HARMONIC POLYNOMIALS AND GENERALIZATIONS OF
OSTROWSKI–GRÜSS TYPE INEQUALITY AND TAYLOR FORMULA

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Abstract. Some generalizations of Ostrowski-Grüss type inequality and Taylor formula are given, by using harmonic sequences of polynomials. We use inequalities for the Čebyšev functional in terms of the first derivative (see [6]), for some new bounds for the remainders.

1. Introduction

Let the polynomials $P_k(t)$, $k \geq 0$ satisfy the following condition

$$P'_k(t) = P_{k-1}(t), \quad k \geq 1; \quad P_0(t) = 1. \tag{1.1}$$

For a sequence $(P_k(t), k \geq 0)$ of polynomials satisfying the condition (1.1), we say that it is a harmonic sequence of polynomials. From (1.1), by an easy induction it follows that every harmonic sequence of polynomials must be of the form

$$P_k(t) = \sum_{i=0}^{k} \frac{c_i}{(k-i)!} t^{k-i}, \quad k \geq 0,$$

where $(c_k, k \geq 0)$ is a sequence of real numbers such that $c_0 = 1$. In fact, $c_k = P_k(0)$, $k \geq 0$. Especially, we have $P_0(t) = 1$, $P_1(t) = t + c_1$, $P_2(t) = \frac{1}{2} t^2 + c_1 t + c_2$.

**Example 1.** For fixed $x \in \mathbb{R}$ define

$$P_k(t) = \frac{1}{k!} (t-x)^k, \quad k \geq 0.$$ 

Then $(P_k(t), k \geq 0)$ is a harmonic sequence of polynomials.

**Example 2.** Similarly, for fixed $x \in \mathbb{R}$ define

$$P_k(t) = \frac{1}{k!} \left( t - \frac{a+x}{2} \right)^k, \quad k \geq 0.$$ 

Then $(P_k(t), k \geq 0)$ is also a harmonic sequence of polynomials.
EXAMPLE 3. Here we have the well known Bernoulli polynomials $B_k(t)$. These polynomials can be defined by the expansion

$$\frac{xe^{tx}}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k(t) x^k}{k!}, \quad |x| < 2\pi, \quad t \in \mathbb{R}.$$  

We have

$$B_0(t) = 1, \quad B_1(t) = t - \frac{1}{2}, \quad B_2(t) = t^2 - t + \frac{1}{6}, \quad B_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t, \cdots.$$  

The numbers $B_k := B_k(0)$ are called Bernoulli numbers. The polynomials $B_k(t)$ and the numbers $B_k$ have many interesting properties. It can be shown that the polynomials $B_k(t)$ are uniquely determined by the following two properties ([1], 23.1.5 and 23.1.6):

$$B'_k(t) = kB_k - \frac{1}{t}, \quad k \in \mathbb{N}; \quad B_0(t) = 1 \quad (1.2)$$

and

$$B_k(t+1) - B_k(t) = kt^{k-1}, \quad k \in \mathbb{N}. \quad (1.3)$$

Let $P_k(t) = \frac{1}{k!}B_k(t), \quad k \geq 0$, then from (1.1) it follows that $(P_k(t), k \geq 0)$ is harmonic sequence of polynomials.

EXAMPLE 4. Instead of Bernoulli polynomials $B_k(t)$ we can have Euler polynomials $E_k(t)$ which have the properties similar to those of Bernoulli polynomials. Euler polynomials can be defined by the expansion

$$\frac{2e^{tx}}{e^x + 1} = \sum_{n=0}^{\infty} \frac{E_k(t) x^k}{k!}, \quad |x| < \pi, \quad t \in \mathbb{R}.$$  

We have

$$E_0(t) = 1, \quad E_1(t) = t - \frac{1}{2}, \quad E_2(t) = t^2 - t, \quad E_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{4} \cdots.$$  

It can be shown that the polynomials $E_k(t)$ are uniquely determined by the following two properties ([1], 23.1.5 and 23.1.6):

$$E'_k(t) = kE_{k-1}(t), \quad k \in \mathbb{N}; \quad E_0(t) = 1 \quad (1.4)$$

and

$$E_k(t+1) + E_k(t) = 2t^k, \quad k \in \mathbb{N}. \quad (1.5)$$

$P_k = \frac{E_k(t)}{k!}, \quad k \geq 0$ is harmonic sequence of polynomials.

REMARK 1. In [4] Cerone defined polynomials $P_k(t)$ as

$$P'_k(t) = \xi_k P_{k-1}(t), \quad P_0(t) = 1, \quad t \in \mathbb{R}. \quad (1.6)$$

When $\xi_k = k$, then such functions satisfying (1.6) were defined by Appell in 1980, [3] and are known as Appell polynomials. For $\xi_k = 1$ we have harmonic polynomials. Polynomials satisfying (1.6) will be termed Appell-like polynomials.
For two Lebesgue integrable functions $f, g : [a, b] \to \mathbb{R}$, consider the Čebyšev functional:

$$T(f, g) := \frac{1}{b - a} \int_a^b f(t)g(t)dt - \frac{1}{b - a} \int_a^b f(t)dt \cdot \frac{1}{b - a} \int_a^b g(t)dt. \quad (1.7)$$

In [6] the authors proved the following theorems:

**THEOREM 1.** Let $f, g : [a, b] \to \mathbb{R}$ be two absolutely continuous functions on $[a, b]$ with

$$(\cdot - a)(b - \cdot)(f')^2, (\cdot - a)(b - \cdot)(g')^2 \in L[a, b].$$

Then we have the inequality

$$|T(f, g)| \leq \frac{1}{\sqrt{2}} |T(f, f)|^{\frac{1}{2}} \frac{1}{\sqrt{b - a}} \left( \int_a^b (x - a)(b - x) [g'(x)]^2 dx \right)^{\frac{1}{2}} \quad (1.8)$$

$$\leq \frac{1}{2(b - a)} \left( \int_a^b (x - a)(b - x) [f'(x)]^2 dx \right)^{\frac{1}{2}} \times \left( \int_a^b (x - a)(b - x) [g'(x)]^2 dx \right)^{\frac{1}{2}}.$$

The constant $\frac{1}{\sqrt{2}}$ and $\frac{1}{2}$ are best possible in (1.8).

**THEOREM 2.** Assume that $g : [a, b] \to \mathbb{R}$ is monotonic nondecreasing on $[a, b]$ and $f : [a, b] \to \mathbb{R}$ is absolutely continuous with $f' \in L_\infty[a, b]$. Then we have the inequality

$$|T(f, g)| \leq \frac{1}{2(b - a)} ||f'||_\infty \int_a^b (x - a)(b - x)dg(x). \quad (1.9)$$

The constant $\frac{1}{2}$ is best possible.

In this paper we will show some generalizations of inequalities of Ostrowski-Grüss type and generalizations of Taylor’s formula using sequences of harmonic polynomials. We will use the above theorems to get some new bounds for the remainders.

2. **On some identities related to Ostrowski inequality**

The well-known Ostrowski inequality (see, for example [18]) states that if $f \in C^1([a, b]), x \in [a, b]$, then

$$\left| \frac{1}{b - a} \int_a^b f(y)dy - f(x) \right| \leq \frac{1}{4} + \frac{(x - (a + b)/2)^2}{(b - a)^2} ||f'||_\infty. \quad (2.1)$$

G. V. Milovanović and J. Pečarić in [17] and A. M. Fink in [14] (see also [18], p. 470) have considered generalizations of Ostrowski inequality in the form

$$\left| \frac{1}{n} \left[ f(x) + \sum_{k=1}^{n-1} F_k(x) \right] - \frac{1}{b - a} \int_a^b f(t)dt \right| \leq K(n, p, x)||f^{(n)}||_p, \quad (2.2)$$
where \( F_k(x) \) is defined by

\[
F_k(x) = \frac{n-k}{k!(b-a)} \left[ f^{(k-1)}(a)(x-a)^k - f^{(k-1)}(b)(x-b)^k \right].
\] (2.3)

For \( n = 1 \) the sum above is defined to be zero.

In fact, G. V. Milovanović and J. Pečarić have proved that ([18]):

\[
K(n, \infty, x) = \frac{(x-a)^{n+1} + (b-x)^{n+1}}{n(n+1)!(b-a)},
\] (2.4)

while A. M. Fink proved that

\[
K(n, p, x) = \frac{([x-a]^{np'+1} + [b-x]^{np'+1})^{1/p'}}{n!(b-a)} B((n-1)p'+1, p'+1)^{1/p'},
\] (2.5)

where \( 1 < p \leq \infty \), \( 1/p + 1/p' = 1 \), \( B \) is the beta function, and

\[
K(n, 1, x) = \frac{(n-1)^{n-1}}{n^n n!(b-a)} \max([x-a]^n, (b-x)^n].
\] (2.6)

In [7], the authors gave some generalizations of previous results:

Let \( (P_n, n \geq 0) \) be a harmonic sequence of polynomials. Furthermore, let \( I \subset \mathbb{R} \) be a segment and \( f : I \to \mathbb{R} \) such that \( f^{(n-1)} \) is Lipschitzian or has bounded variation on \( I \), for some \( n \geq 1 \). Then, using notations

\[
\tilde{F}_k := \frac{(-1)^k(n-k)}{b-a} \left[ P_k(a)f^{(k-1)}(a) - P_k(b)f^{(k-1)}(b) \right]
\]

and

\[
k(t, x) = \begin{cases} t - a, & t \in [a, x] \\ t - b, & t \in (x, b) \end{cases}
\]

the following identity holds:

\[
\frac{1}{n} \left[ f(x) + \sum_{k=1}^{n-1} (-1)^k P_k(x)f^{(k)}(x) + \sum_{k=1}^{n-1} \tilde{F}_k \right] - \frac{1}{b-a} \int_a^b f(t)dt = \frac{(-1)^{n-1}}{n(b-a)} \int_a^b P_{n-1}(t)k(t, x)df^{(n-1)}(t).
\] (2.7)

The sums above are defined to be zero for \( n = 1 \).

For the harmonic sequence of polynomials from Example 1 relation (2.7) becomes

\[
\frac{1}{n} \left[ f(x) + \sum_{k=1}^{n-1} F_k(x) \right] - \frac{1}{b-a} \int_a^b f(t)dt = \frac{1}{n!(b-a)} \int_a^b (x-t)^{n-1}k(t, x)df^{(n-1)}(t),
\] (2.8)
where $F_k(x)$ is defined by (2.3).

Ostrowski inequality has been also generalized by Anastassiou [2]. He proved that if $f \in C^{n+1}([a,b])$ for $n \in \mathbb{N}$ and $x \in [a, b]$ fixed, then

$$
\left| \frac{1}{b-a} \int_a^b f(y)dy - f(x) \right| \leq \frac{1}{b-a} \left[ \sum_{k=1}^n \left| \frac{f^{(k)}(x)}{(k+1)!} \right| (b-x)^{k+1} - (a-x)^{k+1} \right] + \frac{\|f^{(n+1)}\|_{\infty}}{(n+2)!} ((x-a)^{n+2} + (b-x)^{n+2})
$$

This reduces to the Ostrowski inequality in the extreme case $n = 0$, when the sum becomes empty and so vanishes identically.

Another form of this has been obtained by Cerone, Dragomir and Roumeliotis [5], who have shown that

$$
\left| \int_a^b f(t)dt - \sum_{k=0}^{n-1} \frac{(b-x)^{k+1} + (-1)^k(b-a)^{k+1}}{(k+1)!} f^{(k)}(x) \right|
\leq \frac{\|f^{(n)}\|_{\infty}}{(n+1)!} ((x-a)^{n+1} + (b-x)^{n+1}) \leq \frac{\|f^{(n)}\|_{\infty}(b-a)^{n+1}}{(n+1)!}.
$$

In [19] the authors gave further generalizations of the above results. Let the sequel $(P_n(x), (Q_n(x))$ denote sequences of harmonic polynomials. Set

$$S_n(t,x) := \begin{cases} P_n(t), & t \in [a,x] \\ Q_n(t), & t \in (x,b]. \end{cases}
$$

Then,

$$(-1)^n \int_a^b S_n(t,x)df^{(n-1)}(t) = I_n(x),$$

where

$$I_n(x) := \sum_{k=1}^n (-1)^k \left[ Q_k(b)f^{(k-1)}(b) + (P_k(x) - Q_k(x))f^{(k-1)}(x) - P_k(a)f^{(k-1)}(a) \right] + \int_a^b f(t)dt.
$$

With the convention that the empty sum represents zero, the same definition gives $I_0(x) = \int_a^b f(t)dt$.

The special case (see Example 1)

$$P_n(t) := (t-a)^n/n! \quad Q_n(t) := (t-b)^n/n!,
$$

arises in the literature. The corresponding values of $I_n(x)$ for these choices we denote by $I_n^n(x)$, so that

$$I_n^n(x) = \int_a^b f(t)dt + \sum_{k=1}^n \frac{(-1)^k}{k!} \left[ (x-a)^k - (x-b)^k \right] f^{(k-1)}(x).$$
In following (see [21]) we will show a generalizations of the above results using more than two sequences of harmonic polynomials.

Let \( \sigma = \{a = x_0 < x_1 < \ldots x_{n-1} < x_n = b\} \) be a subdivision of the interval \([a, b]\).

Set
\[
S_m(t, \sigma) = \begin{cases} 
P_{1m}(t), & t \in [a, x_1] \\
P_{2m}(t), & t \in (x_1, x_2] \\
\vdots \\
P_{nm}(t), & t \in (x_{n-1}, x_n],
\end{cases}
\]
where \((P_{jm})_m\) are sequences of harmonic polynomials. By successive integration by parts we have whenever the integrals exist that
\[
(-1)^m \int_a^b S_m(t, \sigma) f^{(m-1)}(t) \, dt = \int_a^b f(t) \, dt + \sum_{k=1}^m (-1)^k \left[ P_{nk}(b) f^{(k-1)}(b) + \sum_{j=1}^{n-1} (P_{jk}(x_j) - P_{j+1,k}(x_j)) f^{(k-1)}(x_j) - P_{1k}(a) f^{(k-1)}(a) \right].
\] (2.13)

The right-hand side of the previous identity is denoted by \( I_m(\sigma) \).

3. Generalizations of inequalities of Ostrowski-Grüss type

There have been several extensions of the Ostrowski inequality. It has been shown in [15] that if \( f \) is differentiable on \((a, b)\) and \( f' \) is integrable and satisfies
\[
\gamma \leq f'(t) \leq \Gamma \text{ for all } t \in [a, b],
\]
then
\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt - \frac{f(b) - f(a)}{b-a} \left( x - \frac{a+b}{2} \right) \right| \leq \frac{1}{4\sqrt{3}}(b-a)(\Gamma - \gamma),
\] (3.1)
for all \( x \in [a, b] \). A version of this estimate occurs in [10], but without the \( \sqrt{3} \) on the right.

Here we give generalizations of above result (see [20]) using sequences of harmonic polynomials defining in previous section by (2.10).

THEOREM 3. Suppose \( f : [a, b] \to \mathbb{R} \) is such that \( f^{(n)} \) is integrable with
\[
\gamma \leq f^{(n)}(t) \leq \Gamma \text{ for all } t \in [a, b].
\]

Put
\[
U_n(x) := [Q_{n+1}(b) - Q_{n+1}(x) + P_{n+1}(x) - P_{n+1}(a)]/(b-a).
\]
Then for any \( x \in [a, b] \),
\[
\left| I_n(x) - (-1)^n U_n(x) \left[ f^{(n-1)}(b) - f^{(n-1)}(a) \right] \right| \leq \frac{1}{2} K(\Gamma - \gamma)(b-a),
\]
where

\[ K := \left\{ \frac{1}{b-a} \left[ \int_a^x P_n^2(t) \, dt + \int_x^b Q_n^2(t) \, dt \right] - [U_n(x)]^2 \right\}^{1/2}. \]

**COROLLARY 1.** Under the assumptions of Theorem 3,

\[
\left| \frac{1}{b-a} \int_a^b f(t) \, dt + \sum_{k=1}^n \frac{(-1)^k}{k!(b-a)} [(b-B)^k f^{(k-1)}(b) + ((x-A)^k - (x-B)^k) f^{(k-1)}(x) - (a-A)^k f^{(k-1)}(a)] - \frac{(-1)^n (f^{(n-1)}(b) - f^{(n-1)}(a))}{(n+1)! (b-a)^2} [(b-B)^{n+1} - (x-B)^{n+1}] \right| \leq \frac{1}{2} (\Gamma - \gamma) K_1
\]

for all \( x \in [a, b] \) and \( A, B \in \mathbb{R} \), where

\[ K_1 := \frac{1}{n!} \left( \frac{(x-A)^{2n+1} - (a-A)^{2n+1} + (b-B)^{2n+1} - (x-B)^{2n+1}}{(2n+1)(b-a)} \right)^{1/2}. \]

In [11], Dragomir proved some of above results in terms of the Euclidian norm of \( f^{(n)} \) which is valid for a larger class of mappings, i.e., for the mappings \( f \) for which \( f^{(n)} \) is unbounded on \( (a, b) \) but \( f^{(n)} \in L_2[a, b] \).

**THEOREM 4.** Assume that the mapping \( f : [a, b] \to \mathbb{R} \) is such that \( f^{(n-1)} \) is absolutely continuous on \( [a, b] \) and \( f^{(n)} \in L_2[a, b] \), \( (n \geq 1) \). Then we have the inequality

\[
\left| I_n(x) - (-1)^n (b-a) U_n(x) f^{(n-1)}[a, b] \right| \leq K(b-a) \left[ \frac{1}{(b-a)} \|f^{(n)}\|^2 \right]^{1/2} - \left( f^{(n)}[a, b] \right)^{1/2} \]

(3.2)

for all \( x \in [a, b] \), where \( K, \gamma \) and \( \Gamma \) are as in Theorem 3 and

\[ f^{(n-1)}[a, b] = \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a} \]

is the divided difference.

Now, we will apply the Grüss type inequalities from Theorem 1 and Theorem 2 in pointing out different bounds for the remainder for identities (2.7) and identities (2.11).

Using Theorem 1 for identity (2.7) we get the following Grüss type inequality:
THEOREM 5. Let $f : [a, b] \to \mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous and $(P_n, n \geq 0)$ be a harmonic sequence of polynomials. Then we define the remainder

$$F_n(f; a, b)$$

$$= f(x) + \sum_{k=1}^{n-1} (-1)^k P_k(x) f^{(k)}(x) + \sum_{k=1}^{n-1} F_k - \frac{n}{b-a} \int_a^b f(t) dt$$

$$+ \frac{(-1)^n}{b-a} [P_n(x)(b-a) - P_{n+1}(b) + P_{n+1}(a)] f^{(n-1)}[a, b]$$

and $F_n(f; a, b)$ satisfies the estimation

$$|F_n(f; a, b)|$$

$$\leq \frac{1}{\sqrt{2(b-a)}} \left[ \int_a^b (P_{n-1}(t) k(t,x))^2 dt - \frac{[P_n(x)(b-a) - P_{n+1}(b) + P_{n+1}(a)]}{b-a} \right]^{\frac{1}{2}}$$

$$\times \left( \int_a^b (t-a)(b-t) \left[ f^{(n+1)}(t) \right]^2 dt \right)^{\frac{1}{2}}$$

(3.4)

for any $x \in [a, b]$.

Proof. If we apply Theorem 1 for $f \to k(\cdot, x) P_{n-1}$, $g \to f^{(n)}$, we deduce

$$\left| \frac{1}{b-a} \int_a^b P_{n-1}(t) k(t,x) f^{(n)}(t) dt - \frac{1}{b-a} \int_a^b P_{n-1}(t) k(t,x) dt \cdot \frac{1}{b-a} \int_a^b f^{(n)}(t) dt \right|$$

$$\leq \frac{1}{\sqrt{2}} \left[ T(k(\cdot, x) P_{n-1}, k(\cdot, x) P_{n-1}) \right]^{\frac{1}{2}}$$

$$\times \left( \frac{1}{b-a} \left( \int_a^b (t-a)(b-t) \left[ f^{(n+1)}(t) \right]^2 dt \right) \right)^{\frac{1}{2}},$$

(3.5)

where

$$T(k(\cdot, x) P_{n-1}, k(\cdot, x) P_{n-1}) = \frac{1}{b-a} \int_a^b (P_{n-1}(t) k(t,x))^2 dt - \left[ P_n(x) - \frac{P_{n+1}(b) - P_{n+1}(a)}{b-a} \right]^2.$$ 

Using (2.7) and (3.5), we deduce the representation (3.3) and the bound (3.4). □

REMARK 2. Using Theorem 5 for the harmonic sequence of polynomials from Example 1 we get:

$$F_n(f; a, b)$$

$$= f(x) + \sum_{k=1}^{n-1} (-1)^k P_k(x) f^{(k)}(x) + \sum_{k=1}^{n-1} F_k - \frac{n}{b-a} \int_a^b f(t) dt$$

$$+ \frac{1}{(n+1)!(b-a)} [(x-b)^{n+1} - (x-a)^{n+1}] f^{(n-1)}[a, b]$$

(3.6)
and

\[ |F_n(f; a, b)| \leq \frac{1}{\sqrt{2(b-a)}} \left[ \frac{(x-a)^{2n+1} - (x-b)^{2n+1}}{[(n-1)!]^2(2n+1)(2n-1)n} - \frac{1}{b-a} \left[ \frac{(x-a)^{n+1} - (x-b)^{n+1}}{(n+1)!} \right]^2 \right]^{1/2} \times \left( \int_a^b (t-a)(b-t) \left[ f^{(n+1)}(t) \right]^2 dt \right)^{1/2}.

The following Grüss type inequality also holds.

**Theorem 6.** Let \( f : [a, b] \to \mathbb{R} \) be such that \( f^{(n)} \) is absolutely continuous and \( f^{(n+1)} \geq 0 \) on \([a, b]\) and \((P_n, n \geq 0)\) be a harmonic sequence of polynomials. Then we have the representation (3.3) and the remainder \( F_n(f; a, b) \) satisfies the bound

\[ |F_n(f; a, b)| \leq ||k(\cdot, x)P_{n-2} + P_{n-1}||_\infty \left\{ \frac{f^{(n-1)}(a) + f^{(n-1)}(b)}{2} - f^{(n-2)}[a, b] \right\} \quad (3.7) \]

for any \( x \in [a, b] \).

**Proof.** If we apply Theorem 2 for \( f \to k(\cdot, x)P_{n-1}, \ g \to f^{(n)} \), we deduce

\[ \left| \frac{1}{b-a} \int_a^b P_{n-1}(t)k(t,x)f^{(n)}(t)dt - \frac{1}{b-a} \int_a^b P_{n-1}(t)k(t,x)dt \cdot \frac{1}{b-a} \int_a^b f^{(n)}(t)dt \right| \leq \frac{1}{2(b-a)} ||k(\cdot, x)P_{n-2} + P_{n-1}||_\infty \left( \int_a^b (t-a)(b-t)f^{(n+1)}(t)dt \right)^{1/2}. \quad (3.8) \]

Since

\[ \int_a^b (t-a)(b-t)f^{(n+1)}(t)dt = \int_a^b f^{(n)}(t)[2t - (a+b)]dt = (b-a) \left[ f^{(n-1)}(b) + f^{(n-1)}(a) \right] - 2 \left( f^{(n-2)}(b) - f^{(n-2)}(a) \right). \]

Using the representation (3.3) and the inequality (3.8), we deduce (3.7). \( \square \)

**Remark 3.** Using Theorem 6 for the harmonic sequence of polynomials from Example 1 we get:

\[ |F_n(f; a, b)| \leq \frac{1}{(n-2)!} \max_{x \in [a, b]} \{ |b-x|^{n-1}, |x-a|^{n-1} \} \left\{ \frac{f^{(n-1)}(a) + f^{(n-1)}(b)}{2} - f^{(n-2)}[a, b] \right\}. \]

Using Theorem 1 for identity (2.11) we get the following Grüss type inequality:
THEOREM 7. Let $f : [a, b] \to \mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous and $(P_n, n \geq 0)$ and $(Q_n, n \geq 0)$ be harmonic sequences of polynomials as in (2.10). Then we have

$$A_n(f; a, b) = \sum_{k=1}^{n} (-1)^k \left[ Q_k(b) f^{(k-1)}(b) + (P_k(x) - Q_k(x)) f^{(k-1)}(x) \right]$$

$$-P_k(a) f^{(k-1)}(a) + \int_{a}^{b} f(t) dt$$

$$+(-1)^{n-1} [P_{n+1}(x) - Q_{n+1}(x) + Q_{n+1}(b) - P_{n+1}(a)] f^{(n-1)}[a, b]$$

and $A_n(f; a, b)$ satisfies the estimation

$$|A_n(f; a, b)| \leq \frac{1}{\sqrt{2}} \left[ \int_{a}^{b} S_n^2(t) dt - \frac{[P_{n+1}(x) - Q_{n+1}(x) + Q_{n+1}(b) - P_{n+1}(a)]^2}{b - a} \right]^{\frac{1}{2}}$$

$$\times \left( \int_{a}^{b} (t-a)(b-t) \left[ f^{(n+1)}(t) \right]^2 dt \right)^{\frac{1}{2}}$$

(3.10)

for any $x \in [a, b]$.

Proof. If we apply Theorem 1 for $f \to S_{n-1}(\cdot, x)$, $g \to f^{(n)}$, we deduce

$$\left| \frac{1}{b-a} \int_{a}^{b} S_n(t, x) f^{(n)}(t) dt - \frac{1}{b-a} \int_{a}^{b} S_n(t, x) \cdot \frac{1}{b-a} \int_{a}^{b} f^{(n)}(t) dt \right|$$

$$\leq \frac{1}{\sqrt{2}} \left[ T(S_n(\cdot, x), S_n(\cdot, x)) \right]^{\frac{1}{2}}$$

$$\times \frac{1}{\sqrt{b-a}} \left( \int_{a}^{b} (t-a)(b-t) \left[ f^{(n+1)}(t) \right]^2 dt \right)^{\frac{1}{2}}$$

(3.11)

where

$$T(S_n(\cdot, x), S_n(\cdot, x)) = \frac{1}{b-a} \int_{a}^{b} S_n^2(t, x) dt - \frac{[P_{n+1}(x) - Q_{n+1}(x) + Q_{n+1}(b) - P_{n+1}(a)]^2}{b - a}.$$ 

Using (2.11) and (3.11), we deduce the representation (3.9) and the bound (3.10). □

REMARK 4. Using Theorem 7 for the harmonic sequence of polynomials from Example 1 we get:

$$A_n(f; a, b) = \int_{a}^{b} f(t) dt + \sum_{k=1}^{n} \frac{(-1)^k}{k!} \left[ (x-a)^k - (x-b)^k \right] f^{(k-1)}(x)$$

$$+\frac{(-1)^{n-1}}{(n+1)!} \left[ (x-a)^{n+1} - (x-b)^{n+1} \right] f^{(n-1)}[a, b]$$

(3.12)
and

\[
|A_n(f; a, b)| \leq \frac{1}{\sqrt{2}} \left( \frac{(x-a)^{2n+1} - (x-b)^{2n+1}}{(n!)^2(2n+1)} - \frac{1}{b-a} \left[ \frac{(x-a)^n - (x-b)^n}{(n+1)!} \right]^2 \right)^{1/2} \\
\times \left( \int_a^b (t-a)(b-t) \left[ f^{(n+1)}(t) \right]^2 dt \right)^{1/2}.
\]

The following Grüss type inequality also holds.

**Theorem 8.** Let \( f : [a, b] \to \mathbb{R} \) be such that \( f^{(n)} \) is absolutely continuous and \( f^{(n+1)} \geq 0 \) on \([a, b]\) and \((S_n, n \geq 0)\) be a harmonic sequence of polynomials defined with (2.10). Then we have the representation (3.9) and the remainder \( A_n(f; a, b) \) satisfies the bound

\[
|A_n(f; a, b)| \leq (b-a) ||S_{n-1}||_\infty \left\{ \frac{f^{(n-1)}(a) + f^{(n-1)}(b)}{2} - f^{(n-2)}[a, b] \right\}, \quad (3.13)
\]

for any \( x \in [a, b] \).

**Proof.** If we apply Theorem 2 for \( f \to S_n(\cdot , x) \), \( g \to f^{(n)} \), we deduce

\[
\left| \frac{1}{b-a} \int_a^b S_n(t, x) f^{(n)}(t) dt - \frac{1}{b-a} \int_a^b S_n(t, x) dt \cdot \frac{1}{b-a} \int_a^b f^{(n)}(t) dt \right| \\
\leq \frac{1}{2(b-a)} ||S_{n-1}||_\infty \left( \int_a^b (t-a)(b-t) f^{(n+1)}(t) dt \right)^{1/2}. \quad (3.14)
\]

Using the representation (3.9) and the inequality (3.14), we deduce (3.13). \( \square \)

**Remark 5.** Using Theorem 8 for the harmonic sequence of polynomials from Example 1 we get:

\[
|A_n(f; a, b)| \leq \frac{b-a}{(n-1)!} \max_{x \in [a, b]} \{ |b-x|^{n-1}, |x-a|^{n-1} \} \left\{ \frac{f^{(n-1)}(a) + f^{(n-1)}(b)}{2} - f^{(n-2)}[a, b] \right\}.
\]

4. **On generalizations of Ostrowski-Grüss inequality via Euler harmonic identities**

In this section (see [9]) we use Euler identities involving harmonic sequence of polynomials to show some generalizations of Ostrowski inequality.

Assume that \((P_k(t), k \geq 0)\) is a harmonic sequence of polynomials i.e. the sequence of polynomials satisfying the condition (1.1). Define \( P^*_k(t), k \geq 0 \) to be periodic functions of period 1, related to \( P_k(t), k \geq 0 \) as

\[
P^*_k(t) = P_k(t), \quad 0 \leq t < 1, \quad P^*_k(t+1) = P^*_k(t), \quad t \in \mathbb{R}. \quad (4.1)
\]
Thus, $P_0^*(t) = 1$, while for $k \geq 1$, $P_k^*(t)$ is continuous on $\mathbb{R} \setminus \mathbb{Z}$ and has a jump of
\begin{equation}
\alpha_k = P_k(0) - P_k(1)
\end{equation}
at every integer $t$, whenever $\alpha_k \neq 0$. Note that $\alpha_1 = -1$, since $P_1(t) = t + c_1$, for some $c_1 \in \mathbb{R}$. Also, note that from (1.1) it follows
\begin{equation}
P_k^*(t) = P_{k-1}^*(t), \quad k \geq 1, \quad t \in \mathbb{R} \setminus \mathbb{Z}.
\end{equation}

Let $a, b \in \mathbb{R}$, $a < b$ and $f : [a, b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a function of bounded variation on $[a, b]$ for some $n \geq 1$. For every $x \in [a, b]$ and $1 \leq m \leq n$ we introduce the following notations
\begin{equation}
\tilde{T}_m(x) = \sum_{k=1}^{m} (b - a)^{k-1} P_k \left( \frac{x - a}{b - a} \right) \left[ f^{(k-1)}(b) - f^{(k-1)}(a) \right],
\end{equation}
with convention $\tilde{T}_0(x) = 0$, and
\begin{equation}
\tau_m(x) = \sum_{k=2}^{m} (b - a)^{k-1} \alpha_k f^{(k-1)}(x),
\end{equation}
with convention $\tau_1(x) = 0$.

**Theorem 9.** Let $(P_k, k \geq 0)$ be a harmonic sequence of polynomials and $f : [a, b] \to \mathbb{R}$ such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a, b]$ for some $n \geq 1$. Then for every $x \in [a, b]$ \begin{align*}
f(x) &= \frac{1}{b - a} \int_{a}^{b} f(t) dt + \tilde{T}_n(x) + \tau_n(x) + \tilde{R}_n^1(x), \quad (4.6) \\
f(x) &= \frac{1}{b - a} \int_{a}^{b} f(t) dt + \tilde{T}_{n-1}(x) + \tau_n(x) + \tilde{R}_n^2(x), \quad (4.7)
\end{align*}
where $\tilde{T}_n(x)$ and $\tau_n(x)$ are defined by (4.4) and (4.5), respectively, and \begin{align*}
\tilde{R}_n^1(x) &= -(b - a)^{n-1} \int_{[a, b]} P_n^* \left( \frac{x - t}{b - a} \right) \, df^{(n-1)}(t), \\
\tilde{R}_n^2(x) &= -(b - a)^{n-1} \int_{[a, b]} \left[ P_n^* \left( \frac{x - t}{b - a} \right) - P_n \left( \frac{x - a}{b - a} \right) \right] \, df^{(n-1)}(t).
\end{align*}
Let $(P_k, k \geq 0)$ is as in Example 3, then we get (see [8]):
\begin{align*}
\tilde{T}_m(x) &= \sum_{k=1}^{m} \frac{(b - a)^{k-1}}{k!} B_k \left( \frac{x - a}{b - a} \right) \left[ f^{(k-1)}(b) - f^{(k-1)}(a) \right], \\
\tau_m(x) &= 0, \quad m \leq n, \\
\tilde{R}_n^1(x) &= -(b - a)^{n-1} \int_{[a, b]} B_n^* \left( \frac{x - t}{b - a} \right) \, df^{(n-1)}(t), \\
\tilde{R}_n^2(x) &= -(b - a)^{n-1} \int_{[a, b]} \left[ B_n^* \left( \frac{x - t}{b - a} \right) - B_n \left( \frac{x - a}{b - a} \right) \right] \, df^{(n-1)}(t).
\end{align*}
Also from Example 1 we have
\[ P_k(0) = \frac{1}{k!}(-\gamma)^k, \quad P_k(1) = \frac{1}{k!}(1 - \gamma)^k. \]

Therefore, in this case
\[ \alpha_k = P_k(0) - P_k(1) = \frac{1}{k!} \left[ (-\gamma)^k - (1 - \gamma)^k \right], \quad k \geq 1. \]

Further, we have
\[
\bar{T}_m(x) = \sum_{k=1}^{m} \frac{(b-a)^{k-1}}{k!} \left( \frac{x-a}{b-a} - \gamma \right)^k \left[ f^{(k-1)}(b) - f^{(k-1)}(a) \right]
\]
and
\[
\tau_m(x) = \sum_{k=2}^{m} \frac{(b-a)^{k-1}}{k!} \left[ (-\gamma)^k - (1 - \gamma)^k \right] f^{(k-1)}(x),
\]
for every \( x \in [a,b] \).

Using Theorem 1 for identity (4.6) we get the following Grüss type inequality:

**Theorem 10.** Let \( f : [a,b] \to \mathbb{R} \) be such that \( f^{(n)} \) is absolutely continuous and \((P_n^*, n \geq 0)\) be defined as in (4.1). Then we have
\[
E_n(f;a,b) = f(x) - \frac{1}{b-a} \int_a^b f(t)dt - \bar{T}_n(x) + \tau_n(x)
\]
\[
- (b-a)^n \left[ P_{n+1}^* \left( \frac{x-b}{b-a} \right) - P_{n+1}^* \left( \frac{x-a}{b-a} \right) \right] f^{(n-1)}[a,b]
\]
and \( E_n(f;a,b) \) satisfies the estimation
\[
|E_n(f;a,b)| \leq \frac{(b-a)^{n-1}}{\sqrt{2}} \left[ \int_a^b \left[ P_{n+1}^* \left( \frac{x-t}{b-a} \right) \right]^2 dt - \left[ P_{n+1}^* \left( \frac{x-a}{b-a} \right) - P_{n+1}^* \left( \frac{x-b}{b-a} \right) \right]^2 \right]^\frac{1}{2}
\]
\[
\times \left( \int_a^b (t-a)(b-t) \left[ f^{(n+1)}(t) \right]^2 dt \right)^\frac{1}{2}
\]
for any \( x \in [a,b] \).

**Proof.** If we apply Theorem 1 for \( f \rightarrow P_n^* \left( \frac{x-t}{b-a} \right), \ g \rightarrow f^{(n)} \), we deduce
\[
\left| \frac{1}{b-a} \int_a^b P_n^* \left( \frac{x-t}{b-a} \right) f^{(n)}(t)dt - \frac{1}{b-a} \int_a^b P_n^* \left( \frac{x-t}{b-a} \right) dt \cdot \frac{1}{b-a} \int_a^b f^{(n)}(t)dt \right|
\]
\[
\leq \frac{1}{\sqrt{2}} \left[ T \left( P_n^* \left( \frac{x-\cdot}{b-a} \right), P_n^* \left( \frac{x-\cdot}{b-a} \right) \right) \right]^\frac{1}{2}
\]
\[
\times \frac{1}{\sqrt{b-a}} \left( \int_a^b (t-a)(b-t) \left[ f^{(n+1)}(t) \right]^2 dt \right)^\frac{1}{2},
\]
(4.10)
where
\[
T \left( P_n^* \left( \frac{x - \cdot}{b - a} \right), P_n^* \left( \frac{x - \cdot}{b - a} \right) \right) = \frac{1}{b - a} \int_a^b \left[ P_n^* \left( \frac{x - t}{b - a} \right) \right]^2 dt - \left[ P_{n+1}^* \left( \frac{x - a}{b - a} \right) - P_{n+1}^* \left( \frac{x - b}{b - a} \right) \right]^2 .
\]

Using (4.6) and (4.10), we deduce the representation (4.8) and the bound (4.9).

The following Grüss type inequality also holds.

**Theorem 11.** Let \( f : [a, b] \to \mathbb{R} \) be such that \( f^{(n+1)} \) is absolutely continuous and \( f^{(n)} \geq 0 \) on \([a, b]\) and \((P_n^*, n \geq 0)\) be defined in (4.1). Then we have the representation (4.8) and the remainder \( E_n(f; a, b) \) satisfies the bound

\[
|E_n(f; a, b)| \leq (b - a)^n \left\| P_{n-1}^* \left( \frac{x - \cdot}{b - a} \right) \right\|_\infty \left\{ \frac{f^{(n-1)}(a) + f^{(n-1)}(b)}{2} - f^{(n-2)}[a, b] \right\}
\]

for any \( x \in [a, b] \).

**Proof.** If we apply Theorem 2 for \( f \to P_n^* \left( \frac{x - \cdot}{b - a} \right), \ g \to f^{(n)} \), we deduce

\[
\left| \frac{1}{b - a} \int_a^b P_n^* \left( \frac{x - t}{b - a} \right) f^{(n)}(t) dt - \frac{1}{b - a} \int_a^b P_n^* \left( \frac{x - t}{b - a} \right) dt \cdot \frac{1}{b - a} \int_a^b f^{(n)}(t) dt \right| \leq \frac{1}{2(b - a)} \left\| P_{n-1}^* \left( \frac{x - \cdot}{b - a} \right) \right\|_\infty \left( \int_a^b (t - a)(b - t) f^{(n+1)}(t) dt \right)^{1/2} .
\]

Using the representation (4.8) and the inequality (4.12), we deduce (4.11).

**5. On estimation of the remainder in generalized Taylor’s formula**

S. S. Dragomir in [12] has obtained the following result:

**Theorem 12.** Let \( f : I \to \mathbb{R}, \ I \subset \mathbb{R} \), is a closed interval, \( a \in I \) be such that \( f^{(n)} \) is absolutely continuous. Then we have the Taylor’s perturbed formula:

\[
f(x) = T_n(f; a, x) + \frac{(x - a)^{n+1}}{(n + 1)!} f^{(n)}[a, x] + G_n(f; a, x),
\]

where

\[
T_n(f; a, x) := \sum_{k=0}^n \frac{(x - a)^k}{k!} f^{(k)}(a).
\]

The remainder \( G_n(f; a, x) \) satisfies the estimation:

\[
|G_n(f; a, x)| \leq \frac{(x - a)^{n+1}}{4(n!)} [\Gamma(x) - \gamma(x)],
\]

where \( \Gamma(x) \) is the Euler gamma function and \( \gamma(x) \) is the digamma function.
where
\[
\Gamma(x) := \sup_{t \in [a, x]} f^{(n+1)}(t), \quad \gamma(x) := \inf_{t \in [a, x]} f^{(n+1)}(t) \tag{5.4}
\]
for all \(x \geq a, x \in I\).

In this section we will show improvement and generalization of this (see [16]). At first we consider a formula which can be regarded as generalized Taylor’s formula.

**Theorem 13.** Let \((P_k, k \geq 0)\) be a harmonic sequence of polynomials, Further, let \(I \subset \mathbb{R}\) be a closed interval and \(a \in I\). If \(f : I \to \mathbb{R}\) is any function such that, for some \(n \in \mathbb{N}\), \(f^{(n)}\) is absolutely continuous, then for any \(x \in I\)

\[
f(x) = f(a) + \sum_{k=1}^{n} (-1)^{k+1} \left[ P_k(x)f^{(k)}(x) - P_k(a)f^{(k)}(a) \right] + R_n(f; a, x), \tag{5.5}
\]

where

\[
R_n(f; a, x) = (-1)^n \int_{a}^{x} P_n(t)f^{(n+1)}(t) dt. \tag{5.6}
\]

We can call (5.5) the generalized Taylor’s formula. Namely, if we use polynomials from Example 1 we get the classical Taylor’s formula:

\[
f(x) = f(a) + \sum_{k=1}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) + R_n^T(f; a, x), \tag{5.7}
\]

where

\[
R_n^T(f; a, x) := \frac{1}{n!} \int_{a}^{x} (x-t)^n f^{(n+1)}(t) dt. \tag{5.8}
\]

For Example 2 we have

\[
f(x) = T_n^M(f; a, x) + R_n^M(f; a, x),
\]

where

\[
T_n^M(f; a, x) := f(a) + \sum_{k=1}^{n} \frac{(x-a)^k}{2^k k!} \left[ f^{(k)}(a) - (-1)^k f^{(k)}(x) \right] \tag{5.9}
\]

and

\[
R_n^M(f; a, x) := \frac{(-1)^n}{n!} \int_{a}^{x} \left( t - \frac{a+x}{2} \right)^n f^{(n+1)}(t) dt. \tag{5.10}
\]

Here we give another special case of (5.5) by using Bernoulli polynomials (see Example 3). If we set

\[
P_n(t) = \frac{(x-a)^n}{n!} B_n \left( \frac{t-a}{x-a} \right), \quad n \in \mathbb{N}, \quad P_0(t) = 1,
\]

we can apply (5.5) to obtain

\[
f(x) = f(a) + \sum_{k=1}^{n} (-1)^{k+1} \frac{(x-a)^k}{k!} \left[ B_k(1)f^{(k)}(x) - B_k(0)f^{(k)}(a) \right] + R_n^B(f; a, x),
\]
where
\[ R_n^B(f; a, x) := (-1)^n \frac{(x-a)^n}{n!} \int_a^x B_n \left( \frac{t-a}{x-a} \right) f^{(n+1)}(t) \, dt. \] (5.11)

We have that \( B_n(1) - B_n(0) = 0 \), for \( n \neq 1 \), that is \( B_n(1) = B_n(0) = B_n \), for \( n \neq 1 \). Also, \( B_1(1) = -B_1(0) = 1/2 \) so that we have
\[
f(x) = f(a) + \frac{x-a}{2} \left[ f'(x) + f'(a) \right] + \sum_{k=2}^{n} \frac{(-1)^{k+1} (x-a)^k}{k!} B_k \left[ f^{(k)}(x) - f^{(k)}(a) \right] + R_n^B(f; a, x).
\]

Finally, we can use the fact that \( B_{2k+1} = 0 \) for \( k = 1, 2, \ldots \), \(([1], 23.1.19)\), so that
\[
f(x) = T_n^B(f; a, x) + R_n^B(f; a, x),
\]

where
\[
T_n^B(f; a, x) := f(a) + \frac{x-a}{2} \left[ f'(x) + f'(a) \right] - \sum_{k=1}^{\lfloor z \rfloor} \frac{(x-a)^{2k}}{(2k)!} B_{2k} \left[ f^{(2k)}(x) - f^{(2k)}(a) \right]
\] (5.12)

and \( R_n^B(f; a, x) \) is given by (5.11). (Here, as well as in the rest of section, \([z]\) denotes the greatest integer less than or equal to \( z \).)

Using Euler polynomials from Example 4 we see that
\[ P_n(t) = \frac{(x-a)^n}{n!} E_n \left( \frac{t-a}{x-a} \right), \quad n \in \mathbb{N}, \quad P_0(t) = 1 \]
is a harmonic sequence of polynomials so that (5.5) yields
\[
f(x) = f(a) + \sum_{k=1}^{n} \frac{(-1)^{k+1} (x-a)^k}{k! \lfloor z \rfloor} \left[ E_k(1)f^{(k)}(x) - E_k(0)f^{(k)}(a) \right] + R_n^E(f; a, x),
\]

where
\[
R_n^E(f; a, x) := (-1)^n \frac{(x-a)^n}{n!} \int_a^x E_n \left( \frac{t-a}{x-a} \right) f^{(n+1)}(t) \, dt. \] (5.13)

Further, since \(([1], 23.1.20)\)
\[ E_n(0) = -E_n(1) = -\frac{2}{n+1}(2^{n+1} - 1)B_{n+1}, \quad \text{for} \quad n \in \mathbb{N}, \]

we get
\[
f(x) = f(a) + 2 \sum_{k=1}^{n} \frac{(-1)^{k+1} (x-a)^k (2^{k+1} - 1)}{(k+1)!} B_{k+1} \left[ f^{(k)}(x) + f^{(k)}(a) \right] + R_n^E(f; a, x).
\]
Finally, since $B_{2k+1} = 0$ for $k = 1, 2, \ldots$, we get

$$f(x) = T^E_n (f; a, x) + R^E_n (f; a, x),$$

where

$$T^E_n (f; a, x) := f(a) + 2 \sum_{k=1}^{n+1} \frac{(x-a)^{2k-1}(4^k - 1)}{(2k)!} B_{2k} \left[ f^{(2k-1)}(x) + f^{(2k-1)}(a) \right]$$  \hspace{1cm} (5.14)$$

and $R^E_n (f; a, x)$ is given by (5.13).

Now we show the generalization of the result stated in Theorem 12. As we shall see, this result also improves the estimation (5.3).

**THEOREM 14.** Let $(P_k, k \geq 0)$ be a harmonic sequence of polynomials. Let $I \subset \mathbb{R}$ be a closed interval and $a \in I$. Suppose $f : I \rightarrow \mathbb{R}$ is such that $f^{(n)}$ is absolutely continuous. Then for any $x \in I$ we have the generalized Taylor’s perturbed formula:

$$f(x) = \tilde{T}_n (f; a, x) + (-1)^n [P_{n+1}(x) - P_{n+1}(a)] f^{(n)} [a, x] + \tilde{G}_n (f; a, x),$$  \hspace{1cm} (5.15)$$

where

$$\tilde{T}_n (f; a, x) = f(a) + \sum_{k=1}^{n} (-1)^{k+1} \left[ P_k(x) f^{(k)}(x) - P_k(a) f^{(k)}(a) \right].$$  \hspace{1cm} (5.16)$$

For $x \geq a$ the remainder $\tilde{G}(f; a, x)$ satisfies the estimation

$$|\tilde{G}(f; a, x)| \leq \frac{x-a}{2} \sqrt{T(P_n, P_n)} \left[ \Gamma(x) - \gamma(x) \right],$$  \hspace{1cm} (5.17)$$

where $\Gamma(x)$ and $\gamma(x)$ are defined by (5.4).

In [13] Dragomir improved the inequality (5.17).

**THEOREM 15.** Assume that $(P_n, n \geq 0)$ is a sequence of harmonic polynomials and $f : I \rightarrow \mathbb{R}$ is such that $f^{(n)}$ is absolutely continuous and $f^{(n+1)} \in L^2(I)$. If $x \geq a$, then we have the inequality

$$|\tilde{G}_n (f; a, x)| \leq (x-a) \left[ T(P_n, P_n) \right]^{\frac{1}{2}} \left[ \frac{1}{x-a} \left\| f^{(n+1)} \right\|_2^2 - \left( f^{(n)} [a, x] \right) \right]^{\frac{1}{2}}.$$

Using Theorem 1 for identity (5.5) we get the following Grüss type inequality:

**THEOREM 16.** Let $f : I \rightarrow \mathbb{R}$ be such that $f^{(n+1)}$ is absolutely continuous, $a \in I$ and $(P_n, n \geq 0)$ be a harmonic sequence of polynomials. Then we have

$$RT_n (f; a, x)$$  \hspace{1cm} (5.18)$$

$$= f(x) - f(a) - \sum_{k=1}^{n} (-1)^{k+1} \left[ P_k(x) f^{(k)}(x) - P_k(a) f^{(k)}(a) \right]$$

$$- (-1)^n [P_{n+1}(x) - P_{n+1}(a)] f^{(n)} [a, x]$$
and $RT_n(f; a, x)$ satisfies the estimation

$$|RT_n(f; a, x)| \leq \frac{1}{\sqrt{2}} \left[ \int_a^x P_n^2(t) dt - \frac{[P_{n+1}(x) - P_{n+1}(a)]^2}{x - a} \right]^{1/2} \times \left( \int_a^x (t - a)(x - t) \left[ f^{(n+2)}(t) \right]^2 dt \right)^{1/2}$$

(5.19)

for any $x \in I$.

**Proof.** If we apply Theorem 1 for $f \rightarrow P_n$, $g \rightarrow f^{(n+1)}$, we deduce

$$\left| \frac{1}{x-a} \int_a^x P_n(t) f^{(n+1)}(t) dt - \frac{1}{x-a} \int_a^x P_n(t) dt \cdot \frac{1}{x-a} \int_a^x f^{(n+1)}(t) dt \right|$$

$$\leq \frac{1}{\sqrt{2}} \left[ T(P_n, P_n) \right]^{1/2} \frac{1}{\sqrt{x-a}} \left( \int_a^x (t - a)(x - t) \left[ f^{(n+2)}(t) \right]^2 dt \right)^{1/2},$$

(5.20)

where

$$T(P_n, P_n) = \frac{1}{x-a} \int_a^x P_n^2(t) dt - \frac{[P_{n+1}(x) - P_{n+1}(a)]^2}{x - a}.$$

Using (5.5) and (5.20), we deduce the representation (5.18) and the bound (5.19). □

The following Grüss type inequality also holds.

**Theorem 17.** Let $f : I \rightarrow \mathbb{R}$ be such that $f^{(n+2)}$ is absolutely continuous and $f^{(n+1)} \geq 0$ on $I$ and $(P_n, n \geq 0)$ be a harmonic sequence of polynomials. If $a \in I$ then we have the representation (5.18) and the remainder $RT_n(f; a, x)$ satisfies the bound

$$|RT_n(f; a, x)| \leq (x-a)\|P_{n-1}\|_\infty \left\{ \frac{f^{(n)}(x) + f^{(n)}(a)}{2} - f^{(n-1)}[a,x] \right\},$$

(5.21)

for any $x \in I$.

**Proof.** If we apply Theorem 2 for $f \rightarrow P_n$, $g \rightarrow f^{(n+1)}$, we deduce

$$\left| \frac{1}{x-a} \int_a^x P_n(t) f^{(n+1)}(t) dt - \frac{1}{x-a} \int_a^x P_n(t) dt \cdot \frac{1}{x-a} \int_a^x f^{(n+1)}(t) dt \right|$$

$$\leq \frac{1}{2(x-a)} \|P_{n-1}\|_\infty \left( \int_a^x (t - a)(x - t) f^{(n+2)}(t) dt \right)^{1/2}.$$  

(5.22)

Using the representation (5.18) and the inequality (5.22), we deduce (5.21). □

**Remark 6.** Using Theorem 16 and Theorem 17 for the harmonic sequence of polynomials from Example 1 we get the results from [6].
The results from following lemma are proved in [16].

**LEMMA 1.** (i) If \( P_n(t) = \frac{1}{n!} (t - \frac{a + x}{2})^n \), then

\[
\int_a^x P_n^2(t) dt = \frac{(x - a)^{2n+1}}{(n!)^2 (2n+1) 2^{2n}}
\]

and

\[
\|P_{n-1}\|_\infty = \frac{1}{(n-1)!} \left( \frac{x - a + x}{2} \right)^{n-1} = \frac{(x-a)^{n-1}}{2^{n-1}(n-1)!}.
\]

(ii) Let \( P_n(t) = \frac{(x-a)^n}{n!} B_n \left( \frac{t-a}{x-a} \right) \), where \( B_n(t) \) are Bernoulli polynomials. Then

\[
\int_a^x P_n^2(t) dt = \frac{(x-a)^{2n+1}}{(2n)!} |B_{2n}|.
\]

(iii) Let \( P_n(t) = \frac{(x-a)^n}{n!} E_n \left( \frac{t-a}{x-a} \right) \), where \( E_n(t) \) are Euler polynomials. Then

\[
\int_a^x P_n^2(t) dt = \frac{4(x-a)^{2n+1}(4^{n+1}-1)}{(2n+2)!} |B_{2n+2}|.
\]

**COROLLARY 2.** Let \( f : I \to \mathbb{R} \) be such that \( f^{(n+1)} \) is absolutely continuous, \( a \in I \). Then we have

(i) If \( T_n^M(f; a, x) \) is defined by (5.9), then

\[
f(x) = T_n^M(f; a, x) + \frac{(x-a)^{n+1}(1 + (-1)^n)}{2^{n+1}(n+1)!} f^{(n)}[a, x] + RT_n^M(f; a, x),
\]

\[
|RT_n^M(f; a, x)| \leq \frac{(x-a)^n}{2^n n!} \sqrt{\frac{x-a}{2(2n+1)}} - \frac{(x-a)(1 + (-1)^n)}{4(n+1)^2} \int_a^x (t - a)(x - t) \left[ f^{(n+2)}(t) \right]^2 dt \right)^{1/2},
\]

and for \( f^{(n+2)} \geq 0 \)

\[
|RT_n^M(f; a, x)| \leq \frac{(x-a)^n}{2^{n-1}(n-1)!} \left\{ \frac{f^{(n)}(x) + f^{(n)}(a)}{2} - f^{(n-1)}[a, x] \right\}.
\]

(ii) If \( T_n^B(f; a, x) \) is defined by (5.12), then

\[
f(x) = T_n^B(f; a, x) + RT_n^B(f; a, x).
\]
and
\[
|RT_n^B(f; a, x)| \leq \frac{1}{\sqrt{2}} \left[ \frac{(x-a)^{2n+1}}{(2n)!} |B_{2n}| \right]^{1/2} \left( \frac{\int_a^x (t-a)(x-t) [f^{n+2}(t)]^2 dt}{|B_{2n}|} \right)^{1/2}.
\]

(iii) If \( T_n^E(f; a, x) \) is defined by (5.14), then
\[
f(x) = T_n^E(f; a, x) + (-1)^n \frac{4(x-a)^{n+1}(2n+2-1)}{(n+2)!} B_{n+2} f^{(n)}[a, x] + RT_n^E(f; a, x)
\]
and
\[
|RT_n^E(f; a, x)| \leq 2(x-a)^{n+\frac{1}{2}} \left[ \frac{4^{n+1} - 1}{(2n+2)!} |B_{n+2}| - \frac{4(2n+2-1)^2}{((n+2)!)^2} B_{n+2}^2 \right]^{1/2}
\times \left( \frac{\int_a^x (t-a)(x-t) [f^{n+2}(t)]^2 dt}{|B_{n+2}|} \right)^{1/2}.
\]

Proof. (i) Set \( P_n = \frac{1}{n!} (t - \frac{a+x}{2})^n \). We have
\[
P_{n+1}(x) - P_{n+1}(a) = (-1)^n \frac{(x-a)^{n+1} [1 + (-1)^{n+1}]}{2^{n+1} (n+1)!}.
\]
Now apply Theorem 16, Theorem 17 and Lemma 1(i).

(ii) Set \( P_n(t) = \frac{(x-a)^n}{n!} B_n \left( \frac{t-a}{x-a} \right) \). We have
\[
P_{n+1}(x) - P_{n+1}(a) = \frac{(x-a)^{n+1}}{(n+1)!} \left[ B_{n+1} (1) - B_{n+1} (0) \right] = 0.
\]
Now apply Theorem 16 and Lemma 1(ii).

(iii) Set \( P_n(t) = \frac{(x-a)^n}{n!} E_n \left( \frac{t-a}{x-a} \right) \). We have
\[
P_{n+1}(x) - P_{n+1}(a) = \frac{(x-a)^{n+1}}{(n+1)!} \left[ E_{n+1} (1) - E_{n+1} (0) \right]
= \frac{(x-a)^{n+1}}{(n+1)!} 2E_{n+1}(1) = \frac{4(x-a)^{n+1}}{(n+2)!} \frac{2n+1}{(n+2)} B_{n+2}.
\]
Now apply Theorem 16 and Lemma 1(iii). \( \square \)

5.1. Special case for logarithmic function

Consider the logarithmic function \( f : (0, +\infty) \to \mathbb{R}, f(t) = \ln t \). Then we have
\[
f^{(n)}(t) = \frac{(-1)^{n-1}(n-1)!}{t^n}, \quad t > 0, \quad n \in \mathbb{N}.
\]
Using the results from (5.23) we get the estimation

\[
\int_a^x (t-a)(x-t)[f^{(n+2)}(t)]^2 \, dt = ((n+1)!)^2 \left[ \frac{1}{2n+1} \left( \frac{1}{x^{2n+1}} - \frac{1}{a^{2n+1}} \right) \right. \\
\left. - \frac{a+x}{2(n+1)} \left( \frac{1}{x^{2n+2}} - \frac{1}{a^{2n+2}} \right) + \frac{ax}{2n+3} \left( \frac{1}{x^{2n+3}} - \frac{1}{a^{2n+3}} \right) \right] \tag{5.23}
\]

Now, let us observe four different cases.

**Case 1.** Let \( P_n(t) \) is as in Example 1. An easy calculation gives (see [6])

\[
\ln x = \ln a + \sum_{k=1}^{n} \frac{(-1)^k}{k} \ln^k (a-x) - \frac{1}{2n+1} \left( \frac{1}{x^{2n+1}} - \frac{1}{a^{2n+1}} \right) \\
- \frac{a+x}{2(n+1)} \left( \frac{1}{x^{2n+2}} - \frac{1}{a^{2n+2}} \right) + \frac{ax}{2n+3} \left( \frac{1}{x^{2n+3}} - \frac{1}{a^{2n+3}} \right) \]

Using the results from [6] and (5.23) we get the estimation

\[
|RT_n(\ln; a, x)| \leq \frac{n \cdot n!}{\sqrt{2(2n+1)}} \left| x - a \right|^{n-\frac{1}{2}} \left( \frac{1}{2n+1} \left( \frac{1}{x^{2n+1}} - \frac{1}{a^{2n+1}} \right) \right. \\
\left. - \frac{a+x}{2(n+1)} \left( \frac{1}{x^{2n+2}} - \frac{1}{a^{2n+2}} \right) + \frac{ax}{2n+3} \left( \frac{1}{x^{2n+3}} - \frac{1}{a^{2n+3}} \right) \right)^{\frac{1}{2}}.
\]

**Case 2.** Let \( P_n(t) \) is as in Example 2. In this case we have

\[
\ln x = \ln a + \sum_{k=1}^{n} \frac{(x-a)^k}{k^2} \left[ \frac{1}{x^k} + \frac{(-1)^{k-1}}{a^k} \right] \\
+ \frac{1}{n(n+1)^{2n+1}} \left( \frac{1}{a^n} - \frac{1}{x^n} \right) x-a^n + RT_n^M(\ln; a, x),
\]

where by Corollary 2 (i) and by (5.23) the remainder \( RT_n^M(\ln; a, x) \) satisfies the estimation

\[
|RT_n^M(\ln; a, x)| \leq \frac{(x-a)^n(n+1)}{2^n} \sqrt{\frac{x-a}{2n+1}} \left( \frac{x-a}{4(n+1)^2} \right) \left( \frac{1}{2n+1} \left( \frac{1}{x^{2n+1}} - \frac{1}{a^{2n+1}} \right) \right. \\
\left. - \frac{a+x}{2(n+1)} \left( \frac{1}{x^{2n+2}} - \frac{1}{a^{2n+2}} \right) + \frac{ax}{2n+3} \left( \frac{1}{x^{2n+3}} - \frac{1}{a^{2n+3}} \right) \right)^{\frac{1}{2}}.
\]

**Case 3.** Let \( P_n(t) \) is as in Example 3. We easily calculate

\[
\ln x = \ln a + \frac{x^2 - a^2}{2ax} - \frac{1}{2} \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{B_{2k}}{k} \left( \frac{1}{a^{2k}} - \frac{1}{x^{2k}} \right) (x-a)^{2k} + RT_n^B(\ln; a, x),
\]
where by Corollary 2 (ii) and by (5.23) the remainder \( RT_n^B(\ln; a, x) \) satisfies the estimation

\[
|RT_n^B(\ln; a, x)| \leq \frac{1}{\sqrt{2}} \left[ (x-a)^{2n+1} \frac{(n+1)!}{(2n)!} |B_{2n}| \right]^{1/2} (n+1)! \left( \frac{1}{2n+1} \left( \frac{1}{x^{2n+1}} - \frac{1}{a^{2n+1}} \right) \right)
\]

\[
- \frac{a+x}{2(n+1)} \left( \frac{1}{x^{2n+2}} - \frac{1}{a^{2n+2}} \right) + \frac{ax}{2n+3} \left( \frac{1}{x^{2n+3}} - \frac{1}{a^{2n+3}} \right)^{1/2}.
\]

Case 4. Let \( P_n(t) \) is as in Example 4. We easily calculate

\[
\ln x = \ln a + \sum_{k=1}^{\lceil \frac{n+1}{2} \rceil} \frac{(4^k-1)B_{2k}}{k(2k-1)} \left( \frac{1}{a^{2k-1}} + \frac{1}{x^{2k-1}} \right) (x-a)^{2k-1}
\]

\[
+ \frac{4(2^{n+2}-1)B_{n+2}}{n(n+1)(n+2)} \left( \frac{1}{a^n} - \frac{1}{x^n} \right) (x-a)^n + RT_n^E(\ln; a, x),
\]

where by Corollary 2 (iii) and by (5.23) the remainder \( RT_n^E(\ln; a, x) \) satisfies the estimation

\[
|RT_n^E(\ln; a, x)| \leq 2(x-a)^{n+\frac{1}{2}} \left[ (x-a)^{2n+2} \frac{(n+1)!}{(2n+2)!} |B_{n+2}| - \frac{4(2^{n+2}-1)^2}{((n+2)!)^2} B_{n+2}^2 \right]^{1/2} (n+1)!
\]

\[
\times \left( \frac{1}{2n+1} \left( \frac{1}{x^{2n+1}} - \frac{1}{a^{2n+1}} \right) \right) - \frac{a+x}{2(n+1)} \left( \frac{1}{x^{2n+2}} - \frac{1}{a^{2n+2}} \right) + \frac{ax}{2n+3} \left( \frac{1}{x^{2n+3}} - \frac{1}{a^{2n+3}} \right)^{1/2}.
\]

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