<t<\infty} t^{-\frac{\gamma}{p}} \operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau)\,\tau^{\frac{\gamma}{p}} \leqslant C\,\varphi_2(x,r),$$

where C *does not depend on* x *and* r. *Then the operator* M^P_Ω *is bounded from* $\mathscr{M}_{p,\varphi_1,P}(\mathbb{R}^n)$ *to* $\mathscr{M}_{p,\varphi_2,P}(\mathbb{R}^n)$.

5. Parabolic fractional integral operator with rough kernels in the spaces $M_{p,\varphi,P}$

5.1. Spanne type result

LEMMA 5.1. *Suppose that* $0<\alpha<\gamma$ *and the function* $\Omega\in L_{\frac{\gamma}{\gamma-\alpha}}(S_\rho)$, *is* A_t*-homogeneous of degree zero. Let* $1\leqslant p<\frac{\gamma}{\alpha}$, *and* $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{\gamma}$. *Then for all* $f\in L^{\rm loc}_p(\mathbb{R}^n)$ *there hold the inequalities*

$$\|I^P_{\Omega,\alpha}f\|_{L_q(\mathscr{E}(x_0,r))} \lesssim r^{\frac{\gamma}{q}} \int_{2kr}^{\infty} t^{-\frac{\gamma}{q}-1}\|f\|_{L_p(\mathscr{E}(x_0,t))}dt,\ \ p>1$$

and

$$\|I^P_{\Omega,\alpha}f\|_{WL_q(\mathscr{E}(x_0,r))} \lesssim r^{\frac{\gamma}{q}} \int_{2kr}^{\infty} t^{-\frac{\gamma}{q}-1}\|f\|_{L_1(\mathscr{E}(x_0,t))}dt,\ \ p=1. \tag{5.1}$$

Proof. For a given ball $\mathscr{E}=\mathscr{E}(x_0,r)$ f, we represent f as

$$f=f_1+f_2, \quad f_1(y)=f(y)\chi_{2k\mathscr{E}}(y), \quad f_2(y)=f(y)\chi_{\complement_{(2k\mathscr{E})}}(y), \quad r>0,$$

and have

$$\|I^P_{\Omega,\alpha}f\|_{L_q(\mathscr{E})} \leqslant \|I^P_{\Omega,\alpha}f_1\|_{L_q(\mathscr{E})}+\|I^P_{\Omega,\alpha}f_2\|_{L_q(\mathscr{E})}.$$

Since $f_1\in L_p(\mathbb{R}^n)$, by the boundedness of $I^P_{\Omega,\alpha}$ from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ it follows that

$$\|I^P_{\Omega,\alpha}f_1\|_{L_q(\mathscr{E})} \leqslant \|I^P_{\Omega,\alpha}f_1\|_{L_q(\mathbb{R}^n)} \leqslant C\|f_1\|_{L_p(\mathbb{R}^n)} = C\|f\|_{L_p(2k\mathscr{E})}.$$

Observe that the conditions $x\in\mathscr{E}$, $y\in {}^{\complement}(2k\mathscr{E})$ imply

$$\frac{1}{2k}\rho(x_0-y) \leqslant \rho(x-y) \leqslant \frac{3k}{2}\rho(x_0-y).$$

We then get

$$|I^P_{\Omega,\alpha}f_2(x)| \leqslant 2^{\gamma-\alpha}c_1 \int_{\complement_{(2k\mathscr{E})}} \frac{|f(y)||\Omega(x-y)|}{\rho(x_0-y)^{\gamma-\alpha}}dy.$$

By Fubini's theorem we have

$$\begin{aligned}\int_{\complement_{(2k\mathscr{E})}} \frac{|f(y)||\Omega(x-y)|}{\rho(x_0-y)^{\gamma-\alpha}}dy &\approx \int_{\complement_{(2k\mathscr{E})}} |f(y)||\Omega(x-y)| \int_{\rho(x_0-y)}^{\infty} \frac{dt}{t^{\gamma+1-\alpha}}dy \\ &\approx \int_{2kr}^{\infty}\int_{2kr\leqslant\rho(x_0-y)<t} |f(y)||\Omega(x-y)|dy\frac{dt}{t^{\gamma+1-\alpha}} \\ &\lesssim \int_{2kr}^{\infty}\int_{\mathscr{E}(x_0,t)} |f(y)||\Omega(x-y)|dy\frac{dt}{t^{\gamma+1-\alpha}}.\end{aligned}$$

Applying Hölder's inequality with (3.1) taken into account, we get

$$\begin{aligned}&\int_{\complement_{(2k\mathscr{E})}} \frac{|f(y)||\Omega(x-y)|}{\rho(x_0-y)^{\gamma-\alpha}}dy \\ &\lesssim \int_{2kr}^{\infty} \|f\|_{L_p(\mathscr{E}(x_0,t))}\,\|\Omega(x-\cdot)\|_{L_{\frac{\gamma}{\gamma-\alpha}}(\mathscr{E}(x_0,t))}\,|\mathscr{E}(x_0,t)|^{\frac{\alpha}{\gamma}-\frac{1}{p}}\,\frac{dt}{t^{\gamma+1-\alpha}} \\ &\lesssim \int_{2kr}^{\infty} \|f\|_{L_p(\mathscr{E}(x_0,t))}\,\frac{dt}{t^{\frac{\gamma}{q}+1}}.\end{aligned}$$

Moreover, for all $p\in[1,\infty)$ the inequality

$$\|I^P_{\Omega,\alpha}f_2\|_{L_q(\mathscr{E})} \lesssim r^{\frac{\gamma}{q}} \int_{2kr}^{\infty} \|f\|_{L_p(\mathscr{E}(x_0,t))}\frac{dt}{t^{\frac{\gamma}{q}+1}}. \tag{5.2}$$

is valid. Thus

$$\|I^P_{\Omega,\alpha}f\|_{L_q(\mathscr{E})} \lesssim \|f\|_{L_p(2k\mathscr{E})} + r^{\frac{\gamma}{q}} \int_{2kr}^{\infty} \|f\|_{L_p(\mathscr{E}(x_0,t))}\frac{dt}{t^{\frac{\gamma}{q}+1}}.$$

On the other hand,

$$\|f\|_{L_p(2k\mathscr{E})} \approx r^{\frac{\gamma}{q}} \|f\|_{L_p(2k\mathscr{E})} \int_{2kr}^{\infty} \frac{dt}{t^{\frac{\gamma}{q}+1}}$$
$$\leqslant r^{\frac{\gamma}{q}} \int_{2kr}^{\infty} \|f\|_{L_p(\mathscr{E}(x_0,t))} \frac{dt}{t^{\frac{\gamma}{q}+1}}. \tag{5.3}$$

Thus

$$\|I^P_{\Omega,\alpha} f\|_{L_q(\mathscr{E})} \lesssim r^{\frac{\gamma}{q}} \int_{2kr}^{\infty} \|f\|_{L_p(\mathscr{E}(x_0,t))} \frac{dt}{t^{\frac{\gamma}{q}+1}}.$$

Finally, in the case $p=1$ by the weak $(1,q)$-boundedness of $I^P_{\Omega,\alpha}$ and the inequality (5.3) it follows that

$$\|I^P_{\Omega,\alpha} f_1\|_{WL_q(\mathscr{E})} \leqslant \|I^P_{\Omega,\alpha} f_1\|_{WL_q(\mathbb{R}^n)} \lesssim \|f_1\|_{L_1(\mathbb{R}^n)}$$
$$= \|f\|_{L_1(2k\mathscr{E})} \lesssim r^{\frac{\gamma}{q}} \int_{2kr}^{\infty} \|f\|_{L_1(\mathscr{E}(x_0,t))} \frac{dt}{t^{\frac{\gamma}{q}+1}}. \tag{5.4}$$

Then from (5.2) and (5.4) we get the inequality (5.1). □

THEOREM 5.1. *Suppose that $0<\alpha<\gamma$ and the function $\Omega \in L_{\frac{\gamma}{\gamma-\alpha}}(S_\rho)$ is A_t-homogeneous of degree zero. Let $1 \leqslant p < \frac{\gamma}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\gamma}$, and the pair (φ_1, φ_2) satisfy the condition*

$$\int_r^{\infty} \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau)\tau^{\frac{n}{p}}}{t^{\frac{\gamma}{q}+1}} dt \leqslant C\,\varphi_2(x,r), \tag{5.5}$$

where C does not depend on x and r. Then the operator $I^P_{\Omega,\alpha}$ is bounded from $M_{p,\varphi_1,P}$ to $M_{q,\varphi_2,P}$ for $p>1$ and from $M_{1,\varphi_1,P}$ to $WM_{q,\varphi_2,P}$ for $p=1$.

Proof. By Lemma 5.1 and Theorem 2.5 with $v_2(r) = \varphi_2(x_0,r)^{-1}$, $v_1(r) = \varphi_1(x_0,r)^{-1} r^{-\frac{\gamma}{p}}$ and $w(r) = r^{-\frac{\gamma}{q}}$ we have for $p>1$

$$\|I^P_{\Omega,\alpha} f\|_{M_{q,\varphi_2,P}} \lesssim \sup_{r>0} \varphi_2(x_0,r)^{-1} \int_r^{\infty} \|f\|_{L_p(\mathscr{E}(x_0,t))} \frac{dt}{t^{\frac{\gamma}{q}+1}}$$
$$\lesssim \sup_{r>0} \varphi_1(x_0,r)^{-1} r^{-\frac{\gamma}{p}} \|f\|_{L_p(\mathscr{E}(x_0,r))} = \|f\|_{M_{p,\varphi_1,P}}$$

and for $p=1$

$$\|I^P_{\Omega,\alpha} f\|_{WM_{q,\varphi_2,P}} \lesssim \sup_{r>0} \varphi_2(x_0,r)^{-1} \int_r^{\infty} \|f\|_{L_1(\mathscr{E}(x_0,t))} \frac{dt}{t^{\frac{\gamma}{q}+1}}$$
$$\lesssim \sup_{r>0} \varphi_1(x_0,r)^{-1} r^{-\gamma} \|f\|_{L_p(\mathscr{E}(x_0,r))} = \|f\|_{M_{1,\varphi_1,P}}. \quad \square$$

REMARK 5.2. Note that, in the case $\Omega \equiv 1$ and $P=I$ Theorem 5.1 was proved in [17]. Also in the case $P=I$ Theorem 5.1 was proved in [18]. The condition (5.5) in Theorem 5.1 is weaker than condition (2.4) in Theorem 2.3 (see [17]).

REMARK 5.3. The condition (4.5) is weaker than (5.5). Indeed, (5.5) implies (4.5):

$$\begin{aligned}\varphi_2(x,r) &\gtrsim \int_r^\infty \frac{\operatorname*{ess\,inf}_{t<\tau<\infty}\varphi_1(x,\tau)\tau^{\frac{\gamma}{p}}}{t^{\frac{\gamma}{q}+1}}dt \\ &\gtrsim \int_s^\infty \frac{\operatorname*{ess\,inf}_{t<\tau<\infty}\varphi_1(x,\tau)\tau^{\frac{\gamma}{p}}}{t^{\frac{\gamma}{q}+1}}dt \\ &\gtrsim \operatorname*{ess\,inf}_{s<\tau<\infty}\varphi_1(x,\tau)\tau^{\frac{\gamma}{p}}\int_s^\infty t^{-\frac{\gamma}{q}-1}dt \\ &\approx \frac{\operatorname*{ess\,inf}_{s<\tau<\infty}\varphi_1(x,\tau)\tau^{\frac{\gamma}{p}}}{t^{\frac{\gamma}{q}}},\end{aligned}$$

where we took $s\in(r,\infty)$, so that

$$\sup_{s>r}\frac{\operatorname*{ess\,inf}_{s<\tau<\infty}\varphi_1(x,\tau)\tau^{\frac{\gamma}{p}}}{t^{\frac{\gamma}{q}}} \lesssim \varphi_2(x,r).$$

On the other hand the functions $\varphi_1(x,t)=t^{-\alpha}$ and $\varphi_2(x,t)=1$ satisfy the condition (4.5), but do not satisfy the condition (5.5).

5.2. Adams type result

The following is a result of Adams type ([1]) for the fractional integral operator.

THEOREM 5.2. *Suppose that the function $\Omega\in L_s(S_\rho)$, $1<s\leqslant\infty$ is A_t-homogeneous of degree zero. Let $s'<p<q<\infty$, $0<\alpha<\frac{\gamma}{p}$ and let $\varphi(x,t)$ satisfy the conditions*

$$\sup_{r<t<\infty} t^{-\gamma}\operatorname*{ess\,inf}_{t<\tau<\infty}\varphi(x,\tau)\,\tau^{\gamma}\leqslant C\,\varphi(x,r) \tag{5.6}$$

and

$$\int_r^\infty t^{\alpha-1}\varphi(x,t)^{\frac{1}{p}}dt\leqslant Cr^{-\frac{\alpha p}{q-p}}, \tag{5.7}$$

where C does not depend on $x\in\mathbb{R}^n$ and $r>0$. Then the operator $I^P_{\Omega,\alpha}$ is bounded from $\mathscr{M}_{p,\varphi^{\frac{1}{p}},P}(\mathbb{R}^n)$ to $\mathscr{M}_{q,\varphi^{\frac{1}{q}},P}(\mathbb{R}^n)$.

Proof. Let $f\in\mathscr{M}_{p,\varphi^{\frac{1}{p}},P}(\mathbb{R}^n)$. Write $f=f_1+f_2$, where $f_1=f\chi_{\mathscr{E}(x,2kr)}$ and $f_2=f\chi_{\complement_{(\mathscr{E}(x,2kr))}}$.

For $I^P_{\Omega,\alpha}f_2(y)$ with $y\in\mathscr{E}$ from (5.2) we have

$$\begin{aligned}I^P_{\Omega,\alpha}(f_2)(y) &\lesssim \int_{2kr}^\infty t^{\gamma-\alpha}\int_{\mathscr{E}(x,t)}|f(z)|dz \\ &\lesssim \int_{2kr}^\infty t^{-\frac{\gamma}{q}-1}\|f\|_{L_p(\mathscr{E}(x,t))}dt. \end{aligned}\tag{5.8}$$

Then from (5.8) by the condition (5.7) we get

$$\begin{aligned} I^P_{\Omega,\alpha}f(y) &\lesssim r^\alpha M^P f(y) + \int_{2kr}^{\infty} t^{-\frac{\gamma}{q}-1}\|f\|_{L_p(\mathscr{E}(x,t))}dt \\ &\leqslant r^\alpha M^P_\Omega f(y) + \|f\|_{\mathscr{M}_{p,\varphi^{\frac{1}{p}},P}} \int_{2kr}^{\infty} t^{\alpha-1}\varphi(x,t)^{\frac{1}{p}}dt \\ &\lesssim r^\alpha M^P_\Omega f(y) + r^{-\frac{\alpha p}{q-p}}\|f\|_{\mathscr{M}_{p,\varphi^{\frac{1}{p}},P}}. \end{aligned}$$

Choose now $r = \left(\frac{\|f\|_{\mathscr{M}_{p,\varphi^{1/p},P}}}{M^P_\Omega f(y)}\right)^{\frac{q-p}{\alpha q}}$ and get

$$I^P_{\Omega,\alpha}f(y) \lesssim (M^P_\Omega f(y))^{\frac{p}{q}}\|f\|^{1-\frac{p}{q}}_{\mathscr{M}_{p,\varphi^{\frac{1}{p}},P}}.$$

Hence the statement of the theorem follows in view of the boundedness of the maximal operator M^P_Ω in $\mathscr{M}_{p,\varphi^{\frac{1}{p}},P}(\mathbb{R}^n)$ provided by Theorem 4.2 in virtue of condition (5.6). Therefore,

$$\begin{aligned} \|I^P_{\Omega,\alpha}f\|_{\mathscr{M}_{q,\varphi^{\frac{1}{q}},P}} &= \sup_{x\in\mathbb{R}^n,\, t>0} \varphi(x,t)^{-\frac{1}{q}}t^{-\frac{\gamma}{q}}\|I^P_{\Omega,\alpha}f\|_{L_q(\mathscr{E}(x,t))} \\ &\lesssim \|f\|^{1-\frac{p}{q}}_{\mathscr{M}_{p,\varphi^{\frac{1}{p}},P}} \sup_{x\in\mathbb{R}^n,\, t>0} \varphi(x,t)^{-\frac{1}{q}}t^{-\frac{\gamma}{q}}\|M^P_\Omega f\|^{\frac{p}{q}}_{L_p(\mathscr{E}(x,t))} \\ &= \|f\|^{1-\frac{p}{q}}_{\mathscr{M}_{p,\varphi^{\frac{1}{p}},P}} \left(\sup_{x\in\mathbb{R}^n,\, t>0} \varphi(x,t)^{-\frac{1}{p}}t^{-\frac{\gamma}{p}}\|M^P_\Omega f\|_{L_p(\mathscr{E}(x,t))}\right)^{\frac{p}{q}} \\ &= \|f\|^{1-\frac{p}{q}}_{\mathscr{M}_{p,\varphi^{\frac{1}{p}},P}} \|M^P_\Omega f\|^{\frac{p}{q}}_{\mathscr{M}_{p,\varphi^{\frac{1}{p}},P}} \\ &\lesssim \|f\|_{\mathscr{M}_{p,\varphi^{\frac{1}{p}},P}}. \quad \square \end{aligned}$$

In the case $\varphi(x,r) = r^{\lambda-\gamma}$, $0<\lambda<\gamma$ from Theorem 5.2 we get the following Adams type result [1] for the parabolic fractional maximal and integral operators with rough kernels.

COROLLARY 5.2. *Suppose that the function $\Omega \in L_s(S_\rho)$, $1<s\leqslant\infty$ is A_t-homogeneous of degree zero. Let $0<\alpha<\gamma$, $s'<p<q<\infty$, $0<\lambda<\gamma-\alpha p$ and $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{\gamma-\lambda}$. Then the operators $M^P_{\Omega,\alpha}$ and $I^P_{\Omega,\alpha}$ are bounded from $\mathscr{M}_{p,\lambda,P}(\mathbb{R}^n)$ to $\mathscr{M}_{q,\lambda,P}(\mathbb{R}^n)$.*

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