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DUNKL–WILLIAMS TYPE INEQUALITIES FOR OPERATORS

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Abstract. The purpose of this paper is to discuss inequalities related to operator versions of the classical Dunkl-Williams inequality. We obtain refinements of some operator inequalities presented by Zou, He and Qaisar [Linear Algebra Appl. 438 (2013) 436–442].

1. Introduction

In this note we mainly adopt the notation and terminology in [7]. Throughout this note, we assume that $p,q \in \mathbb{R}$ with $\frac{1}{p}+\frac{1}{q}=1$.

In 1964, Dunkl and Williams [3] proved that the inequality

$$\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\| \leqslant \frac{4\|x-y\|}{\|x\|+\|y\|} \tag{1.1}$$

holds for all nonzero elements x,y in a normed linear space X. Pečarić and Rajić [6] obtained a refinement of (1.1): For all nonzero elements x,y in a normed linear space X,

$$\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\| \leqslant \frac{\sqrt{2\|x-y\|^2+2(\|x\|-\|y\|)^2}}{\max\{\|x\|,\|y\|\}}. \tag{1.2}$$

These authors also gave an operator-valued version of (1.2), which says that if $A,B \in B(H)$ with $|A|$ and $|B|$ are invertible and $p,q>1$, then

$$\left|A|A|^{-1}-B|B|^{-1}\right|^2 \leqslant |A|^{-1}\left(p|A-B|^2+q(|A|-|B|)^2\right)|A|^{-1}. \tag{1.3}$$

Saito and Tominaga [7] presented a generalization of (1.3), which states that if $A,B \in B(H)$ with polar decomposition $A=U|A|$, $B=V|B|$ and $p,q>1$, then

$$|(U-V)|A||^2 \leqslant p|A-B|^2+q(|A|-|B|)^2. \tag{1.4}$$

Recently, Zou, He and Qaisar [9] proved that if $1<p\leqslant 2$, then

$$|(U-V)|A||^2+\frac{2}{p}|(1-p)(A-B)-V(|A|-|B|)|^2 \leqslant p|A-B|^2+q(|A|-|B|)^2, \tag{1.5}$$

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Paper JMI-09-28

if $p > 2$, then

$$|(U-V)|A||^2 + \frac{2}{q}|A-B-(1-q)V(|A|-|B|)|^2 \leqslant p|A-B|^2 + q(|A|-|B|)^2. \quad (1.6)$$

Inequalities (1.5) and (1.6) are improvements of inequality (1.4). Meanwhile, we also shown that if $1 < p \leqslant 2$, then

$$p|A-B|^2 + q|V(|A|-|B|)|^2 - \frac{2}{q}|A-B+(q-1)V(|A|-|B|)|^2 \leqslant |(U-V)|A||^2, \quad (1.7)$$

if $p > 2$, then

$$p|A-B|^2 + q|V(|A|-|B|)|^2 - \frac{2}{p}|(p-1)(A-B)+V(|A|-|B|)|^2 \leqslant |(U-V)|A||^2. \quad (1.8)$$

Inequalities (1.7) and (1.8) are lower bounds for $|(U-V)|A||^2$.

In this note, we present refinements of inequalities (1.5), (1.6), (1.7) and (1.8).

2. Main results

Now, we give refinements of inequalities (1.5) and (1.6). To do this, we need the following lemma [4].

LEMMA 2.1. *Let $T_1, T_2 \in B(H)$. Then for $t \neq 0$,*

$$|T_1 - T_2|^2 + \frac{1}{t}|tT_1 + T_2|^2 = (1+t)|T_1|^2 + \left(1 + \frac{1}{t}\right)|T_2|^2. \quad (2.1)$$

For more information on the equivalent forms of equality (2.1) and their applications the reader is referred to [1, 2, 5, 8, 10].

THEOREM 2.1. *Let $A, B \in B(H)$ with polar decomposition $A = U|A|$ and $B = V|B|$. If $q > 0$, then*

$$|(U-V)|A||^2 + \frac{q}{p}|(1-p)(A-B) - V(|A|-|B|)|^2 \leqslant p|A-B|^2 + q(|A|-|B|)^2. \quad (2.2)$$

If $q < 0$, then

$$|(U-V)|A||^2 + \frac{q}{p}|(1-p)V(|A|-|B|) - (A-B)|^2 \leqslant p(|A|-|B|)^2 + q|A-B|^2.$$

Proof. Note that

$$|(U-V)|A||^2 = |A-B-V(|A|-|B|)|^2.$$

Putting $T_1 = A-B$, $T_2 = V(|A|-|B|)$, $p = 1+t$ and $q = 1+\frac{1}{t}$ in (2.1), we get

$$|(U-V)|A||^2 + \frac{q}{p}|(1-p)(A-B) - V(|A|-|B|)|^2 = p|A-B|^2 + q|V(|A|-|B|)|^2. \quad (2.3)$$

Since $V^*V \leqslant I$ and $q > 0$, we have

$$p\,|A-B|^2 + q\,|V(|A|-|B|)|^2 \leqslant p\,|A-B|^2 + q\,(|A|-|B|)^2. \tag{2.4}$$

Combining with (2.3) and (2.4), we obtain

$$|(U-V)\,|A||^2 + \frac{q}{p}\,|(1-p)(A-B) - V(|A|-|B|)|^2 \leqslant p\,|A-B|^2 + q\,(|A|-|B|)^2.$$

Putting $T_1 = V(|A|-|B|)$, $T_2 = A-B$, $p = 1+t$ and $q = 1+\dfrac{1}{t}$ in (2.1), we get

$$|(U-V)\,|A||^2 + \frac{q}{p}\,|(1-p)V(|A|-|B|) - (A-B)|^2 = p\,|V(|A|-|B|)|^2 + q\,|A-B|^2. \tag{2.5}$$

Since $q < 0$ implies $0 < p < 1$, we have

$$p\,|V(|A|-|B|)|^2 + q\,|A-B|^2 \leqslant p\,(|A|-|B|)^2 + q\,|A-B|^2. \tag{2.6}$$

It follows from (2.5) and (2.6) that

$$|(U-V)\,|A||^2 + \frac{q}{p}\,|(1-p)V(|A|-|B|) - (A-B)|^2 \leqslant p\,(|A|-|B|)^2 + q\,|A-B|^2.$$

This completes the proof. $\square$

REMARK 2.1. Simple calculations show that if $1 < p \leqslant 2$, then $q > 2$. So, inequality (2.2) is a refinement of inequality (1.5). Note that

$$\frac{1}{t}\,|tT_1 + T_2|^2 = t\left|T_1 + \frac{1}{t}T_2\right|^2. \tag{2.7}$$

So, inequality (2.2) can be rewritten as

$$|(U-V)\,|A||^2 + \frac{p}{q}\,|A-B-(1-q)V(|A|-|B|)|^2 \leqslant p\,|A-B|^2 + q\,(|A|-|B|)^2,$$

which is an improvement of inequality (1.6).

Next, we present some lower bounds for $|(U-V)\,|A||^2$.

THEOREM 2.2. *Let $A, B \in B(H)$ with polar decomposition $A = U\,|A|$ and $B = V\,|B|$. If $p < 0$, then*

$$p\,(|A|-|B|)^2 + q\,|A-B|^2 - \frac{q}{p}\,|(1-p)V(|A|-|B|) - (A-B)|^2 \leqslant |(U-V)\,|A||^2. \tag{2.8}$$

If $0 < p < 1$, then

$$p\,|A-B|^2 + q\,(|A|-|B|)^2 - \frac{q}{p}\,|(1-p)(A-B) - V(|A|-|B|)|^2 \leqslant |(U-V)\,|A||^2. \tag{2.9}$$

If $1 < p \leqslant 2$, *then*

$$p|A-B|^2 + q|V(|A|-|B|)|^2 - \frac{p+2}{2q}|A-B-(1-q)V(|A|-|B|)|^2 \leqslant |(U-V)|A||^2. \tag{2.10}$$

If $p > 2$, *then*

$$p|A-B|^2 + q|V(|A|-|B|)|^2 - \frac{q+2}{2p}|(1-p)(A-B)-V(|A|-|B|)|^2 \leqslant |(U-V)|A||^2. \tag{2.11}$$

Proof. Firstly, we prove inequality (2.8). Since $V^*V \leqslant I$ and $p < 0$, we have

$$p|V(|A|-|B|)|^2 + q|A-B|^2 \geqslant p(|A|-|B|)^2 + q|A-B|^2 \tag{2.12}$$

It follows from (2.5) and (2.12) that

$$p(|A|-|B|)^2 + q|A-B|^2 - \frac{q}{p}|(1-p)V(|A|-|B|)-(A-B)|^2 \leqslant |(U-V)|A||^2.$$

Next, we prove inequality (2.9). Let $0 < p < 1$, then $q < 0$. So, we have

$$p|A-B|^2 + q|V(|A|-|B|)|^2 \geqslant p|A-B|^2 + q(|A|-|B|)^2. \tag{2.13}$$

Combining with (2.3) and (2.13), we obtain

$$p|A-B|^2 + q(|A|-|B|)^2 - \frac{q}{p}|(1-p)(A-B)-V(|A|-|B|)|^2 \leqslant |(U-V)|A||^2.$$

Finally, we prove inequalities (2.10) and (2.11). It follows from (2.1) and (2.7) that

$$|T_1 - T_2|^2 + t\left|T_1 + \frac{1}{t}T_2\right|^2 = (1+t)|T_1|^2 + \left(1+\frac{1}{t}\right)|T_2|^2. \tag{2.14}$$

Taking the sum of (2.1) and (2.14) and putting $T_1 = A-B$, $T_2 = V(|A|-|B|)$, $p = 1+t$ and $q = 1+\frac{1}{t}$, we obtain

$$\begin{aligned} p|A-B|^2 + q|V(|A|-|B|)|^2 = {} & |(U-V)|A||^2 \\ & + \frac{q}{2p}|(1-p)(A-B)-V(|A|-|B|)|^2 \\ & + \frac{p}{2q}|A-B-(1-q)V(|A|-|B|)|^2. \end{aligned} \tag{2.15}$$

For $1 < p \leqslant 2$, by inequality (2.15), we have

$$\begin{aligned} p|A-B|^2 + q|V(|A|-|B|)|^2 \leqslant {} & |(U-V)|A||^2 \\ & + \frac{q}{2p}|(1-p)(A-B)-V(|A|-|B|)|^2 \\ & + \frac{1}{q}|A-B-(1-q)V(|A|-|B|)|^2, \end{aligned}$$

which is equivalent to

$$p\,|A-B|^2+q\,|V\,(|A|-|B|)|^2-\frac{p+2}{2q}\,|A-B-(1-q)\,V\,(|A|-|B|)|^2\leqslant|(U-V)\,|A||^2\,.$$

Since $p\geqslant 2$ implies $1<q\leqslant 2$, by inequality (2.15), we also have

$$\begin{aligned}p\,|A-B|^2+q\,|V\,(|A|-|B|)|^2 \leqslant\ & |(U-V)\,|A||^2\\ &+\frac{1}{p}\,|(1-p)\,(A-B)-V\,(|A|-|B|)|^2\\ &+\frac{p}{2q}\,|A-B-(1-q)\,V\,(|A|-|B|)|^2\,,\end{aligned}$$

which is equivalent to

$$p\,|A-B|^2+q\,|V\,(|A|-|B|)|^2-\frac{q+2}{2p}\,|(1-p)\,(A-B)-V\,(|A|-|B|)|^2\leqslant|(U-V)\,|A||^2\,.$$

This completes the proof. □

REMARK 2.2. Inequalities (2.10) and (2.11) are refinements of inequalities (1.7) and (1.8) respectively.

REMARK 2.3. Since $q>0$ implies $p>1$ or $p<0$ and $q<0$ implies $0<p<1$, combining with Theorems 2.1 and 2.2, we have the following results.

If $p<0$, then

$$\begin{aligned}&p\,(|A|-|B|)^2+q\,|A-B|^2-\frac{q}{p}\,|(1-p)\,V\,(|A|-|B|)-(A-B)|^2\\ &\leqslant|(U-V)\,|A||^2\\ &\leqslant p\,|A-B|^2+q\,(|A|-|B|)^2-\frac{q}{p}\,|(1-p)\,(A-B)-V\,(|A|-|B|)|^2\,.\end{aligned}$$

If $0<p<1$, then

$$\begin{aligned}&p\,|A-B|^2+q\,(|A|-|B|)^2-\frac{q}{p}\,|(1-p)\,(A-B)-V\,(|A|-|B|)|^2\\ &\leqslant|(U-V)\,|A||^2\\ &\leqslant p\,(|A|-|B|)^2+q\,|A-B|^2-\frac{q}{p}\,|(1-p)\,V\,(|A|-|B|)-(A-B)|^2\,.\end{aligned}$$

If $1<p\leqslant 2$, then

$$\begin{aligned}&p\,|A-B|^2+q\,|V\,(|A|-|B|)|^2-\frac{p+2}{2q}\,|A-B-(1-q)\,V\,(|A|-|B|)|^2\\ &\leqslant|(U-V)\,|A||^2\\ &\leqslant p\,|A-B|^2+q\,(|A|-|B|)^2-\frac{q}{p}\,|(1-p)\,(A-B)-V\,(|A|-|B|)|^2\,.\end{aligned}$$

If $p > 2$, then

$$\begin{aligned}
&p|A-B|^2 + q|V(|A|-|B|)|^2 - \frac{q+2}{2p}|(1-p)(A-B) - V(|A|-|B|)|^2 \\
&\leqslant |(U-V)|A||^2 \\
&\leqslant p|A-B|^2 + q(|A|-|B|)^2 - \frac{q}{p}|(1-p)(A-B) - V(|A|-|B|)|^2 .
\end{aligned}$$

REMARK 2.4. Theorems 2.1 and 2.2 of this note are also complements of Theorems 2.2 and 2.3 in [9], because Theorems 2.2 and 2.3 of [9] did not consider the case $p < 1$.

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