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COMPARISON OF DIFFERENCES AMONG POWER MEANS $Q_{r,\alpha}(a,b,\mathbf{x})s$

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Abstract. It is shown that the differences of the power means $Q_{r,\alpha}(a,b,\mathbf{x})s$ associated to the distinct sequences of weights are comparable, in terms of some constants depending on the smallest and largest quotients of the weights. Applications to the theory of operator inequalities are given.

1. Introduction

It is known that the weighted power means $M_t(\mathbf{x},\alpha)$ of the numbers x_i with weights α_i are defined as

$$\begin{cases} M_t(\mathbf{x},\alpha) = \left(\sum_{i=1}^{n} \alpha_i x_i^t\right)^{1/t}, & \text{if} \quad t \neq 0 \\ M_0(\mathbf{x},\alpha) = \prod_{i=1}^{n} x_i^{\alpha_i}. \end{cases}$$

It follows from the Jensen inequality that if $s \leqslant t$, then

$$M_s(\mathbf{x},\alpha) \leqslant M_t(\mathbf{x},\alpha).$$

In paper [9], the authors proved the following theorem.

THEOREM 1.1. ([9]) *If* $\alpha_i, \beta_i > 0$ *such that* $\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \beta_i = 1$, *and* $x_i > 0$ $(i = 1,\ldots,n)$, *then, for* $n \geqslant 2$, $s \leqslant t$, *and* $0 < t \leqslant 1$, *we have*

$$\min_{k=1,\cdots,n}\left\{\frac{\alpha_k}{\beta_k}\right\}^{\frac{1}{t}} \left(M_t(\mathbf{x},\beta) - M_s(\mathbf{x},\beta)\right) \leqslant M_t(\mathbf{x},\alpha) - M_s(\mathbf{x},\alpha)$$
$$\leqslant \max_{k=1,\cdots,n}\left\{\frac{\alpha_k}{\beta_k}\right\}^{\frac{1}{t}} \left(M_t(\mathbf{x},\beta) - M_s(\mathbf{x},\beta)\right). \quad (1)$$

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If we let $t = 1$ and $s = 0$ in Theorem 1.1, we get the self-bounds of differences between arithmetic and geometric means which were obtained by Aldaz [2]:

$$\min_{k=1,\cdots,n}\left\{\frac{\alpha_k}{\beta_k}\right\}\left(\sum_{i=1}^{n}\beta_i x_i-\prod_{i=1}^{n}x_i^{\beta_i}\right)\leqslant\sum_{i=1}^{n}\alpha_i x_i-\prod_{i=1}^{n}x_i^{\alpha_i}$$
$$\leqslant\max_{k=1,\cdots,n}\left\{\frac{\alpha_k}{\beta_k}\right\}\left(\sum_{i=1}^{n}\beta_i x_i-\prod_{i=1}^{n}x_i^{\beta_i}\right).$$

In [6], A. McD. Mercer investigated another family of power means as follows.

DEFINITION 1.2. ([6]) Suppose that $0<a<b$, $a\leqslant x_1\leqslant\cdots\leqslant x_n\leqslant b$ and α_i $(i=1,\ldots,n)$ are positive weights with $\sum\alpha_i=1$. The power means are defined by

$$\begin{cases} Q_{r,\alpha}(a,b,\mathbf{x}) = (a^r+b^r-M_r^r(\mathbf{x},\alpha))^{1/r} \quad \text{for} \quad r\neq 0,\\ Q_{0,\alpha}(a,b,\mathbf{x}) = \dfrac{ab}{M_0(\mathbf{x},\alpha)}. \end{cases}$$

For the power means, Mercer proved the following theorem.

THEOREM 1.3. ([6]) *For* $s<t$, $Q_{s,\alpha}(a,b,\mathbf{x})\leqslant Q_{t,\alpha}(a,b,\mathbf{x})$.

In this paper, we will study the power means $Q_{r,\alpha}(a,b,\mathbf{x})s$ from Definition 1.2 and will give a comparison of these power means associated to different sequences of weights.

The following is the well-known Jensen's inequality.

THEOREM 1.4. (Jensen's inequality) *A real valued function f defined on I is convex if and only if*

$$f\left(\sum_{i=1}^{n}p_i x_i\right)\leqslant\sum_{i=1}^{n}p_i f(x_i),$$

for all $x_1,x_2,\ldots,x_n\in I$ *and* $p_1,p_2,\ldots,p_n\in(0,1)$ *with* $\sum_{i=1}^{n}p_i=1$.

To get our purpose realized, we will use the following Jensen-Mercer inequality, which was proved by Mercer in [7].

THEOREM 1.5. ([7]) *Suppose that* $0<a<b$, $a\leqslant x_1\leqslant\cdots\leqslant x_n\leqslant b$ *and* α_i $(i=1,\ldots,n)$ *are non-negative weights with* $\sum_{i=1}^{n}\alpha_i=1$. *If f is convex on* $[a,b]$, *then*

$$f\left(a+b-\sum_{i=1}^{n}\alpha_i x_i\right)\leqslant f(a)+f(b)-\sum_{i=1}^{n}\alpha_i f(x_i). \tag{2}$$

In [1], it was shown that inequality (2) remains valid even when the condition on non-negativity of weights is relaxed. Actually, the authors of [1] proved the following result.

THEOREM 1.6. ([1]) *Suppose* $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in I^n$ *is a monotonic* n*-tuple, and the real numbers* $w_1, w_2, \ldots, w_n$ *satisfy* $W_k := \sum_{i=1}^{k} w_i$, $k = 1,2,\ldots,n$, *with* $0 \leqslant W_k \leqslant W_n$, $k = 1,2,\ldots,n-1$, *and* $W_n > 0$. *Then every convex function* $f : I \to \mathbb{R}$ *verifies the inequality:*

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \leqslant \frac{1}{W_n}\sum_{i=1}^{n} w_i f(x_i).$$

For a further generalization we refer the reader to [8]. Moreover, from [4, 5] we can see more information on the related studies.

2. Main result and proof

Our main result in this paper is the following theorem.

THEOREM 2.1. *If* $\alpha_i, \beta_i > 0$ *such that* $\sum\limits_{i=1}^{n} \alpha_i = \sum\limits_{i=1}^{n} \beta_i = 1$, *and* $0 < a < b$, $a \leqslant x_1 \leqslant \cdots \leqslant x_n \leqslant b$, *then for* $0 < t \leqslant 1$, $s \leqslant t$,

$$\begin{aligned}
&\min_{k=1,\cdots,n}\left\{\frac{\alpha_k}{\beta_k}\right\}^{\frac{1}{t}}\left(Q_{t,\beta}(a,b,\mathbf{x}) - Q_{s,\beta}(a,b,\mathbf{x})\right)\\
&\leqslant Q_{t,\alpha}(a,b,\mathbf{x}) - Q_{s,\alpha}(a,b,\mathbf{x})\\
&\leqslant \max_{k=1,\cdots,n}\left\{\frac{\alpha_k}{\beta_k}\right\}^{\frac{1}{t}}\left(Q_{t,\beta}(a,b,\mathbf{x}) - Q_{s,\beta}(a,b,\mathbf{x})\right).
\end{aligned} \tag{3}$$

Proof. Firstly, we suppose $s \neq 0$. Then the second inequality in (3) is equivalent to

$$\begin{aligned}
(a^s + b^s - M_s^s(\mathbf{x},\beta))^{\frac{1}{s}} \leqslant \left(a^t + b^t - M_t^t(\mathbf{x},\beta)\right)^{\frac{1}{t}} - \min_{k=1,\cdots,n}\left\{\frac{\beta_k}{\alpha_k}\right\}^{\frac{1}{t}}\left(a^t + b^t - M_t^t(\mathbf{x},\alpha)\right)^{\frac{1}{t}}\\
+ \min_{k=1,\cdots,n}\left\{\frac{\beta_k}{\alpha_k}\right\}^{\frac{1}{t}}(a^s + b^s - M_s^s(\mathbf{x},\alpha))^{\frac{1}{s}}.
\end{aligned} \tag{4}$$

Write

$$A = a^t + b^t - M_t^t(\mathbf{x},\beta),$$

$$B = \min_{k=1,\cdots,n}\left\{\frac{\beta_k}{\alpha_k}\right\}(a^s + b^s - M_s^s(\mathbf{x},\alpha))^{\frac{t}{s}},$$

$$C = \min_{k=1,\cdots,n}\left\{\frac{\beta_k}{\alpha_k}\right\}\left(a^t + b^t - M_t^t(\mathbf{x},\alpha)\right).$$

$$m = \min_{k=1,\cdots,n}\left\{\frac{\beta_k}{\alpha_k}\right\}.$$

When $0 < s \leqslant t$ or $s < 0$, the function $f(x) = x^{\frac{t}{s}}$ satisfies $f''(x) \geqslant 0$ on $[a,b]$. This means that $f(x) = x^{\frac{t}{s}}$ is convex on $[a,b]$. Thus, applying Theorem 1.5 we have

$$\begin{aligned}(a^s+b^s-M_s^s(\mathbf{x},\alpha))^{\frac{t}{s}} &= \left(a^s+b^s-\sum_{i=1}^n \alpha_i x_i^s\right)^{\frac{t}{s}} \\ &\leqslant (a^s)^{\frac{t}{s}}+(b^s)^{\frac{t}{s}}-\sum_{i=1}^n \alpha_i (x_i^s)^{\frac{t}{s}} \\ &= a^t+b^t-\sum_{i=1}^n \alpha_i x_i^t \\ &= \left(a^t+b^t-M_t^t(\mathbf{x},\alpha)\right).\end{aligned}$$

Hence,

$$B \leqslant C.$$

Noting that

$$a^t \leqslant x_i^t \leqslant b^t \qquad \text{for all } i=1,\cdots,n,$$

we have

$$(1-m)a^t \leqslant \sum_{i=1}^n (\beta_i - m\alpha_i)x_i^t \leqslant (1-m)b^t.$$

Hence,

$$a^t+b^t-\sum_{i=1}^n \beta_i x_i^t - ma^t - mb^t + m\sum_{i=1}^n \alpha_i x_i^t = (1-m)a^t+(1-m)b^t-\sum_{i=1}^n (\beta_i-m\alpha_i)x_i^t \geqslant 0,$$

i.e.

$$C \leqslant A.$$

Then, for $t \in (0,1]$, by using Theorem 1.5, we get

$$(A+B-C)^{\frac{1}{t}} \leqslant A^{\frac{1}{t}}+B^{\frac{1}{t}}-C^{\frac{1}{t}}.$$

i.e.

$$\begin{aligned}&\left[a^t+b^t-M_t^t(\mathbf{x},\beta)+\min_{k=1,\cdots,n}\left\{\frac{\beta_k}{\alpha_k}\right\}(a^s+b^s-M_s^s(\mathbf{x},\alpha))^{\frac{t}{s}}\right. \\ &\qquad \left. -\min_{k=1,\cdots,n}\left\{\frac{\beta_k}{\alpha_k}\right\}\left(a^t+b^t-M_t^t(\mathbf{x},\alpha)\right)\right]^{\frac{1}{t}} \\ &\leqslant \left(a^t+b^t-M_t^t(\mathbf{x},\beta)\right)^{\frac{1}{t}} - \min_{k=1,\cdots,n}\left\{\frac{\beta_k}{\alpha_k}\right\}^{\frac{1}{t}}\left(a^t+b^t-M_t^t(\mathbf{x},\alpha)\right)^{\frac{1}{t}} \\ &\qquad + \min_{k=1,\cdots,n}\left\{\frac{\beta_k}{\alpha_k}\right\}^{\frac{1}{t}}(a^s+b^s-M_s^s(\mathbf{x},\alpha))^{\frac{1}{s}}.\end{aligned}$$

On the other hand, we obtain

$$\begin{aligned}
&\left(a^s+b^s-M_s^s(\mathbf{x},\beta)\right)^{\frac{1}{s}}\\
&=\left(\left(a^s+b^s-\sum_{i=1}^{n}\beta_i x_i^s\right)^{\frac{t}{s}}\right)^{\frac{1}{t}} \qquad (5)\\
&=\left\{\left[a^s(1-m)+b^s(1-m)-\sum_{i=1}^{n}(\beta_i-m\alpha_i)x_i^s+m\left[\left(a^s+b^s-\sum_{i=1}^{n}\alpha_i x_i^s\right)^{\frac{1}{s}}\right]^s\right]^{\frac{t}{s}}\right\}^{\frac{1}{t}}.
\end{aligned}$$

Since

$$a \leqslant \left(a^s+b^s-\sum_{i=1}^{n}\alpha_i x_i^s\right)^{\frac{1}{s}} \leqslant b,$$

if we put $a=x_0$, $b=x_{n+1}$ then there exists a number $0\leqslant k\leqslant n$ such that

$$x_k \leqslant \left(a^s+b^s-\sum_{i=1}^{n}\alpha_i x_i^s\right)^{\frac{1}{s}} \leqslant x_{k+1}.$$

We reorder

$$a=x_0,x_1,\dots,x_k,\left(a^s+b^s-\sum_{i=1}^{n}\alpha_i x_i^s\right)^{\frac{1}{s}},x_{k+1},\dots,x_n,b=x_{n+1}$$

by putting

$$\begin{cases} y_i = x_{i-1} & i=1,\dots,k+1,\\ y_{k+2} = Q_{s,\alpha}(a,b,\mathbf{x}), & \\ y_{i+2} = x_i & i=k+1,\dots,n+1. \end{cases}$$

Let w_i, $(i=1,2,\dots,n+3)$ be the weight corresponding to y_i (equivalently y_i^s). Then we can easily get

$$0\leqslant W_k\leqslant W_{n+3}, \quad k=1,2,\dots,n+2, \quad \text{and} \quad W_{n+3}=1>0.$$

Since $x^{\frac{t}{s}}$ is convex for $s\leqslant t$ $(s\neq 0)$, we obtain by using Theorem 1.6 on y_i^s in (5),

$$\begin{aligned}
&\left\{\left[a^s(1-m)+b^s(1-m)-\sum_{i=1}^{n}(\beta_i-m\alpha_i)x_i^s+m\left[\left(a^s+b^s-\sum_{i=1}^{n}\alpha_i x_i^s\right)^{\frac{1}{s}}\right]^s\right]^{\frac{t}{s}}\right\}^{\frac{1}{t}}\\
&\leqslant\left\{a^t(1-m)+b^t(1-m)-\sum_{i=1}^{n}(\beta_i-m\alpha_i)x_i^t+m\left[\left(a^s+b^s-\sum_{i=1}^{n}\alpha_i x_i^s\right)\right]^{\frac{t}{s}}\right\}^{\frac{1}{t}}
\end{aligned}$$

$$\leqslant \left[a^t + b^t - M_t^t(\mathbf{x},\beta) + \min_{k=1,\cdots,n}\left\{\frac{\beta_k}{\alpha_k}\right\}\left(a^s + b^s - M_s^s(\mathbf{x},\alpha)\right)^{\frac{t}{s}} \right.$$
$$\left. - \min_{k=1,\cdots,n}\left\{\frac{\beta_k}{\alpha_k}\right\}\left(a^t + b^t - M_t^t(\mathbf{x},\alpha)\right)\right]^{\frac{1}{t}}.$$

Thus the second inequality is true.

To obtain the first inequality, we multiply both sides of the second inequality by $\min_{k=1,\cdots,n}\left\{\frac{\beta_k}{\alpha_k}\right\}^{\frac{1}{t}}$, and note that it is just the first inequality with the roles of the $\alpha's$ and the $\beta's$ interchanged.

Consequently, we have proved that, for $0 < t \leqslant 1$, $s \leqslant t$ $(s \neq 0)$,

$$\begin{aligned}
&\min_{k=1,\cdots,n}\left\{\frac{\alpha_k}{\beta_k}\right\}^{\frac{1}{t}}\left(\left(a^t + b^t - M_t^t(\mathbf{x},\beta)\right)^{\frac{1}{t}} - \left(a^s + b^s - M_s^s(\mathbf{x},\beta)\right)^{\frac{1}{s}}\right)\\
&\leqslant \left(a^t + b^t - M_t^t(\mathbf{x},\alpha)\right)^{\frac{1}{t}} - \left(a^s + b^s - M_s^s(\mathbf{x},\alpha)\right)^{\frac{1}{s}}\\
&\leqslant \max_{k=1,\cdots,n}\left\{\frac{\alpha_k}{\beta_k}\right\}^{\frac{1}{t}}\left(\left(a^t + b^t - M_t^t(\mathbf{x},\beta)\right)^{\frac{1}{t}} - \left(a^s + b^s - M_s^s(\mathbf{x},\beta)\right)^{\frac{1}{s}}\right).
\end{aligned} \tag{6}$$

It is known that

$$\lim_{r\to 0}\left(a^r + b^r - M_r^r(\mathbf{x},\alpha)\right)^{1/r} = \frac{ab}{M_0(\mathbf{x},\alpha)}.$$

Therefore, letting $s \to 0$ in (6), we get

$$\begin{aligned}
&\min_{k=1,\cdots,n}\left\{\frac{\alpha_k}{\beta_k}\right\}^{\frac{1}{t}}\left(\left(a^t + b^t - M_t^t(\mathbf{x},\beta)\right)^{\frac{1}{t}} - \frac{ab}{M_0(\mathbf{x},\beta)}\right)\\
&\leqslant \left(a^t + b^t - M_t^t(\mathbf{x},\alpha)\right)^{\frac{1}{t}} - \frac{ab}{M_0(\mathbf{x},\beta)}\\
&\leqslant \max_{k=1,\cdots,n}\left\{\frac{\alpha_k}{\beta_k}\right\}^{\frac{1}{t}}\left(\left(a^t + b^t - M_t^t(\mathbf{x},\beta)\right)^{\frac{1}{t}} - \frac{ab}{M_0(\mathbf{x},\beta)}\right).
\end{aligned} \tag{7}$$

(7) means that in the case of $s = 0$, (3) is also true under the assumptions of Theorem 2.1. □

REMARK 2.2. Theorem 2.1 is a generalization of previous results in [2, 5, 8].

Moreover, if we let $t = 1$ and $s = 0$ in Theorem 2.1, then we get the following corollary directly. The corollary shows us a new relation between the arithmetic means $\sum_{i=1}^n \alpha_i x_i$ and the geometric means $\prod x_i^{\alpha_i}$, which is different from that given by [2, Theorem 2.1].

COROLLARY 2.3. *If* $\alpha_i, \beta_i > 0$ *such that* $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i = 1$, *and* $0 < a < b$,

$a \leqslant x_1 \leqslant \cdots \leqslant x_n \leqslant b$, *then*

$$\min_{k=1,\cdots,n}\left\{\frac{\alpha_k}{\beta_k}\right\}\left(a+b-\sum_{i=1}^{n}\beta_i x_i-\frac{ab}{\prod x_i^{\beta_i}}\right) \leqslant a+b-\sum_{i=1}^{n}\alpha_i x_i-\frac{ab}{\prod x_i^{\alpha_i}}$$
$$\leqslant \max_{k=1,\cdots,n}\left\{\frac{\alpha_k}{\beta_k}\right\}\left(a+b-\sum_{i=1}^{n}\beta_i x_i-\frac{ab}{\prod x_i^{\beta_i}}\right).$$

Since

$$Q_{-1,\alpha}(a,b,\mathbf{x})=\left(a^{-1}+b^{-1}-\sum_{i=1}^{n}\alpha_i x_i^{-1}\right)^{-1}.$$

If we take $t=1$ and $s=-1$ in Theorem 2.1, then we get the following corollary. The corollary presents new inequalities, which are not implied by [9, Theorem 1.1].

COROLLARY 2.4. *If* $\alpha_i,\beta_i>0$ *such that* $\sum_{i=1}^{n}\alpha_i=\sum_{i=1}^{n}\beta_i=1$, *and* $0<a<b$, $a \leqslant x_1 \leqslant \cdots \leqslant x_n \leqslant b$, *then*

$$\min_{k=1,\cdots,n}\left\{\frac{\alpha_k}{\beta_k}\right\}\left[\left(a+b-\sum_{i=1}^{n}\beta_i x_i\right)-\left(a^{-1}+b^{-1}-\sum_{i=1}^{n}\beta_i x_i^{-1}\right)^{-1}\right]$$
$$\leqslant \left[\left(a+b-\sum_{i=1}^{n}\alpha_i x_i\right)-\left(a^{-1}+b^{-1}-\sum_{i=1}^{n}\alpha_i x_i^{-1}\right)^{-1}\right]$$
$$\leqslant \max_{k=1,\cdots,n}\left\{\frac{\alpha_k}{\beta_k}\right\}\left[\left(a+b-\sum_{i=1}^{n}\beta_i x_i\right)-\left(a^{-1}+b^{-1}-\sum_{i=1}^{n}\beta_i x_i^{-1}\right)^{-1}\right].$$

3. Applications

In this section, we apply our results to the theory of operator inequalities and give some new operator inequalities.

Let $\mathscr{H}$ be a Hilbert space, $\mathfrak{B}(\mathscr{H})$ the space of all bounded linear operators on $\mathscr{H}$, and $\mathfrak{B}_h(\mathscr{H})$ the semi-space in $\mathfrak{B}(\mathscr{H})$ of all self-adjoint operators. Moreover, let $\mathfrak{B}^{+}(\mathscr{H})$ and $\mathfrak{B}^{++}(\mathscr{H})$ denote the sets of all positive and positive invertible operators in $\mathfrak{B}_h(\mathscr{H})$ respectively. The weighted operator arithmetic mean ∇_ν for $\nu\in[0,1]$ and $A,B\in\mathfrak{B}^{+}(\mathscr{H})$, is defined as

$$A\nabla_\nu B=(1-\nu)A+\nu B.$$

It is well-know (cf., e.g., [3]) that if $X\in\mathfrak{B}_h(\mathscr{H})$ with spectrum $Sp(X)$ and f,g are continuous real-valued function on $Sp(X)$, then

$$f(t)\geqslant g(t),\quad t\in Sp(X),\qquad \text{implies that}\qquad f(X)\geqslant g(X). \tag{8}$$

THEOREM 3.1. *Let* $A,B \in \mathfrak{B}^{++}(\mathscr{H})$ *satisfy* $0 < aA \leqslant A \leqslant B \leqslant bA$ *for some positive numbers* $0 < a \leqslant 1 \leqslant b$. *Then, for* $\nu, \mu \in [0,1]$,

$$\begin{aligned}
&\min\left\{\frac{1-\nu}{1-\mu}, \frac{\nu}{\mu}\right\}\left(aA + bA - A\nabla_\mu B - ab \cdot A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{-\mu}A^{\frac{1}{2}}\right)\\
&\leqslant aA + bA - A\nabla_\nu B - ab \cdot A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{-\nu}A^{\frac{1}{2}}\\
&\leqslant \max\left\{\frac{1-\nu}{1-\mu}, \frac{\nu}{\mu}\right\}\left(aA + bA - A\nabla_\mu B - ab \cdot A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{-\mu}A^{\frac{1}{2}}\right).
\end{aligned}$$

Proof. It follows from Corollary 2.3 that for $0 < a \leqslant 1 \leqslant x \leqslant b$,

$$\begin{aligned}
&\min\left\{\frac{1-\nu}{1-\mu}, \frac{\nu}{\mu}\right\}\left(a + b - [(1-\mu) + \mu x] - ab \cdot (x)^{-\mu}\right)\\
&\leqslant a + b - [(1-\nu) + \nu x] - ab \cdot (x)^{-\nu}\\
&\leqslant \max\left\{\frac{1-\nu}{1-\mu}, \frac{\nu}{\mu}\right\}\left(a + b - [(1-\mu) + \mu x] - ab \cdot (x)^{-\mu}\right).
\end{aligned}$$

Hence by (8) for $X \in \mathfrak{B}^{++}(\mathscr{H})$ satisfying $1 \leqslant X \leqslant b$, i.e. $Sp(X) \subset [1,b]$, we have

$$\begin{aligned}
&\min\left\{\frac{1-\nu}{1-\mu}, \frac{\nu}{\mu}\right\}\left(a + b - [(1-\mu) + \mu X] - ab \cdot (X)^{-\mu}\right)\\
&\leqslant a + b - [(1-\nu) + \nu X] - ab \cdot (X)^{-\nu}\\
&\leqslant \max\left\{\frac{1-\nu}{1-\mu}, \frac{\nu}{\mu}\right\}\left(a + b - [(1-\mu) + \mu X] - ab \cdot (X)^{-\mu}\right). \qquad (9)
\end{aligned}$$

By our assumption, we see that

$$0 < a \leqslant 1 \leqslant A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leqslant b.$$

This together with (9) implies that

$$\begin{aligned}
&\min\left\{\frac{1-\nu}{1-\mu}, \frac{\nu}{\mu}\right\}\left(a + b - [(1-\mu) + \mu(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})] - ab \cdot (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{-\mu}\right)\\
&\leqslant a + b - [(1-\nu) + \nu(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})] - ab \cdot (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{-\nu}\\
&\leqslant \max\left\{\frac{1-\nu}{1-\mu}, \frac{\nu}{\mu}\right\}\left(a + b - [(1-\mu) + \mu(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})] - ab \cdot (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{-\mu}\right). \quad (10)
\end{aligned}$$

Multiplying on both sides of (10) by $A^{\frac{1}{2}}$ we get

$$\begin{aligned}
&\min\left\{\frac{1-\nu}{1-\mu}, \frac{\nu}{\mu}\right\}\left(aA + bA - [(1-\mu)A + \mu B] - ab \cdot A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{-\mu}A^{\frac{1}{2}}\right)\\
&\leqslant \left(aA + bA - [(1-\nu)A + \nu B] - ab \cdot A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{-\nu}A^{\frac{1}{2}}\right)\\
&\leqslant \max\left\{\frac{1-\nu}{1-\mu}, \frac{\nu}{\mu}\right\}\left(aA + bA - [(1-\mu)A + \mu B] - ab \cdot A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{-\mu}A^{\frac{1}{2}}\right).
\end{aligned}$$

THEOREM 3.2. *Let $A,B \in \mathfrak{B}^{++}(\mathscr{H})$ satisfying $0 < aA \leqslant A \leqslant B \leqslant bA$ for some positive numbers $0 < a \leqslant 1 \leqslant b$. Then for $\nu, \mu \in [0,1]$,*

$$\begin{aligned}
&\min\left\{\frac{1-\nu}{1-\mu}, \frac{\nu}{\mu}\right\}\left[aA + bA - A\nabla_\mu B - \left(a^{-1}A^{-1} + b^{-1}A^{-1} - A^{-1}\nabla_\mu B^{-1}\right)^{-1}\right] \\
&\leqslant \left[aA + bA - A\nabla_\nu B - \left(a^{-1}A^{-1} + b^{-1}A^{-1} - A^{-1}\nabla_\nu B^{-1}\right)^{-1}\right] \\
&\leqslant \max\left\{\frac{1-\nu}{1-\mu}, \frac{\nu}{\mu}\right\}\left[aA + bA - A\nabla_\mu B - \left(a^{-1}A^{-1} + b^{-1}A^{-1} - A^{-1}\nabla_\mu B^{-1}\right)^{-1}\right].
\end{aligned}$$

Proof. In view of Corollary 2.4, we know that for $0 < a \leqslant 1 \leqslant x \leqslant b$,

$$\begin{aligned}
&\min\left\{\frac{1-\nu}{1-\mu}, \frac{\nu}{\mu}\right\}\left(a + b - [(1-\mu) + \mu x] - [a^{-1} + b^{-1} - ((1-\mu) + \mu x^{-1})]^{-1}\right) \\
&\leqslant \left(a + b - [(1-\nu) + \nu x] - [a^{-1} + b^{-1} - ((1-\nu) + \nu x^{-1})]^{-1}\right) \\
&\leqslant \max\left\{\frac{1-\nu}{1-\mu}, \frac{\nu}{\mu}\right\}\left(a + b - [(1-\mu) + \mu x] - [a^{-1} + b^{-1} - ((1-\mu) + \mu x^{-1})]^{-1}\right).
\end{aligned}$$

By (8) we see that for $X \in \mathfrak{B}^{++}(\mathscr{H})$ with $1 \leqslant X \leqslant b$,

$$\begin{aligned}
&\min\left\{\frac{1-\nu}{1-\mu}, \frac{\nu}{\mu}\right\}\left(a+b-[(1-\mu)+\mu X]-[a^{-1}+b^{-1}-((1-\mu)+\mu X^{-1})]^{-1}\right) \\
&\leqslant \left(a + b - [(1-\nu) + \nu X] - [a^{-1} + b^{-1} - ((1-\nu) + \nu X^{-1})]^{-1}\right) \\
&\leqslant \max\left\{\frac{1-\nu}{1-\mu}, \frac{\nu}{\mu}\right\}\left(a+b-[(1-\mu)+\mu X]-[a^{-1}+b^{-1}-((1-\mu)+\mu X^{-1})]^{-1}\right). \qquad (11)
\end{aligned}$$

Our assumptions show that

$$0 < a \leqslant 1 \leqslant A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leqslant b.$$

Putting $X = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ in (11) and then multiplying $A^{\frac{1}{2}}$ from both sides, we obtain

$$\begin{aligned}
&\min\left\{\frac{1-\nu}{1-\mu}, \frac{\nu}{\mu}\right\}\left(aA+bA-A\nabla_\mu B-[a^{-1}A^{-1}+b^{-1}A^{-1}-((1-\mu)A^{-1}+\mu B^{-1})]^{-1}\right) \\
&\leqslant \left(aA + bA - A\nabla_\nu B - [a^{-1}A^{-1} + b^{-1}A^{-1} - ((1-\nu)A^{-1} + \nu B^{-1})]^{-1}\right) \\
&\leqslant \max\left\{\frac{1-\nu}{1-\mu}, \frac{\nu}{\mu}\right\}\left(aA+bA-A\nabla_\mu B-[a^{-1}A^{-1}+b^{-1}A^{-1}-((1-\mu)A^{-1}+\mu B^{-1})]^{-1}\right).
\end{aligned}$$

□

REMARK 3.3. The operator inequalities given in Theorems 3.1 and 3.2 are new.

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