Journal of Mathematical Inequalities
Volume 9, Number 1 (2015), 297–319 doi:10.7153/jmi-09-25

HARMONIC POLYNOMIALS AND GENERALIZATIONS OF OSTROWSKI–GRÜSS TYPE INEQUALITY AND TAYLOR FORMULA

K. M. AWAN, J. PEČARIĆ AND A. VUKELIĆ

(*Communicated by A. Aglić Aljinović*)

Abstract. Some generalizations of Ostrowski-Grüss type inequality and Taylor formula are given, by using harmonic sequences of polynomials. We use inequalities for the Čebyšev functional in terms of the first derivative (see [6]), for some new bounds for the remainders.

1. Introduction

Let the polynomials $P_k(t),\ k \geqslant 0$ satisfy the following condition

$$P_k'(t) = P_{k-1}(t), \quad k \geqslant 1; \quad P_0(t) = 1. \tag{1.1}$$

For a sequence $(P_k(t),\ k \geqslant 0)$ of polynomials satisfying the condition (1.1), we say that it is a harmonic sequence of polynomials. From (1.1), by an easy induction it follows that every harmonic sequence of polynomials must be of the form

$$P_k(t) = \sum_{i=0}^{k} \frac{c_i}{(k-i)!} t^{k-i}, \quad k \geqslant 0,$$

where $(c_k, k \geqslant 0)$ is a sequence of real numbers such that $c_0 = 1$. In fact, $c_k = P_k(0)$, $k \geqslant 0$. Especially, we have $P_0(t) = 1$, $P_1(t) = t + c_1$, $P_2(t) = \frac{1}{2}t^2 + c_1 t + c_2$.

EXAMPLE 1. For fixed $x \in \mathbb{R}$ define

$$P_k(t) = \frac{1}{k!}(t-x)^k, \quad k \geqslant 0.$$

Then $(P_k(t), k \geqslant 0)$ is a harmonic sequence of polynomials.

EXAMPLE 2. Similarly, for fixed $x \in \mathbb{R}$ define

$$P_k(t) = \frac{1}{k!}\left(t - \frac{a+x}{2}\right)^k, \quad k \geqslant 0.$$

Then $(P_k(t), k \geqslant 0)$ is also a harmonic sequence of polynomials.

Mathematics subject classification (2010): 26D15, 26D20, 26D99.
Keywords and phrases: Čebyšev functional, harmonic polynomials, Ostrowski-Grüss inequality, Taylor formula.

EXAMPLE 3. Here we have the well known Bernoulli polynomials $B_k(t)$. These polynomials can be defined by the expansion

$$\frac{xe^{tx}}{e^x-1}=\sum_{k=0}^{\infty}\frac{B_k(t)}{k!}x^k, \quad |x|<2\pi, \quad t\in\mathbb{R}.$$

We have

$$B_0(t)=1, \quad B_1(t)=t-\frac{1}{2}, \quad B_2(t)=t^2-t+\frac{1}{6}, \quad B_3(t)=t^3-\frac{3}{2}t^2+\frac{1}{2}t, \cdots.$$

The numbers $B_k := B_k(0)$ are called Bernoulli numbers. The polynomials $B_k(t)$ and the numbers B_k have many interesting properties. It can be shown that the polynomials $B_k(t)$ are uniquely determined by the following two properties ([[1], 23.1.5 and 23.1.6]):

$$B_k'(t)=kB_{k-1}(t), \quad k\in\mathbb{N}; \quad B_0(t)=1 \tag{1.2}$$

and

$$B_k(t+1)-B_k(t)=kt^{k-1}, \quad k\in\mathbb{N}. \tag{1.3}$$

Let $P_k(t)=\frac{1}{k!}B_k(t),\ k\geqslant 0$, then from (1.1) it follows that $(P_k(t), k\geqslant 0)$ is harmonic sequence of polynomials.

EXAMPLE 4. Instead of Bernoulli polynomials $B_k(t)$ we can have Euler polynomials $E_k(t)$ which have the properties similar to those of Bernoulli polynomials. Euler polynomials can be defined by the expansion

$$\frac{2e^{tx}}{e^x+1}=\sum_{n=0}^{\infty}\frac{E_k(t)}{k!}x^k, \quad |x|<\pi, \quad t\in\mathbb{R}.$$

We have

$$E_0(t)=1, \quad E_1(t)=t-\frac{1}{2}, \quad E_2(t)=t^2-t, \quad E_3(t)=t^3-\frac{3}{2}t^2+\frac{1}{4}, \cdots.$$

It can be shown that the polynomials $E_k(t)$ are uniquely determined by the following two properties ([[1], 23.1.5 and 23.1.6]):

$$E_k'(t)=kE_{k-1}(t),\ k\in\mathbb{N}; \quad E_0(t)=1 \tag{1.4}$$

and

$$E_k(t+1)+E_k(t)=2t^k,\ \ k\in\mathbb{N}. \tag{1.5}$$

$P_k=\frac{E_k(t)}{k!}$, $k\geqslant 0$ is harmonic sequence of polynomials.

REMARK 1. In [4] Cerone defined polynomials $P_k(t)$ as

$$P_k'(t)=\xi_k P_{k-1}(t), \quad P_0(t)=1, \quad t\in\mathbb{R}. \tag{1.6}$$

When $\xi_k=k$, then such functions satisfying (1.6) were defined by Appell in 1980, [3] and are known as Appell polynomials. For $\xi_k=1$ we have harmonic polynomials. Polynomials satisfying (1.6) will be termed Appell-like polynomials.

For two Lebesgue integrable functions $f,g:[a,b]\to\mathbb{R}$, consider the Čebyšev functional:

$$T(f,g):=\frac{1}{b-a}\int_a^b f(t)g(t)dt-\frac{1}{b-a}\int_a^b f(t)dt\cdot\frac{1}{b-a}\int_a^b g(t)dt. \tag{1.7}$$

In [6] the authors proved the following theorems:

THEOREM 1. *Let $f,g:[a,b]\to\mathbb{R}$ be two absolutely continuous functions on $[a,b]$ with*

$$(\cdot-a)(b-\cdot)(f')^2,(\cdot-a)(b-\cdot)(g')^2\in L[a,b].$$

Then we have the inequality

$$\begin{aligned}|T(f,g)| &\leqslant \frac{1}{\sqrt{2}}[T(f,f)]^{\frac{1}{2}}\frac{1}{\sqrt{b-a}}\left(\int_a^b (x-a)(b-x)\left[g'(x)\right]^2dx\right)^{\frac{1}{2}} \\ &\leqslant \frac{1}{2(b-a)}\left(\int_a^b (x-a)(b-x)\left[f'(x)\right]^2dx\right)^{\frac{1}{2}} \\ &\quad\times\left(\int_a^b (x-a)(b-x)\left[g'(x)\right]^2dx\right)^{\frac{1}{2}}.\end{aligned} \tag{1.8}$$

The constant $\frac{1}{\sqrt{2}}$ and $\frac{1}{2}$ are best possible in (1.8).

THEOREM 2. *Assume that $g:[a,b]\to\mathbb{R}$ is monotonic nondecreasing on $[a,b]$ and $f:[a,b]\to\mathbb{R}$ is absolutely continuous with $f'\in L_\infty[a,b]$. Then we have the inequality*

$$|T(f,g)|\leqslant\frac{1}{2(b-a)}||f'||_\infty\int_a^b (x-a)(b-x)dg(x). \tag{1.9}$$

The constant $\frac{1}{2}$ is best possible.

In this paper we will show some generalizations of inequalities of Ostrowski-Grüss type and generalizations of Taylor's formula using sequences of harmonic polynomials. We will use the above theorems to get some new bounds for the remainders.

2. On some identities related to Ostrowski inequality

The well-known Ostrowski inequality (see, for example [18]) states that if $f\in C^1([a,b])$, $x\in[a,b]$, then

$$\left|\frac{1}{b-a}\int_a^b f(y)dy-f(x)\right|\leqslant\left(\frac{1}{4}+\frac{(x-(a+b)/2)^2}{(b-a)^2}\right)\|f'\|_\infty. \tag{2.1}$$

G. V. Milovanović and J. Pečarić in [17] and A. M. Fink in [14] (see also [18], p. 470) have considered generalizations of Ostrowski inequality in the form

$$\left|\frac{1}{n}\left[f(x)+\sum_{k=1}^{n-1}F_k(x)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|\leqslant K(n,p,x)\|f^{(n)}\|_p, \tag{2.2}$$

where $F_k(x)$ is defined by

$$F_k(x) = \frac{n-k}{k!(b-a)}\left[f^{(k-1)}(a)(x-a)^k - f^{(k-1)}(b)(x-b)^k\right]. \tag{2.3}$$

For $n = 1$ the sum above is defined to be zero.

In fact, G. V. Milovanović and J. Pečarić have proved that ([18]):

$$K(n,\infty,x) = \frac{(x-a)^{n+1}+(b-x)^{n+1}}{n(n+1)!(b-a)}, \tag{2.4}$$

while A. M. Fink proved that

$$K(n,p,x) = \frac{[(x-a)^{np'+1}+(b-x)^{np'+1}]^{1/p'}}{n!(b-a)}B((n-1)p'+1,p'+1)^{1/p'}, \tag{2.5}$$

where $1 < p \leqslant \infty$, $1/p+1/p' = 1$, B is the beta function, and

$$K(n,1,x) = \frac{(n-1)^{n-1}}{n^n n!(b-a)}\max[(x-a)^n,(b-x)^n]. \tag{2.6}$$

In [7], the authors gave some generalizations of previous results:

Let $(P_n, n \geqslant 0)$ be a harmonic sequence of polynomials. Furthermore, let $I \subset \mathbb{R}$ be a segment and $f : I \to \mathbb{R}$ such that $f^{(n-1)}$ is Lipschitzian or has bounded variation on I, for some $n \geqslant 1$. Then, using notations

$$\tilde{F}_k := \frac{(-1)^k(n-k)}{b-a}\left[P_k(a)f^{(k-1)}(a) - P_k(b)f^{(k-1)}(b)\right]$$

and

$$k(t,x) = \begin{cases} t-a, & t \in [a,x] \\ t-b, & t \in (x,b] \end{cases},$$

the following identity holds:

$$\frac{1}{n}\left[f(x) + \sum_{k=1}^{n-1}(-1)^k P_k(x) f^{(k)}(x) + \sum_{k=1}^{n-1}\tilde{F}_k\right] - \frac{1}{b-a}\int_a^b f(t)dt \tag{2.7}$$
$$= \frac{(-1)^{n-1}}{n(b-a)}\int_a^b P_{n-1}(t)k(t,x)df^{(n-1)}(t).$$

The sums above are defined to be zero for $n = 1$.

For the harmonic sequence of polynomials from Example 1 relation (2.7) becomes

$$\frac{1}{n}\left[f(x) + \sum_{k=1}^{n-1}F_k(x)\right] - \frac{1}{b-a}\int_a^b f(t)dt \tag{2.8}$$
$$= \frac{1}{n!(b-a)}\int_a^b (x-t)^{n-1}k(t,x)df^{(n-1)}(t),$$

where $F_k(x)$ is defined by (2.3).

Ostrowski inequality has been also generalized by Anastassiou [2]. He proved that if $f \in C^{n+1}([a,b])$ for $n \in N$ and $x \in [a,b]$ fixed, then

$$\left|\frac{1}{b-a}\int_a^b f(y)dy - f(x)\right| \leqslant \frac{1}{b-a}\left[\sum_{k=1}^{n}\frac{|f^{(k)}(x)|}{(k+1)!}\left|(b-x)^{k+1}-(a-x)^{k+1}\right| \right.$$
$$\left. +\frac{\|f^{(n+1)}\|_\infty}{(n+2)!}\left((x-a)^{n+2}+(b-x)^{n+2}\right)\right].$$

This reduces to the Ostrowski inequality in the extreme case $n = 0$, when the sum becomes empty and so vanishes identically.

Another form of this has been obtained by Cerone, Dragomir and Roumeliotis [5], who have shown that

$$\left|\int_a^b f(t)dt - \sum_{k=0}^{n-1}\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}f^{(k)}(x)\right|$$
$$\leqslant \frac{\|f^{(n)}\|_\infty}{(n+1)!}\left|(x-a)^{n+1}+(b-x)^{n+1}\right| \leqslant \frac{\|f^{(n)}\|_\infty(b-a)^{n+1}}{(n+1)!}. \tag{2.9}$$

In [19] the authors gave further generalizations of the above results.

Let the sequel $(P_n(x)),(Q_n(x))$ denote sequences of harmonic polynomials. Set

$$S_n(t,x) := \begin{cases} P_n(t), & t \in [a,x] \\ Q_n(t), & t \in (x,b]. \end{cases} \tag{2.10}$$

Then,

$$(-1)^n\int_a^b S_n(t,x)df^{(n-1)}(t) = I_n(x), \tag{2.11}$$

where

$$I_n(x) := \sum_{k=1}^{n}(-1)^k\left[Q_k(b)f^{(k-1)}(b)+(P_k(x)-Q_k(x))f^{(k-1)}(x)\right.$$
$$\left. -P_k(a)f^{(k-1)}(a)\right] + \int_a^b f(t)dt.$$

With the convention that the empty sum represents zero, the same definition gives $I_0(x) = \int_a^b f(t)dt$.

The special case (see Example 1)

$$P_n(t) := (t-a)^n/n! \qquad Q_n(t) := (t-b)^n/n!, \tag{2.12}$$

arises in the literature. The corresponding values of $I_n(x)$ for these choices we denote by $I_n^*(x)$, so that

$$I_n^*(x) = \int_a^b f(t)dt + \sum_{k=1}^{n}\frac{(-1)^k}{k!}\left[(x-a)^k-(x-b)^k\right]f^{(k-1)}(x).$$

In following (see [21]) we will show a generalizations of the above results using more than two sequences of harmonic polynomials.

Let $\sigma = \{a = x_0 < x_1 < \ldots x_{n-1} < x_n = b\}$ be a subdivision of the interval $[a,b]$. Set

$$S_m(t,\sigma) = \begin{cases} P_{1m}(t),\ t \in [a,x_1] \\ P_{2m}(t),\ t \in (x_1,x_2] \\ \vdots \\ P_{nm}(t),\ t \in (x_{n-1},x_n], \end{cases}$$

where $(P_{jm})_m$ are sequences of harmonic polynomials. By successive integration by parts we have whenever the integrals exist that

$$(-1)^m \int_a^b S_m(t,\sigma)df^{(m-1)}(t) = \int_a^b f(t)dt + \sum_{k=1}^{m}(-1)^k \Big[P_{nk}(b)f^{(k-1)}(b) + \sum_{j=1}^{n-1}(P_{jk}(x_j) - P_{j+1,k}(x_j))f^{(k-1)}(x_j) - P_{1k}(a)f^{(k-1)}(a)\Big]. \tag{2.13}$$

The right-hand side of the previous identity is denoted by $I_m(\sigma)$.

3. Generalizations of inequalities of Ostrowski-Grüss type

There have been several extensions of the Ostrowski inequality. It has been shown in [15] that if f is differentiable on (a,b) and f' is integrable and satisfies

$$\gamma \leqslant f'(t) \leqslant \Gamma \text{ for all } t \in [a,b],$$

then

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right)\right| \leqslant \frac{1}{4\sqrt{3}}(b-a)(\Gamma-\gamma), \tag{3.1}$$

for all $x \in [a,b]$. A version of this estimate occurs in [10], but without the $\sqrt{3}$ on the right.

Here we give generalizations of above result (see [20]) using sequences of harmonic polynomials defining in previous section by (2.10).

THEOREM 3. *Suppose $f : [a,b] \to \mathbb{R}$ is such that $f^{(n)}$ is integrable with*

$$\gamma \leqslant f^{(n)}(t) \leqslant \Gamma \textit{ for all } t \in [a,b].$$

Put

$$U_n(x) := [Q_{n+1}(b) - Q_{n+1}(x) + P_{n+1}(x) - P_{n+1}(a)]/(b-a).$$

Then for any $x \in [a,b]$,

$$\left| I_n(x) - (-1)^n U_n(x)\left[f^{(n-1)}(b) - f^{(n-1)}(a)\right]\right| \leqslant \frac{1}{2}K(\Gamma-\gamma)(b-a),$$

where

$$K := \left\{ \frac{1}{b-a} \left[\int_a^x P_n^2(t)dt + \int_x^b Q_n^2(t)dt \right] - [U_n(x)]^2 \right\}^{\frac{1}{2}}.$$

COROLLARY 1. *Under the assumptions of Theorem 3,*

$$\Bigg| \frac{1}{b-a} \int_a^b f(t)dt + \sum_{k=1}^{n} \frac{(-1)^k}{k!(b-a)} \Big[(b-B)^k f^{(k-1)}(b) + ((x-A)^k - (x-B)^k) f^{(k-1)}(x) - (a-A)^k f^{(k-1)}(a) \Big] - \frac{(-1)^n (f^{(n-1)}(b) - f^{(n-1)}(a))}{(n+1)!(b-a)^2} [(b-B)^{n+1} - (x-B)^{n+1} + (x-A)^{n+1} - (a-A)^{n+1}] \Bigg| \leqslant \frac{1}{2}(\Gamma - \gamma) K_1$$

for all $x \in [a,b]$ *and* $A, B \in \mathbb{R}$*, where*

$$K_1 := \frac{1}{n!} \left(\frac{(x-A)^{2n+1} - (a-A)^{2n+1} + (b-B)^{2n+1} - (x-B)^{2n+1}}{(2n+1)(b-a)} - \left(\frac{(b-B)^{n+1} - (x-B)^{n+1} + (x-A)^{n+1} - (a-A)^{n+1}}{(n+1)(b-a)} \right)^2 \right)^{1/2}.$$

In [11], Dragomir proved some of above results in terms of the Euclidian norm of $f^{(n)}$ which is valid for a larger class of mappings, i.e., for the mappings f for which $f^{(n)}$ is unbounded on (a,b) but $f^{(n)} \in L_2[a,b]$.

THEOREM 4. *Assume that the mapping* $f : [a,b] \to \mathbb{R}$ *is such that* $f^{(n-1)}$ *is absolutely continuous on* $[a,b]$ *and* $f^{(n)} \in L_2[a,b]$, $(n \geqslant 1)$. *Then we have the inequality*

$$\left| I_n(x) - (-1)^n (b-a) U_n(x) f^{(n-1)}[a,b] \right| \leqslant K(b-a) \left[\frac{1}{(b-a)} \|f^{(n)}\|_2^2 - \left(f^{(n)}[a,b] \right)^2 \right]^{\frac{1}{2}} \tag{3.2}$$

for all $x \in [a,b]$, *where* K, γ *and* Γ *are as in Theorem 3 and*

$$f^{(n-1)}[a,b] = \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a}$$

is the divided difference.

Now, we will apply the Grüss type inequalities from Theorem 1 and Theorem 2 in pointing out different bounds for the remainder for identities (2.7) and identities (2.11).

Using Theorem 1 for identity (2.7) we get the following Grüss type inequality:

THEOREM 5. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous and $(P_n, n\geqslant 0)$ be a harmonic sequence of polynomials. Then we define the remainder*

$$
\begin{aligned}
&F_n(f;a,b) \\
&= f(x)+\sum_{k=1}^{n-1}(-1)^k P_k(x)f^{(k)}(x)+\sum_{k=1}^{n-1}\tilde{F}_k-\frac{n}{b-a}\int_a^b f(t)dt \\
&\quad+\frac{(-1)^n}{b-a}\left[P_n(x)(b-a)-P_{n+1}(b)+P_{n+1}(a)\right]f^{(n-1)}[a,b]
\end{aligned}
\tag{3.3}
$$

and $F_n(f;a,b)$ satisfies the estimation

$$
\begin{aligned}
&|F_n(f;a,b)| \\
&\leqslant \frac{1}{\sqrt{2}(b-a)}\left[\int_a^b (P_{n-1}(t)k(t,x))^2\,dt-\frac{[P_n(x)(b-a)-P_{n+1}(b)+P_{n+1}(a)]^2}{b-a}\right]^{\frac{1}{2}} \\
&\quad\times\left(\int_a^b (t-a)(b-t)\left[f^{(n+1)}(t)\right]^2dt\right)^{\frac{1}{2}}
\end{aligned}
\tag{3.4}
$$

for any $x\in[a,b]$.

Proof. If we apply Theorem 1 for $f\to k(\cdot,x)P_{n-1},\ g\to f^{(n)}$, we deduce

$$
\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b P_{n-1}(t)k(t,x)f^{(n)}(t)dt-\frac{1}{b-a}\int_a^b P_{n-1}(t)k(t,x)dt\cdot\frac{1}{b-a}\int_a^b f^{(n)}(t)dt\right| \\
&\leqslant \frac{1}{\sqrt{2}}\left[T\left(k(\cdot,x)P_{n-1},k(\cdot,x)P_{n-1}\right)\right]^{\frac{1}{2}} \\
&\times\frac{1}{\sqrt{b-a}}\left(\int_a^b (t-a)(b-t)\left[f^{(n+1)}(t)\right]^2dt\right)^{\frac{1}{2}},
\end{aligned}
\tag{3.5}
$$

where

$$
T\left(k(\cdot,x)P_{n-1},k(\cdot,x)P_{n-1}\right)=\frac{1}{b-a}\int_a^b (P_{n-1}(t)k(t,x))^2\,dt-\left[P_n(x)-\frac{P_{n+1}(b)-P_{n+1}(a)}{b-a}\right]^2.
$$

Using (2.7) and (3.5), we deduce the representation (3.3) and the bound (3.4). □

REMARK 2. Using Theorem 5 for the harmonic sequence of polynomials from Example 1 we get:

$$
\begin{aligned}
&F_n(f;a,b) \\
&= f(x)+\sum_{k=1}^{n-1}(-1)^k P_k(x)f^{(k)}(x)+\sum_{k=1}^{n-1}F_k-\frac{n}{b-a}\int_a^b f(t)dt \\
&\quad+\frac{1}{(n+1)!(b-a)}\left[(x-b)^{n+1}-(x-a)^{n+1}\right]f^{(n-1)}[a,b]
\end{aligned}
\tag{3.6}
$$

and

$$
\begin{aligned}
&|F_n(f;a,b)| \\
&\leqslant \frac{1}{\sqrt{2}(b-a)}\left[\frac{(x-a)^{2n+1}-(x-b)^{2n+1}}{[(n-1)!]^2(2n+1)(2n-1)n}-\frac{1}{b-a}\left[\frac{(x-a)^{n+1}-(x-b)^{n+1}}{(n+1)!}\right]^2\right]^{\frac{1}{2}} \\
&\quad \times\left(\int_a^b (t-a)(b-t)\left[f^{(n+1)}(t)\right]^2 dt\right)^{\frac{1}{2}}.
\end{aligned}
$$

The following Grüss type inequality also holds.

THEOREM 6. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous and $f^{(n+1)}\geqslant 0$ on $[a,b]$ and $(P_n, n\geqslant 0)$ be a harmonic sequence of polynomials. Then we have the representation (3.3) and the remainder $F_n(f;a,b)$ satisfies the bound*

$$
|F_n(f;a,b)| \leqslant ||k(\cdot,x)P_{n-2}+P_{n-1}||_\infty \left\{\frac{f^{(n-1)}(a)+f^{(n-1)}(b)}{2}-f^{(n-2)}[a,b]\right\} \tag{3.7}
$$

for any $x\in[a,b]$.

Proof. If we apply Theorem 2 for $f\to k(\cdot,x)P_{n-1},\ g\to f^{(n)}$, we deduce

$$
\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b P_{n-1}(t)k(t,x)f^{(n)}(t)dt-\frac{1}{b-a}\int_a^b P_{n-1}(t)k(t,x)dt\cdot\frac{1}{b-a}\int_a^b f^{(n)}(t)dt\right| \\
&\leqslant \frac{1}{2(b-a)}||k(\cdot,x)P_{n-2}+P_{n-1}||_\infty\left(\int_a^b (t-a)(b-t)f^{(n+1)}(t)dt\right)^{1/2}.
\end{aligned} \tag{3.8}
$$

Since

$$
\begin{aligned}
&\int_a^b (t-a)(b-t)f^{(n+1)}(t)dt \\
&=\int_a^b f^{(n)}(t)[2t-(a+b)]dt \\
&=(b-a)\left[f^{(n-1)}(b)+f^{(n-1)}(a)\right]-2\left(f^{(n-2)}(b)-f^{(n-2)}(a)\right).
\end{aligned}
$$

Using the representation (3.3) and the inequality (3.8), we deduce (3.7). □

REMARK 3. Using Theorem 6 for the harmonic sequence of polynomials from Example 1 we get:

$$
|F_n(f;a,b)| \leqslant \frac{1}{(n-2)!}\max_{x\in[a,b]}\{|b-x|^{n-1},|x-a|^{n-1}\}\left\{\frac{f^{(n-1)}(a)+f^{(n-1)}(b)}{2}-f^{(n-2)}[a,b]\right\}.
$$

Using Theorem 1 for identity (2.11) we get the following Grüss type inequality:

THEOREM 7. *Let* $f:[a,b]\to\mathbb{R}$ *be such that* $f^{(n)}$ *is absolutely continuous and* $(P_n, n\geqslant 0)$ *and* $(Q_n, n\geqslant 0)$ *be harmonic sequences of polynomials as in (2.10). Then we have*

$$\begin{aligned} &A_n(f;a,b) \\ &= \sum_{k=1}^{n}(-1)^k\Big[Q_k(b)f^{(k-1)}(b)+(P_k(x)-Q_k(x))f^{(k-1)}(x) \\ &-P_k(a)f^{(k-1)}(a)\Big]+\int_a^b f(t)dt \\ &\quad +(-1)^{n-1}\left[P_{n+1}(x)-Q_{n+1}(x)+Q_{n+1}(b)-P_{n+1}(a)\right]f^{(n-1)}[a,b] \end{aligned} \tag{3.9}$$

and $A_n(f;a,b)$ *satisfies the estimation*

$$\begin{aligned} &|A_n(f;a,b)| \\ &\leqslant \frac{1}{\sqrt{2}}\left[\int_a^b S_n^2(t)dt-\frac{\left[P_{n+1}(x)-Q_{n+1}(x)+Q_{n+1}(b)-P_{n+1}(a)\right]^2}{b-a}\right]^{\frac{1}{2}} \\ &\quad\times\left(\int_a^b (t-a)(b-t)\left[f^{(n+1)}(t)\right]^2dt\right)^{\frac{1}{2}} \end{aligned} \tag{3.10}$$

for any $x\in[a,b]$.

Proof. If we apply Theorem 1 for $f\to S_{n-1}(\cdot,x)$, $g\to f^{(n)}$, we deduce

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b S_n(t,x)f^{(n)}(t)dt-\frac{1}{b-a}\int_a^b S_n(t,x)\cdot\frac{1}{b-a}\int_a^b f^{(n)}(t)dt\right| \\ &\leqslant \frac{1}{\sqrt{2}}\left[T\left(S_n(\cdot,x),S_n(\cdot,x)\right)\right]^{\frac{1}{2}} \\ &\quad\times\frac{1}{\sqrt{b-a}}\left(\int_a^b (t-a)(b-t)\left[f^{(n+1)}(t)\right]^2dt\right)^{\frac{1}{2}}, \end{aligned} \tag{3.11}$$

where

$$T\left(S_n(\cdot,x),S_n(\cdot,x)\right)=\frac{1}{b-a}\int_a^b S_n^2(t,x)dt-\left[\frac{P_{n+1}(x)-Q_{n+1}(x)+Q_{n+1}(b)-P_{n+1}(a)}{b-a}\right]^2.$$

Using (2.11) and (3.11), we deduce the representation (3.9) and the bound (3.10). □

REMARK 4. Using Theorem 7 for the harmonic sequence of polynomials from Example 1 we get:

$$\begin{aligned} &A_n(f;a,b) \\ &= \int_a^b f(t)dt+\sum_{k=1}^{n}\frac{(-1)^k}{k!}\left[(x-a)^k-(x-b)^k\right]f^{(k-1)}(x) \\ &\quad +\frac{(-1)^{n-1}}{(n+1)!}\left[(x-a)^{n+1}-(x-b)^{n+1}\right]f^{(n-1)}[a,b] \end{aligned} \tag{3.12}$$

and

$$\begin{aligned}&|A_n(f;a,b)|\\ &\leqslant \frac{1}{\sqrt{2}}\left[\frac{(x-a)^{2n+1}-(x-b)^{2n+1}}{(n!)^2(2n+1)}-\frac{1}{b-a}\left[\frac{(x-a)^{n+1}-(x-b)^{n+1}}{(n+1)!}\right]^2\right]^{\frac{1}{2}}\\ &\quad\times\left(\int_a^b (t-a)(b-t)\left[f^{(n+1)}(t)\right]^2 dt\right)^{\frac{1}{2}}.\end{aligned}$$

The following Grüss type inequality also holds.

THEOREM 8. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous and $f^{(n+1)}\geqslant 0$ on $[a,b]$ and $(S_n, n\geqslant 0)$ be a harmonic sequence of polynomials defined with (2.10). Then we have the representation (3.9) and the remainder $A_n(f;a,b)$ satisfies the bound*

$$|A_n(f;a,b)|\leqslant (b-a)||S_{n-1}||_\infty\left\{\frac{f^{(n-1)}(a)+f^{(n-1)}(b)}{2}-f^{(n-2)}[a,b]\right\},\quad (3.13)$$

for any $x\in[a,b]$.

Proof. If we apply Theorem 2 for $f\to S_n(\cdot,x),\ g\to f^{(n)}$, we deduce

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b S_n(t,x)f^{(n)}(t)dt-\frac{1}{b-a}\int_a^b S_n(t,x)dt\cdot\frac{1}{b-a}\int_a^b f^{(n)}(t)dt\right|\\ &\leqslant \frac{1}{2(b-a)}||S_{n-1}||_\infty\left(\int_a^b (t-a)(b-t)f^{(n+1)}(t)dt\right)^{1/2}.\end{aligned}\quad (3.14)$$

Using the representation (3.9) and the inequality (3.14), we deduce (3.13). □

REMARK 5. Using Theorem 8 for the harmonic sequence of polynomials from Example 1 we get:

$$|A_n(f;a,b)|\leqslant\frac{b-a}{(n-1)!}\max_{x\in[a,b]}\{|b-x|^{n-1},|x-a|^{n-1}\}\left\{\frac{f^{(n-1)}(a)+f^{(n-1)}(b)}{2}-f^{(n-2)}[a,b]\right\}.$$

4. On generalizations of Ostrowski-Grüss inequality via Euler harmonic identities

In this section (see [9]) we use Euler identities involving harmonic sequence of polynomials to show some generalizations of Ostrowski inequality.

Assume that $(P_k(t), k\geqslant 0)$ is a harmonic sequence of polynomials i.e. the sequence of polynomials satisfying the condition (1.1). Define $P_k^*(t),\ k\geqslant 0$ to be periodic functions of period 1, related to $P_k(t),\ k\geqslant 0$ as

$$P_k^*(t)=P_k(t),\quad 0\leqslant t<1,\qquad P_k^*(t+1)=P_k^*(t),\quad t\in\mathbb{R}.\quad (4.1)$$

Thus, $P_0^*(t) = 1$, while for $k \geqslant 1$, $P_k^*(t)$ is continuous on $\mathbb{R} \setminus \mathbb{Z}$ and has a jump of

$$\alpha_k = P_k(0) - P_k(1) \tag{4.2}$$

at every integer t, whenever $\alpha_k \neq 0$. Note that $\alpha_1 = -1$, since $P_1(t) = t + c_1$, for some $c_1 \in \mathbb{R}$. Also, note that from (1.1) it follows

$$P_k^{*\prime}(t) = P_{k-1}^*(t), \quad k \geqslant 1, \quad t \in \mathbb{R} \setminus \mathbb{Z}. \tag{4.3}$$

Let $a, b \in \mathbb{R}$, $a < b$ and $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a function of bounded variation on $[a,b]$ for some $n \geqslant 1$. For every $x \in [a,b]$ and $1 \leqslant m \leqslant n$ we introduce the following notations

$$\tilde{T}_m(x) = \sum_{k=1}^{m} (b-a)^{k-1} P_k\left(\frac{x-a}{b-a}\right)\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right], \tag{4.4}$$

with convention $\tilde{T}_0(x) = 0$, and

$$\tau_m(x) = \sum_{k=2}^{m} (b-a)^{k-1} \alpha_k f^{(k-1)}(x), \tag{4.5}$$

with convention $\tau_1(x) = 0$.

THEOREM 9. *Let $(P_k, k \geqslant 0)$ be a harmonic sequence of polynomials and $f : [a,b] \to \mathbb{R}$ such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $n \geqslant 1$. Then for every $x \in [a,b]$*

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t + \tilde{T}_n(x) + \tau_n(x) + \tilde{R}_n^1(x), \tag{4.6}$$

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t + \tilde{T}_{n-1}(x) + \tau_n(x) + \tilde{R}_n^2(x), \tag{4.7}$$

where $\tilde{T}_n(x)$ and $\tau_n(x)$ are defined by (4.4) and (4.5), respectively, and

$$\tilde{R}_n^1(x) = -(b-a)^{n-1} \int_{[a,b]} P_n^*\left(\frac{x-t}{b-a}\right) df^{(n-1)}(t),$$

$$\tilde{R}_n^2(x) = -(b-a)^{n-1} \int_{[a,b]} \left[P_n^*\left(\frac{x-t}{b-a}\right) - P_n\left(\frac{x-a}{b-a}\right)\right] df^{(n-1)}(t).$$

Let $(P_k, k \geqslant 0)$ is as in Example 3, then we get (see [8]):

$$\tilde{T}_m(x) = \sum_{k=1}^{m} \frac{(b-a)^{k-1}}{k!} B_k\left(\frac{x-a}{b-a}\right)\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right],$$

$$\tau_m(x) = 0, \ m \leqslant n,$$

$$\tilde{R}_n^1(x) = -\frac{(b-a)^{n-1}}{n!} \int_{[a,b]} B_n^*\left(\frac{x-t}{b-a}\right) df^{(n-1)}(t),$$

$$\tilde{R}_n^2(x) = -\frac{(b-a)^{n-1}}{n!} \int_{[a,b]} \left[B_n^*\left(\frac{x-t}{b-a}\right) - B_n\left(\frac{x-a}{b-a}\right)\right] df^{(n-1)}(t).$$

Also from Example 1 we have

$$P_k(0) = \frac{1}{k!}(-\gamma)^k, \qquad P_k(1) = \frac{1}{k!}(1-\gamma)^k.$$

Therefore, in this case

$$\alpha_k = P_k(0) - P_k(1) = \frac{1}{k!}\left[(-\gamma)^k - (1-\gamma)^k\right], \quad k \geqslant 1.$$

Further, we have

$$\tilde{T}_m(x) = \sum_{k=1}^{m} \frac{(b-a)^{k-1}}{k!}\left(\frac{x-a}{b-a} - \gamma\right)^k \left[f^{(k-1)}(b) - f^{(k-1)}(a)\right]$$

and

$$\tau_m(x) = \sum_{k=2}^{m} \frac{(b-a)^{k-1}}{k!}\left[(-\gamma)^k - (1-\gamma)^k\right] f^{(k-1)}(x),$$

for every $x \in [a,b]$.

Using Theorem 1 for identity (4.6) we get the following Grüss type inequality:

THEOREM 10. *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous and $(P_n^*, n \geqslant 0)$ be defined as in (4.1). Then we have*

$$\begin{aligned} &E_n(f;a,b) \\ &= f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - \tilde{T}_n(x) + \tau_n(x) \\ &\quad -(b-a)^n\left[P_{n+1}^*\left(\frac{x-b}{b-a}\right) - P_{n+1}^*\left(\frac{x-a}{b-a}\right)\right] f^{(n-1)}[a,b] \end{aligned} \tag{4.8}$$

and $E_n(f;a,b)$ satisfies the estimation

$$\begin{aligned} &|E_n(f;a,b)| \\ &\leqslant \frac{(b-a)^{n-1}}{\sqrt{2}}\left[\int_a^b \left[P_n^*\left(\frac{x-t}{b-a}\right)\right]^2 dt - \left[P_{n+1}^*\left(\frac{x-a}{b-a}\right) - P_{n+1}^*\left(\frac{x-b}{b-a}\right)\right]^2\right]^{\frac{1}{2}} \\ &\quad \times \left(\int_a^b (t-a)(b-t)\left[f^{(n+1)}(t)\right]^2 dt\right)^{\frac{1}{2}} \end{aligned} \tag{4.9}$$

for any $x \in [a,b]$.

Proof. If we apply Theorem 1 for $f \to P_n^*\left(\frac{x-\cdot}{b-a}\right)$, $g \to f^{(n)}$, we deduce

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b P_n^*\left(\frac{x-t}{b-a}\right) f^{(n)}(t)dt - \frac{1}{b-a}\int_a^b P_n^*\left(\frac{x-t}{b-a}\right)dt \cdot \frac{1}{b-a}\int_a^b f^{(n)}(t)dt\right| \\ &\leqslant \frac{1}{\sqrt{2}}\left[T\left(P_n^*\left(\frac{x-\cdot}{b-a}\right), P_n^*\left(\frac{x-\cdot}{b-a}\right)\right)\right]^{\frac{1}{2}} \\ &\quad \times \frac{1}{\sqrt{b-a}}\left(\int_a^b (t-a)(b-t)\left[f^{(n+1)}(t)\right]^2 dt\right)^{\frac{1}{2}}, \end{aligned} \tag{4.10}$$

where

$$T\left(P_n^*\left(\frac{x-\cdot}{b-a}\right), P_n^*\left(\frac{x-\cdot}{b-a}\right)\right) = \frac{1}{b-a}\int_a^b \left[P_n^*\left(\frac{x-t}{b-a}\right)\right]^2 dt - \left[P_{n+1}^*\left(\frac{x-a}{b-a}\right) - P_{n+1}^*\left(\frac{x-b}{b-a}\right)\right]^2.$$

Using (4.6) and (4.10), we deduce the representation (4.8) and the bound (4.9). □

The following Grüss type inequality also holds.

THEOREM 11. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n+1)}$ is absolutely continuous and $f^{(n)} \geqslant 0$ on $[a,b]$ and $(P_n^*, n\geqslant 0)$ be defined in (4.1). Then we have the representation (4.8) and the remainder $E_n(f;a,b)$ satisfies the bound*

$$|E_n(f;a,b)| \leqslant (b-a)^n \left\|P_{n-1}^*\left(\frac{x-\cdot}{b-a}\right)\right\|_\infty \left\{\frac{f^{(n-1)}(a)+f^{(n-1)}(b)}{2} - f^{(n-2)}[a,b]\right\} \tag{4.11}$$

for any $x\in[a,b]$.

Proof. If we apply Theorem 2 for $f \to P_n^*\left(\frac{x-\cdot}{b-a}\right)$, $g\to f^{(n)}$, we deduce

$$\left|\frac{1}{b-a}\int_a^b P_n^*\left(\frac{x-t}{b-a}\right) f^{(n)}(t)dt - \frac{1}{b-a}\int_a^b P_n^*\left(\frac{x-t}{b-a}\right)dt\cdot\frac{1}{b-a}\int_a^b f^{(n)}(t)dt\right| \leqslant \frac{1}{2(b-a)}\left\|P_{n-1}^*\left(\frac{x-\cdot}{b-a}\right)\right\|_\infty \left(\int_a^b (t-a)(b-t)f^{(n+1)}(t)dt\right)^{1/2}. \tag{4.12}$$

Using the representation (4.8) and the inequality (4.12), we deduce (4.11). □

5. On estimation of the remainder in generalized Taylor's formula

S. S. Dragomir in [12] has obtained the following result:

THEOREM 12. *Let $f:I\to\mathbb{R}$, $I\subset\mathbb{R}$, is a closed interval, $a\in I$ be such that $f^{(n)}$ is absolutely continuous. Then we have the Taylor's perturbed formula:*

$$f(x) = T_n(f;a,x) + \frac{(x-a)^{n+1}}{(n+1)!} f^{(n)}[a,x] + G_n(f;a,x), \tag{5.1}$$

where

$$T_n(f;a,x) := \sum_{k=0}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a). \tag{5.2}$$

The remainder $G_n(f;a,x)$ satisfies the estimation:

$$|G_n(f;a,x)| \leqslant \frac{(x-a)^{n+1}}{4(n!)}[\Gamma(x)-\gamma(x)], \tag{5.3}$$

where

$$\Gamma(x) := \sup_{t\in[a,x]} f^{(n+1)}(t), \quad \gamma(x) := \inf_{t\in[a,x]} f^{(n+1)}(t) \tag{5.4}$$

for all $x \geqslant a,\ x \in I$.

In this section we will show improvement and generalization of this (see [16]). At first we consider a formula which can be regarded as generalized Taylor's formula.

THEOREM 13. *Let* $(P_k, k \geqslant 0)$ *be a harmonic sequence of polynomials, Further, let* $I \subset \mathbb{R}$ *be a closed interval and* $a \in I$. *If* $f : I \to \mathbb{R}$ *is any function such that, for some* $n \in \mathbb{N}$, $f^{(n)}$ *is absolutely continuous, then for any* $x \in I$

$$f(x) = f(a) + \sum_{k=1}^{n} (-1)^{k+1} \left[P_k(x) f^{(k)}(x) - P_k(a) f^{(k)}(a)\right] + R_n(f;a,x), \tag{5.5}$$

where

$$R_n(f;a,x) = (-1)^n \int_a^x P_n(t) f^{(n+1)}(t)dt. \tag{5.6}$$

We can call (5.5) the generalized Taylor's formula. Namely, if we use polynomials from Example 1 we get the classical Taylor's formula:

$$f(x) = f(a) + \sum_{k=1}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) + R_n^T(f;a,x), \tag{5.7}$$

where

$$R_n^T(f;a,x) := \frac{1}{n!} \int_a^x (x-t)^n f^{(n+1)}(t)dt. \tag{5.8}$$

For Example 2 we have

$$f(x) = T_n^M(f;a,x) + R_n^M(f;a,x),$$

where

$$T_n^M(f;a,x) := f(a) + \sum_{k=1}^{n} \frac{(x-a)^k}{2^k k!} \left[f^k(a) - (-1)^k f^k(x)\right] \tag{5.9}$$

and

$$R_n^M(f;a,x) := \frac{(-1)^n}{n!} \int_a^x \left(t - \frac{a+x}{2}\right)^n f^{(n+1)}(t)dt. \tag{5.10}$$

Here we give another special case of (5.5) by using Bernoulli polynomials (see Example 3). If we set

$$P_n(t) = \frac{(x-a)^n}{n!} B_n\left(\frac{t-a}{x-a}\right), \quad n \in \mathbb{N}, \quad P_0(t) = 1,$$

we can apply (5.5) to obtain

$$f(x) = f(a) + \sum_{k=1}^{n} (-1)^{k+1} \frac{(x-a)^k}{k!} \left[B_k(1) f^{(k)}(x) - B_k(0) f^{(k)}(a)\right] + R_n^B(f;a,x),$$

where

$$R_n^B(f;a,x) := (-1)^n \frac{(x-a)^n}{n!} \int_a^x B_n\left(\frac{t-a}{x-a}\right) f^{(n+1)}(t)dt. \tag{5.11}$$

We have that $B_n(1) - B_n(0) = 0$, for $n \neq 1$, that is $B_n(1) = B_n(0) = B_n$, for $n \neq 1$. Also, $B_1(1) = -B_1(0) = 1/2$ so that we have

$$\begin{aligned} f(x) &= f(a) + \frac{x-a}{2}\left[f'(x) + f'(a)\right] \\ &\quad + \sum_{k=2}^{n} (-1)^{k+1} \frac{(x-a)^k}{k!} B_k \left[f^{(k)}(x) - f^{(k)}(a)\right] + R_n^B(f;a,x). \end{aligned}$$

Finally, we can use the fact that $B_{2k+1} = 0$ for $k = 1, 2, \cdots$, ([[1], 23.1.19]), so that

$$f(x) = T_n^B(f;a,x) + R_n^B(f;a,x),$$

where

$$T_n^B(f;a,x) := f(a) + \frac{x-a}{2}\left[f'(x) + f'(a)\right] - \sum_{k=1}^{\left[\frac{n}{2}\right]} \frac{(x-a)^{2k}}{(2k)!} B_{2k} \left[f^{(2k)}(x) - f^{(2k)}(a)\right] \tag{5.12}$$

and $R_n^B(f;a,x)$ is given by (5.11). (Here, as well as in the rest of section, $[z]$ denotes the greatest integer less than or equal to z.)

Using Euler polynomials from Example 4 we see that

$$P_n(t) = \frac{(x-a)^n}{n!} E_n\left(\frac{t-a}{x-a}\right), \quad n \in \mathbb{N}, \quad P_0(t) = 1$$

is a harmonic sequence of polynomials so that (5.5) yields

$$f(x) = f(a) + \sum_{k=1}^{n} (-1)^{k+1} \frac{(x-a)^k}{k!} \left[E_k(1) f^{(k)}(x) - E_k(0) f^{(k)}(a)\right] + R_n^E(f;a,x),$$

where

$$R_n^E(f;a,x) := (-1)^n \frac{(x-a)^n}{n!} \int_a^x E_n\left(\frac{t-a}{x-a}\right) f^{(n+1)}(t)dt. \tag{5.13}$$

Further, since ([[1], 23.1.20])

$$E_n(0) = -E_n(1) = -\frac{2}{n+1}(2^{n+1} - 1)B_{n+1}, \quad \text{for} \quad n \in \mathbb{N},$$

we get

$$\begin{aligned} f(x) &= f(a) + 2\sum_{k=1}^{n} (-1)^{k+1} \frac{(x-a)^k (2^{k+1}-1)}{(k+1)!} B_{k+1} \left[f^{(k)}(x) + f^{(k)}(a)\right] \\ &\quad + R_n^E(f;a,x). \end{aligned}$$

Finally, since $B_{2k+1} = 0$ for $k = 1, 2, \cdots$, we get

$$f(x) = T_n^E(f;a,x) + R_n^E(f;a,x),$$

where

$$T_n^E(f;a,x) := f(a) + 2 \sum_{k=1}^{\left[\frac{n+1}{2}\right]} \frac{(x-a)^{2k-1}(4^k-1)}{(2k)!} B_{2k} \left[f^{(2k-1)}(x) + f^{(2k-1)}(a) \right] \quad (5.14)$$

and $R_n^E(f;a,x)$ is given by (5.13).

Now we show the generalization of the result stated in Theorem 12. As we shall see, this result also improves the estimation (5.3).

THEOREM 14. *Let $(P_k, k \geqslant 0)$ be a harmonic sequence of polynomials. Let $I \subset \mathbb{R}$ be a closed interval and $a \in I$. Suppose $f : I \to \mathbb{R}$, is such that $f^{(n)}$ is absolutely continuous. Then for any $x \in I$ we have the generalized Taylor's perturbed formula:*

$$f(x) = \tilde{T}_n(f;a,x) + (-1)^n \left[P_{n+1}(x) - P_{n+1}(a) \right] f^{(n)}[a,x] + \tilde{G}_n(f;a,x), \quad (5.15)$$

where

$$\tilde{T}_n(f;a,x) = f(a) + \sum_{k=1}^{n} (-1)^{k+1} \left[P_k(x) f^{(k)}(x) - P_k(a) f^{(k)}(a) \right]. \quad (5.16)$$

For $x \geqslant a$ the remainder $\tilde{G}(f;a,x)$ satisfies the estimation

$$\left| \tilde{G}(f;a,x) \right| \leqslant \frac{x-a}{2} \sqrt{T(P_n,P_n)} [\Gamma(x) - \gamma(x)], \quad (5.17)$$

where $\Gamma(x)$ and $\gamma(x)$ are defined by (5.4)

In [13] Dragomir improved the inequality (5.17).

THEOREM 15. *Assume that $(P_n, n \geqslant 0)$ is a sequence of harmonic polynomials and $f : I \to \mathbb{R}$ is such that $f^{(n)}$ is absolutely continuous and $f^{(n+1)} \in L_2(I)$. If $x \geqslant a$, then we have the inequality*

$$\left| \tilde{G}_n(f;a,x) \right| \leqslant (x-a) [T(P_n,P_n)]^{\frac{1}{2}} \left[\frac{1}{x-a} \| f^{(n+1)} \|_2^2 - \left(f^{(n)}[a,x] \right)^2 \right]^{\frac{1}{2}}.$$

Using Theorem 1 for identity (5.5) we get the following Grüss type inequality:

THEOREM 16. *Let $f : I \to \mathbb{R}$ be such that $f^{(n+1)}$ is absolutely continuous, $a \in I$ and $(P_n, n \geqslant 0)$ be a harmonic sequence of polynomials. Then we have*

$$\begin{aligned} &RT_n(f;a,x) \quad (5.18) \\ &= f(x) - f(a) - \sum_{k=1}^{n} (-1)^{k+1} \left[P_k(x) f^{(k)}(x) - P_k(a) f^{(k)}(a) \right] \\ &\quad -(-1)^n \left[P_{n+1}(x) - P_{n+1}(a) \right] f^{(n)}[a,x] \end{aligned}$$

and $RT_n(f;a,x)$ satisfies the estimation

$$
\begin{aligned}
&|RT_n(f;a,x)| \\
&\leqslant \frac{1}{\sqrt{2}}\left[\int_a^x P_n^2(t)dt - \frac{[P_{n+1}(x)-P_{n+1}(a)]^2}{x-a}\right]^{\frac{1}{2}} \\
&\quad \times \left(\int_a^x (t-a)(x-t)\left[f^{(n+2)}(t)\right]^2 dt\right)^{\frac{1}{2}}
\end{aligned} \tag{5.19}
$$

for any $x \in I$.

Proof. If we apply Theorem 1 for $f \to P_n$, $g \to f^{(n+1)}$, we deduce

$$
\begin{aligned}
&\left|\frac{1}{x-a}\int_a^x P_n(t)f^{(n+1)}(t)dt - \frac{1}{x-a}\int_a^x P_n(t)dt \cdot \frac{1}{x-a}\int_a^x f^{(n+1)}(t)dt\right| \\
&\leqslant \frac{1}{\sqrt{2}}\left[T(P_n,P_n)\right]^{\frac{1}{2}} \frac{1}{\sqrt{x-a}}\left(\int_a^x (t-a)(x-t)\left[f^{(n+2)}(t)\right]^2 dt\right)^{\frac{1}{2}},
\end{aligned} \tag{5.20}
$$

where

$$
T(P_n,P_n) = \frac{1}{x-a}\int_a^x P_n^2(t)dt - \left[\frac{P_{n+1}(x)-P_{n+1}(a)}{x-a}\right]^2.
$$

Using (5.5) and (5.20), we deduce the representation (5.18) and the bound (5.19). □

The following Grüss type inequality also holds.

THEOREM 17. *Let $f : I \to \mathbb{R}$ be such that $f^{(n+2)}$ is absolutely continuous and $f^{(n+1)} \geqslant 0$ on I and $(P_n, n \geqslant 0)$ be a harmonic sequence of polynomials. If $a \in I$ then we have the representation (5.18) and the remainder $RT_n(f;a,x)$ satisfies the bound*

$$
|RT_n(f;a,x)| \leqslant (x-a)||P_{n-1}||_\infty \left\{\frac{f^{(n)}(x)+f^{(n)}(a)}{2} - f^{(n-1)}[a,x]\right\}, \tag{5.21}
$$

for any $x \in I$.

Proof. If we apply Theorem 2 for $f \to P_n$, $g \to f^{(n+1)}$, we deduce

$$
\begin{aligned}
&\left|\frac{1}{x-a}\int_a^x P_n(t)f^{(n+1)}(t)dt - \frac{1}{x-a}\int_a^x P_n(t)dt \cdot \frac{1}{x-a}\int_a^x f^{(n+1)}(t)dt\right| \\
&\leqslant \frac{1}{2(x-a)}||P_{n-1}||_\infty \left(\int_a^x (t-a)(x-t)f^{(n+2)}(t)dt\right)^{1/2}.
\end{aligned} \tag{5.22}
$$

Using the representation (5.18) and the inequality (5.22), we deduce (5.21). □

REMARK 6. Using Theorem 16 and Theorem 17 for the harmonic sequence of polynomials from Example 1 we get the results from [6].

The results from following lemma are proved in [16].

LEMMA 1. *(i) If* $P_n(t) = \frac{1}{n!}\left(t - \frac{a+x}{2}\right)^n$*, then*

$$\int_a^x P_n^2(t)dt = \frac{(x-a)^{2n+1}}{(n!)^2(2n+1)2^{2n}}$$

and

$$\|P_{n-1}\|_\infty = \frac{1}{(n-1)!}\left(x - \frac{a+x}{2}\right)^{n-1} = \frac{(x-a)^{n-1}}{2^{n-1}(n-1)!}.$$

(ii) Let $P_n(t) = \frac{(x-a)^n}{n!}B_n\left(\frac{t-a}{x-a}\right)$*, where* $B_n(t)$ *are Bernoulli polynomials. Then*

$$\int_a^x P_n^2(t)dt = \frac{(x-a)^{2n+1}}{(2n)!}\left|B_{2n}\right|.$$

(iii) Let $P_n(t) = \frac{(x-a)^n}{n!}E_n\left(\frac{t-a}{x-a}\right)$*, where* $E_n(t)$ *are Euler polynomials. Then*

$$\int_a^x P_n^2(t)dt = \frac{4(x-a)^{2n+1}(4^{n+1}-1)}{(2n+2)!}\left|B_{2n+2}\right|.$$

COROLLARY 2. *Let* $f: I \to \mathbb{R}$ *be such that* $f^{(n+1)}$ *is absolutely continuous,* $a \in I$*. Then we have*

(i) If $T_n^M(f;a,x)$ *is defined by (5.9), then*

$$f(x) = T_n^M(f;a,x) + \frac{(x-a)^{n+1}(1+(-1)^n)}{2^{n+1}(n+1)!}f^{(n)}[a,x] + RT_n^M(f;a,x),$$

$$\left|RT_n^M(f;a,x)\right| \leqslant \frac{(x-a)^n}{2^n n!}\sqrt{\frac{x-a}{2(2n+1)} - \frac{(x-a)(1+(-1)^n)}{4(n+1)^2}} \times \left(\int_a^x (t-a)(x-t)\left[f^{(n+2)}(t)\right]^2 dt\right)^{1/2},$$

and for $f^{(n+2)} \geqslant 0$

$$\left|RT_n^M(f;a,x)\right| \leqslant \frac{(x-a)^n}{2^{n-1}(n-1)!}\left\{\frac{f^{(n)}(x)+f^{(n)}(a)}{2} - f^{(n-1)}[a,x]\right\}.$$

(ii) If $T_n^B(f;a,x)$ *is defined by (5.12), then*

$$f(x) = T_n^B(f;a,x) + RT_n^B(f;a,x)$$

and

$$\left|RT_n^B(f;a,x)\right| \leqslant \frac{1}{\sqrt{2}}\left[\frac{(x-a)^{2n+1}}{(2n)!}\left|B_{2n}\right|\right]^{1/2}\left(\int_a^x (t-a)(x-t)\left[f^{n+2}(t)\right]^2 dt\right)^{1/2}.$$

(iii) If $T_n^E(f;a,x)$ is defined by (5.14), then

$$f(x) = T_n^E(f;a,x) + (-1)^n \frac{4(x-a)^{n+1}(2^{n+2}-1)}{(n+2)!} B_{n+2} f^{(n)}[a,x] + RT_n^E(f;a,x)$$

and

$$\left|RT_n^E(f;a,x)\right| \leqslant 2(x-a)^{n+\frac{1}{2}}\left[\frac{4^{n+1}-1}{(2n+2)!}\left|B_{n+2}\right| - \frac{4(2^{n+2}-1)^2}{((n+2)!)^2}B_{n+2}^2\right]^{1/2} \times\left(\int_a^x (t-a)(x-t)\left[f^{n+2}(t)\right]^2 dt\right)^{1/2}.$$

Proof. (i) Set $P_n = \frac{1}{n!}\left(t - \frac{a+x}{2}\right)^n$. We have

$$P_{n+1}(x) - P_{n+1}(a) = (-1)^n \frac{(x-a)^{n+1}\left[1+(-1)^{n+1}\right]}{2^{n+1}(n+1)!}.$$

Now apply Theorem 16, Theorem 17 and Lemma 1(i).

(ii) Set $P_n(t) = \frac{(x-a)^n}{n!}B_n\left(\frac{t-a}{x-a}\right)$. We have

$$P_{n+1}(x) - P_{n+1}(a) = \frac{(x-a)^{n+1}}{(n+1)!}\left[B_{n+1}(1) - B_{n+1}(0)\right] = 0.$$

Now apply Theorem 16 and Lemma 1(ii).

(iii) Set $P_n(t) = \frac{(x-a)^n}{n!}E_n\left(\frac{t-a}{x-a}\right)$. We have

$$\begin{aligned} P_{n+1}(x) - P_{n+1}(a) &= \frac{(x-a)^{n+1}}{(n+1)!}\left[E_{n+1}(1) - E_{n+1}(0)\right] \\ &= \frac{(x-a)^{n+1}}{(n+1)!}2E_{n+1}(1) = \frac{4(x-a)^{n+1}}{(n+2)!}\left(2^{n+2}-1\right)B_{n+2}. \end{aligned}$$

Now apply Theorem 16 and Lemma 1(iii). □

5.1. Special case for logarithmic function

Consider the logarithmic function $f:(0,+\infty)\to\mathbb{R}$, $f(t)=\ln t$. Then we have

$$f^{(n)}(t) = \frac{(-1)^{n-1}(n-1)!}{t^n}, \quad t>0, \quad n\in\mathbb{N}.$$

So that

$$\int_a^x (t-a)(x-t)\left[f^{(n+2)}(t)\right]^2 dt \tag{5.23}$$
$$= ((n+1)!)^2 \left[\frac{1}{2n+1}\left(\frac{1}{x^{2n+1}}-\frac{1}{a^{2n+1}}\right) - \frac{a+x}{2(n+1)}\left(\frac{1}{x^{2n+2}}-\frac{1}{a^{2n+2}}\right)+\frac{ax}{2n+3}\left(\frac{1}{x^{2n+3}}-\frac{1}{a^{2n+3}}\right)\right]$$

Now, let us observe four different cases.

Case 1. Let $P_n(t)$ is as in Example 1. An easy calculation gives (see [6])

$$\ln x = \ln a + \sum_{k=1}^{n}(-1)^{k+1}\frac{(x-a)^k}{ka^k} + \frac{(a-x)^n}{n(n+1)}\left(\frac{1}{a^n}-\frac{1}{x^n}\right)+RT_n(\ln;a,x).$$

Using the results from [6] and (5.23) we get the estimation

$$|RT_n(\ln;a,x)| \leqslant \frac{n\cdot n!}{\sqrt{2(2n+1)}}|x-a|^{n-\frac{1}{2}}\left(\frac{1}{2n+1}\left(\frac{1}{x^{2n+1}}-\frac{1}{a^{2n+1}}\right) - \frac{a+x}{2(n+1)}\left(\frac{1}{x^{2n+2}}-\frac{1}{a^{2n+2}}\right)+\frac{ax}{2n+3}\left(\frac{1}{x^{2n+3}}-\frac{1}{a^{2n+3}}\right)\right)^{\frac{1}{2}}.$$

Case 2. Let $P_n(t)$ is as in Example 2. In this case we have

$$\ln x = \ln a + \sum_{k=1}^{n}\frac{(x-a)^k}{k2^k}\left[\frac{1}{x^k}+\frac{(-1)^{k-1}}{a^k}\right] + \frac{1+(-1)^n}{n(n+1)2^{n+1}}\left(\frac{1}{a^n}-\frac{1}{x^n}\right)(x-a)^n + RT_n^M(\ln;a,x),$$

where by Corollary 2 (i) and by (5.23) the remainder $RT_n^M(\ln;a,x)$ satisfies the estimation

$$|RT_n^M(\ln;a,x)| \leqslant \frac{(x-a)^n(n+1)}{2^n}\sqrt{\frac{x-a}{2(2n+1)}-\frac{(x-a)[1+(-1)^n]}{4(n+1)^2}}\left(\frac{1}{2n+1}\left(\frac{1}{x^{2n+1}}-\frac{1}{a^{2n+1}}\right) - \frac{a+x}{2(n+1)}\left(\frac{1}{x^{2n+2}}-\frac{1}{a^{2n+2}}\right)+\frac{ax}{2n+3}\left(\frac{1}{x^{2n+3}}-\frac{1}{a^{2n+3}}\right)\right)^{\frac{1}{2}}.$$

Case 3. Let $P_n(t)$ is as in Example 3. We easily calculate

$$\ln x = \ln a + \frac{x^2-a^2}{2ax} - \frac{1}{2}\sum_{k=1}^{\lfloor \frac{n}{2}\rfloor}\frac{B_{2k}}{k}\left(\frac{1}{a^{2k}}-\frac{1}{x^{2k}}\right)(x-a)^{2k} + RT_n^B(\ln;a,x),$$

where by Corollary 2 (ii) and by (5.23) the remainder $RT_n^B(\ln;a,x)$ satisfies the estimation

$$
\begin{aligned}
&|RT_n^B(\ln;a,x)\\
&\leqslant \frac{1}{\sqrt{2}}\left[\frac{(x-a)^{2n+1}(n+1)!}{(2n)!}|B_{2n}|\right]^{\frac{1}{2}}(n+1)!\left(\frac{1}{2n+1}\left(\frac{1}{x^{2n+1}}-\frac{1}{a^{2n+1}}\right)\right.\\
&\qquad\left.-\frac{a+x}{2(n+1)}\left(\frac{1}{x^{2n+2}}-\frac{1}{a^{2n+2}}\right)+\frac{ax}{2n+3}\left(\frac{1}{x^{2n+3}}-\frac{1}{a^{2n+3}}\right)\right)^{\frac{1}{2}}.
\end{aligned}
$$

Case 4. Let $P_n(t)$ is as in Example 4. We easily calculate

$$
\begin{aligned}
\ln x = \ln a &+ \sum_{k=1}^{\lfloor\frac{n+1}{2}\rfloor}\frac{(4^k-1)B_{2k}}{k(2k-1)}\left(\frac{1}{a^{2k-1}}+\frac{1}{x^{2k-1}}\right)(x-a)^{2k-1}\\
&+\frac{4(2^{n+2}-1)B_{n+2}}{n(n+1)(n+2)}\left(\frac{1}{a^n}-\frac{1}{x^n}\right)(x-a)^n+RT_n^E(\ln;a,x),
\end{aligned}
$$

where by Corollary 2 (iii) and by (5.23) the remainder $RT_n^E(\ln;a,x)$ satisfies the estimation

$$
\begin{aligned}
&|RT_n^E(\ln;a,x)\\
&\leqslant 2(x-a)^{n+\frac{1}{2}}\left[\frac{4^{n+1}-1}{(2n+2)!}|B_{n+2}|-\frac{4(2^{n+2}-1)^2}{((n+2)!)^2}B_{n+2}^2\right]^{1/2}(n+1)!\\
&\quad\times\left(\frac{1}{2n+1}\left(\frac{1}{x^{2n+1}}-\frac{1}{a^{2n+1}}\right)-\frac{a+x}{2(n+1)}\left(\frac{1}{x^{2n+2}}-\frac{1}{a^{2n+2}}\right)\right.\\
&\quad\left.+\frac{ax}{2n+3}\left(\frac{1}{x^{2n+3}}-\frac{1}{a^{2n+3}}\right)\right)^{\frac{1}{2}}.
\end{aligned}
$$

REFERENCES

[1] M. ABRAMOWITZ AND I. A. STEGUN (Eds.), *Handbook of mathematical functions with formulae, graphs and mathematical tables*, National Bureau of Standards, Applied Math. Series 55, 4th printing, Washington 1965.

[2] G. A. ANASTASSIOU, *Ostrowski type inequalities*, Proc. Amer. Math. Soc. **123** (1995), 3775–3781.

[3] P. APPELL, *Sur une classe de polnômes*, Ann. Sci. Ecole Norm. Sup. **9**, 2 (1880), 119–144.

[4] P. CERONE, *Three point rules in numerical integration*, RGMIA, Research Report Collection **3**, 2 (2000), 299–312.

[5] P. CERONE, S. S. DRAGOMIR AND J. ROUMELIOTIS, *Some Ostrowski's type inequalitiies for n-times differentiable mappings and applications*, RGMIA, Research Report Collection **1**, 1 (1988).

[6] P. CERONE AND S. S. DRAGOMIR, *Some new bounds for the Čebyšev functional in terms of the first derivative and applications*, J. Math. Ineq. **8**, 1 (2014), 159–170.

[7] LJ. DEDIĆ, M. MATIĆ AND J. PEČARIĆ, *On generalizations of Ostrowski inequality for Lipschitz functions and functions of bounded variation*, Math. Inequal. Appl. **3**, 1 (2000), 1–14.

[8] LJ. DEDIĆ, M. MATIĆ AND J. PEČARIĆ, *On generalizations of Ostrowski inequality via some Euler-type identities*, Math. Inequal. Appl. **3**, 3 (2000), 337–353.

[9] LJ. DEDIĆ, M. MATIĆ, J. PEČARIĆ AND A. VUKELIĆ, *On generalizations of Ostrowski inequality via some Euler harmonic identities*, J. Inequal. Appl. **7**, 6 (2002), 787–805.

[10] S. S. DRAGOMIR AND S. WANG, *An inequality of Ostrowski-Grüs type and its applications to the estimation of error bounds for some special means and for some numerical quadrature rules*, Comput. Math. Appl. **33**, 11 (1997), 16–20.

[11] S. S. DRAGOMIR, *Better bounds in some Ostrowski-Grüss type inequalities*, RGMIA, Research Report Collection **3**, 1 (2000), 25–31.

[12] S. S. DRAGOMIR, *New estimation of the Remainder in Taylor's Formula Using Grüss Type Inequalities and Applications*, Math. Inequal. Appl. **2**, 2 (1999), 183–193.

[13] S. S. DRAGOMIR, *An improvement of the remainder estimate in the generalised Taylor formula*, RGMIA, Research Report Collection **3**, 1 (2000), 3–8.

[14] A. M. FINK, *Bounds on the deviation of a function from its averages*, Czechoslovak Math. J. **42**, 117 (1992), 289–310.

[15] M. MATIĆ, J. PEČARIĆ AND N. UJEVIĆ, *Improvement and further generalizations of some inequalities of Ostrowski-Grüss type*, Computer. Math. Appl. **39** (2000), 161–175.

[16] M. MATIĆ, J. PEČARIĆ AND N. UJEVIĆ, *On new estimation of the remainder in generalized Taylor's formula*, Math. Inequal. Appl. **2**, 3 (1999), 343–361.

[17] G. V. MILOVANOVIĆ AND J. E. PEČARIĆ, *On generalizations of the inequality of A. Ostrowski and some related applications*, Univ. Beograd. Publ. Eleektrotehn. Fak., Ser. Mat. Fiz., No. 544 – No. 576 (1976).

[18] D. S. MITRINOVIĆ, J. PEČARIĆ AND A. M. FINK, *Inequalities Involving Functions and their Integrals and Derivatives*, Kluwer Acad. Publ., Dordrecht, 1991.

[19] C. E. M. PEARCE, J. PEČARIĆ, *On Anastassiou's generalizations of the Ostrowski inequality and related results*, J. Comput. Anal. Appl. **2**, 3 (2000), 215–235.

[20] C. E. M. PEARCE, J. PEČARIĆ, S. VAROŠANEC, *Generalizations of some inequalities of Ostrowski-Grüs type*, Math. Ineq. Appl. **3**, 1 (2000), 25–34.

[21] J. PEČARIĆ, S. VAROŠANEC, *Harmonic Polynomials and generalizations of Ostrowski inequality with applications in numerical integration*, Nonlin. Anal. **47** (2001), 2365–2374.

(Received January 10, 2014)

K. M. Awan
Department of Mathematics, University of Sargodha
Sargodha, Pakistan
e-mail: khalid819@uos.edu.pk

J. Pečarić
Faculty of Textile Technology, University of Zagreb
Pierottijeva 6
10 000 Zagreb, Croatia
e-mail: pecaric@element.hr

A. Vukelić
Faculty of Food Technology and Biotechnology
University of Zagreb
Pierottijeva 6
10 000 Zagreb, Croatia
e-mail: avukelic@pbf.hr