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SOME NOTES ON GREEN–OSHER'S INEQUALITY

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Abstract. In this note we will first focus our attention on the equality case of Green-Osher's inequality and show that its equality holds if and only if the curve γ is a circle under the assumption that the function $F(x)$ is strictly convex on $(0,+\infty)$. As applications of Green-Osher's inequality we will then give some new geometric inequalities about convex plane curves. Finally, we will derive an upper bound estimate of Green-Osher's difference via the method of deforming convex curves into finite circles.

1. Introduction

The classical isoperimetric inequality is probably the best known geometric inequality which states that for a simple closed curve γ in the Euclidean plane $\mathbb{R}^2$ of length L and enclosing a region of area A, one gets that

$$L^2 - 4\pi A \geqslant 0,$$

and the equality holds if and only if γ is a circle. This fact was known to the ancient Greeks, mathematical proof was only given, however, in the 19th century by Steiner [21] and Edler [5]. Since then there have been many proofs, sharpened forms, generalizations, and applications of it, see, e.g., [1], [4], [12], [13], [14], [16], [17], [20], [22], etc., and the literature therein. Usually, variants of the classical isoperimetric inequality do not involve integral of curvature of the plane curve. In the 1980's, however, Gage [6] has shown an "isoperimetric inequality" which involves the integration of the squared curvature of the curve, that is,

$$\int_\gamma \kappa^2 ds \geqslant \frac{\pi L}{A}, \tag{1.1}$$

where κ, L and A denote the curvature of γ, its length and area, respectively. Gage also presented an example of H. Jacobowitz which shows that inequality (1.1) does not always hold for the bone shaped non-convex curves. Inequality (1.1) plays an essential role in the study of curve evolution problems in the plane (see [7], [8], [9], [11]).

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In [10], Green and Osher have given a more general theorem: *Let* γ *be a* C^2 *closed, strictly convex curve,* $F(x)$ *be a convex function on* $(0,+\infty)$*, then*

$$\frac{1}{2\pi}\int_{S^1} F(\rho(\theta))d\theta \geqslant \frac{1}{2}\left[F(-t_1)+F(-t_2)\right], \tag{1.2}$$

where $\rho(\theta)$ *is the radius of curvature of* γ*,* t_1 *and* t_2 *denote the two roots of Steiner polynomial of the domain enclosed by* γ*.* As a direct corollary of Green-Osher's theorem, the famous Gage inequality (1.1) can be easily obtained by taking $F(x)=\frac{1}{x}$ in (1.2). Moreover, Green and Osher in [10] have shown the following geometric inequalities by some special choices of the convex function $F(x)$,

$$\int_\gamma \kappa^3 ds \geqslant \frac{L^2\pi - 2A\pi^2}{A^2},$$
$$\int_\gamma \kappa^4 ds \geqslant \frac{L^3\pi - 3AL\pi^2}{A^3}.$$

They have also realized that the equality in (1.2) holds when γ is a circle. As an isoperimetric-type inequality, one should prove that if the equality holds in (1.2), then γ must be a circle. The first task of this note is to focus our attention on the equality case in (1.2) and prove that if $F(x)$ is a strictly convex function then that the equality in (1.2) holds can derive that γ is a circle.

As applications of the Green-Osher inequality we will give some new geometric inequalities about convex plane curves in the third section of this note. From (1.2), we know that it is not so easy to give an upper estimate of the quantity $\frac{1}{2\pi}\int_{S^1} F(\rho(\theta))d\theta$, we turn to consider the quantity

$$\frac{1}{2\pi}\int_{S^1} F(\rho(\theta))d\theta - \frac{1}{2}\left[F(-t_1)+F(-t_2)\right],$$

which is called by us *Green-Osher's difference* in the following text. Hence, (1.2) is equivalent to the fact that Green-Osher's difference is non-negative and it becomes an interesting question to give an upper bound for Green-Osher's difference. In the last part of this paper, motivated by the unit-speed outward normal flow, we investigate a curve flow, which deforms a given convex curve to a circle, to ensure upper bound estimate of Green-Osher's difference.

2. The equality case of Green-Osher's inequality

In this section, we will prove that the equality sign in Green-Osher's inequality (1.2) holds if and only if the convex curve γ is a circle when $F(x)$ is strictly convex. From the proof of Theorem 2.2 below, one can see that the assumption of the strict convexity of the function $F(x)$ is necessary.

Now, we introduce some basic preliminaries. Let $t_1 \geqslant t_2$ be the two roots of the Steiner polynomial

$$A(t) = A + Lt + \pi t^2$$

of γ, i.e.,

$$t_1 = -\frac{L}{2\pi} + \frac{u}{2\pi}, \quad t_2 = -\frac{L}{2\pi} - \frac{u}{2\pi},$$

where $u = \sqrt{L^2 - 4\pi A} \geqslant 0$. Let r_i, r_e be the radii of the inscribed and circumscribed discs of the domain enclosed by γ, respectively. Let κ be the curvature of γ, then $\rho = \frac{1}{\kappa}$ is the radius of curvature of γ. Denote by ρ_{max} and ρ_{min} the maximum and minimum values of ρ, respectively. If γ is a strictly convex and non-circular curve, then

$$-\rho_{max} < t_2 < -r_e < -\frac{L}{2\pi} < -r_i < t_1 < -\rho_{min}. \tag{2.1}$$

The proof of (2.1) can be found in [10]. And it is obvious that the above quantities are all equal if γ is a circle.

THEOREM 2.1. *Let γ be a closed, strictly convex C^2 curve, and $F(x)$ be a strictly convex function* (*i.e.,* $F''(x) > 0$) *on* $(0, +\infty)$. *Then*

$$\frac{1}{2\pi}\int_{S^1} F(\rho(\theta))d\theta \geqslant \frac{1}{2}\left[F(-t_1) + F(-t_2)\right], \tag{2.2}$$

and the equality in the above Green-Osher's inequality holds if and only if γ is a circle.

Proof. From Green-Osher's theorem ([10]), (2.2) holds automatically under the condition that $F(x)$ is strictly convex. If γ is a circle, it is clear that the equality holds. On the other hand, if we prove that $\frac{1}{2\pi}\int_{S^1} F(\rho(\theta))d\theta > \frac{1}{2}[F(\rho_1) + F(\rho_2)]$ when γ is not a circle, then the work is done. Therefore, we have to finish the following theorem. $\square$

THEOREM 2.2. *If γ is a closed, strictly convex and non-circular C^2 curve in the plane, then*

$$\frac{1}{2\pi}\int_{S^1} F(\rho(\theta))d\theta > \frac{1}{2}\left[F(-t_1) + F(-t_2)\right]. \tag{2.3}$$

Before proving Theorem 2.2, we need to recall some definitions and basic lemmata.

DEFINITION 2.3. ([10]) Consider

$$\sup\left\{\int_I \rho(\theta)d\theta | I \subset S^1, \int_I d\theta = \pi\right\}.$$

Let I_1 denote a subset of S^1 with measure π and realizing the above supremum, and let I_2 be its complement. There exists an $a \in \mathbb{R}^+$ such that

$$I_1 \subseteq \{\theta | \rho(\theta) \geqslant a\}, \quad I_2 \subseteq \{\theta | \rho(\theta) \leqslant a\}.$$

Set

$$\rho_1 = \frac{1}{\pi}\int_{I_1}\rho(\theta)d\theta, \quad \rho_2 = \frac{1}{\pi}\int_{I_2}\rho(\theta)d\theta,$$

then

$$\rho_1+\rho_2 = \frac{L}{\pi}, \quad \rho_1 \geqslant \rho_2.$$

LEMMA 2.4. *If γ is a closed, strictly convex and non-circular curve in the plane, then*

$$\rho_1 > \rho_2.$$

Proof. By Definition 2.3, $\rho_1 \geqslant \rho_2$. To prove this lemma, we need only to prove γ is a circle when $\rho_1 = \rho_2$. If $\rho_1 = \rho_2$, then for any $I \subset S^1$, and $\int_I d\theta = \pi$,

$$\int_I \rho(\theta)d\theta = \frac{L}{2}. \tag{2.4}$$

Set

$$A = \left\{\theta|\rho(\theta) > \frac{L}{2\pi}\right\}, \quad B = \left\{\theta|\rho(\theta) < \frac{L}{2\pi}\right\}, \quad C = S^1 \setminus (A\cup B),$$

then $\int_A d\theta < \pi$ and $\int_B d\theta < \pi$. Next, we have to prove $A=\emptyset$ and $B=\emptyset$. If $A\neq\emptyset$, then there exists an interval $C' \subset C$ such that $\int_{A\cup C'} d\theta = \pi$ or $\int_{B\cup C'} d\theta = \pi$. Without loss of generality, we set $\int_{A\cup C'} d\theta = \pi$, then

$$\int_{A\cup C'}\rho(\theta)d\theta > \frac{L}{2\pi}m(A) + \frac{L}{2\pi}(\pi - m(A)) = \frac{L}{2},$$

which contradicts to (2.4). Analogously, it can be deduced that $B=\emptyset$. □

The following Lemmata 2.5 and 2.6 have appeared in Green-Osher [10] and Pan-Yang [18]. Since Lemma 2.7 (1)(2)(3) are direct corollaries of the convexity or strict convexity of $F(x)$, we omitted the detailed proofs.

LEMMA 2.5. ([18]) *If γ is a closed, strictly convex and non-circular curve in the plane, then*

$$\rho_1 > -t_2.$$

LEMMA 2.6. ([10]) *Let $F(x)$ be a (strictly) convex function on $(0,+\infty)$, then*

$$\frac{1}{2\pi}\int_{S^1} F(\rho(\theta))d\theta \geqslant \frac{1}{2}[F(\rho_1)+F(\rho_2)].$$

LEMMA 2.7. (1) *If $F(x)$ is strictly convex function on $(0,+\infty)$, then for arbitrary c and $b>a>0$,*

$$F(c+b)+F(c-b) > F(c+a)+F(c-a).$$

(2) *If* $F(x)$ *is (strictly) convex function on* $(0,+\infty)$*, then for arbitrary* b *and* $a \geqslant 0$,

$$F(b+a)+F(b-a) \geqslant 2F(b).$$

(3) *If* $F(x)$ *is strictly convex function on* $(0,+\infty)$*, then for arbitrary* b *and* $a > 0$,

$$F(b+a)+F(b-a) > 2F(b).$$

Proof of Theorem 2.2. From Lemma 2.6,

$$\frac{1}{2\pi}\int_{S^1} F(\rho(\theta))d\theta \geqslant \frac{1}{2}[F(\rho_1)+F(\rho_2)].$$

If γ is a non-circular curve, then $u=\sqrt{L^2-4\pi A}>0$. By Lemma 2.4, there exists a $b>0$ such that $\rho_1=\frac{L}{2\pi}+b$, $\rho_2=\frac{L}{2\pi}-b$. Furthermore, from Lemma 2.5, $b>\frac{u}{2\pi}>0$. Now, by Lemma 2.7(1),

$$\begin{aligned}F(\rho_1)+F(\rho_2) &= F\left(\frac{L}{2\pi}+b\right)+F\left(\frac{L}{2\pi}-b\right) > F\left(\frac{L}{2\pi}+\frac{u}{2\pi}\right)+F\left(\frac{L}{2\pi}-\frac{u}{2\pi}\right)\\ &= F(-t_1)+F(-t_2).\end{aligned}$$

Hence,

$$\frac{1}{2\pi}\int_{S^1} F(\rho(\theta))d\theta \geqslant \frac{1}{2}[F(\rho_1)+F(\rho_2)] > \frac{1}{2}[F(-t_1)+F(-t_2)],$$

which completes the proof. □

3. Application of Green-Osher's inequality

In this section, we will give some new geometric inequalities as applications of the Green-Osher inequality.

THEOREM 3.1. *Let* γ *be a closed, strictly convex* C^2 *curve with length* L*, area* A *and curvature* κ*, if* $n\in\mathbb{N}^+$*, then*

$$\int_0^{2\pi} \kappa^n d\theta \geqslant \frac{\pi\left[L^n+\left(2^{n-1}-1\right)\left(\sqrt{L^2-4\pi A}\right)^n\right]}{2^{n-1}A^n}, \tag{3.1}$$

$$\int_0^{2\pi} \frac{1}{\kappa^n}d\theta \geqslant \frac{L^n+\left(2^{n-1}-1\right)\left(\sqrt{L^2-4\pi A}\right)^n}{(2\pi)^{n-1}}. \tag{3.2}$$

Moreover, the equality signs in (3.1)–(3.2) *hold if and only if* γ *is a circle.*

REMARK 1. In (3.1), when $n=1$, it becomes Gage's inequality [6], and in (3.2), when $n=2$, it turns into

$$\int_0^{2\pi}\frac{d\theta}{\kappa^2} = \oint_\gamma \frac{ds}{\kappa} \geqslant \frac{L^2-2\pi A}{\pi},$$

which is appeared in [18].

To prove Theorem 3.1, we need the following lemma which is motivated by Burago-Zelgaller's monograph [2].

LEMMA 3.2. *If* $x \geqslant a \geqslant 0$, $n \in \mathbb{N}^+$, *then*

$$(x+a)^n + (x-a)^n \geqslant 2\left[x^n + \left(2^{n-1}-1\right)a^n\right]. \tag{3.3}$$

Proof. It is obvious that (3.3) holds when $a = 0$ or $a > 0$, $n = 1$. For the case of $a > 0$ and $n \geqslant 2$, let us consider functions

$$\varphi_n(x) = (x-a)^n - x^n + nax^{n-1}, \quad \psi_n(x) = (x+a)^n - x^n - nax^{n-1}.$$

If $\varphi_n(x)$ and $\psi_n(x)$ are increasing on $[a, +\infty)$, then

$$\begin{aligned}
(x-a)^n &= \varphi_n(x) + x^n - nx^{n-1}a \geqslant \varphi_n(a) + x^n - nx^{n-1}a = (n-1)a^n + x^n - nx^{n-1}a, \\
(x+a)^n &= \psi_n(x) + x^n + nx^{n-1}a \geqslant \psi_n(a) + x^n + nx^{n-1}a = (2^n - n - 1)a^n + x^n + nx^{n-1}a,
\end{aligned}$$

and thus

$$(x+a)^n + (x-a)^n \geqslant 2\left[x^n + \left(2^{n-1}-1\right)a^n\right].$$

Therefore, we need only to prove that functions $\varphi_n(x)$ and $\psi_n(x)$ are increasing on $[a, +\infty)$, which can be finished by induction over n for $n \geqslant 2$.

It is clear that $\psi_2(x) = a^2$ and $\psi_3(x) = 3xa^2 + a^3$ are increasing on $[a, +\infty)$. Now assume that $\psi_{n-1}(x)$ is increasing on $[a, +\infty)$ which implies that $\psi_{n-1}(x) \geqslant \psi_{n-1}(a)$. Since $\psi_n'(x) = n\psi_{n-1}(x)$ and $\psi_{n-1}(a) = (2^{n-1} - n)a^{n-1} > 0$, one gets $\psi_n'(x) = n\psi_{n-1}(x) \geqslant n\psi_{n-1}(a) > 0$, then $\psi_n(x)$ is increasing on $[a, +\infty)$ when $n \geqslant 2$, $a > 0$. Similarly, $\varphi_n(x)$ is also increasing on $[a, +\infty)$ when $n \geqslant 2$, $a > 0$. $\square$

Proof of Theorem 3.1. From Theorem 2.1, we have known that if $F(x)$ is strictly convex function on $(0, +\infty)$, then

$$\frac{1}{2\pi}\int_{S^1} F(\rho(\theta))d\theta \geqslant \frac{1}{2}\left[F(-t_1) + F(-t_2)\right].$$

Taking $F(x) = \frac{1}{x^n}$ or $F(x) = x^n$ gives us

$$\frac{1}{2\pi}\int_0^{2\pi} \kappa^n d\theta \geqslant \frac{1}{2}\left[\left(-\frac{1}{t_1}\right)^n + \left(-\frac{1}{t_2}\right)^n\right] = \frac{1}{2}\frac{(L+u)^n + (L-u)^n}{2^n A^n}$$

or

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{\kappa^n} d\theta \geqslant \frac{1}{2}\left[(-t_1)^n + (-t_2)^n\right] = \frac{1}{2}\frac{(L+u)^n + (L-u)^n}{(2\pi)^n},$$

where $u = \sqrt{L^2 - 4\pi A} \geqslant 0$. From Lemma 3.2 it follows that

$$\int_0^{2\pi} \kappa^n d\theta \geqslant \frac{\pi[L^n + (2^{n-1}-1)(\sqrt{L^2-4\pi A})^n]}{2^{n-1}A^n},$$

$$\int_0^{2\pi} \frac{1}{\kappa^n} d\theta \geqslant \frac{L^n + (2^{n-1}-1)(\sqrt{L^2-4\pi A})^n}{(2\pi)^{n-1}}.$$

It is clear that the equality signs in (3.1)–(3.2) hold when γ is a circle. Subsequently, it is enough to prove the above two inequalities are strict when γ is not a circle. Again, choosing $F(x) = \frac{1}{x^n}$ or $F(x) = x^n$ in (2.3) and using (3.3) can yields

$$\int_0^{2\pi} \kappa^n d\theta > \frac{\pi[(L+u)^n + (L-u)^n]}{2^n A^n} \geqslant \frac{\pi[L^n + (2^{n-1}-1)(\sqrt{L^2-4\pi A})^n]}{2^{n-1} A^n},$$
$$\int_0^{2\pi} \frac{1}{\kappa^n} d\theta > \frac{\pi[(L+u)^n + (L-u)^n]}{(2\pi)^n} \geqslant \frac{L^n + (2^{n-1}-1)(\sqrt{L^2-4\pi A})^n}{(2\pi)^{n-1}},$$

where $u = \sqrt{L^2-4\pi A} > 0$ (γ is not a circle). $\square$

THEOREM 3.3. *Let γ be a closed, strictly convex C^2 curve with curvature κ, length L and area A, if $n \geqslant 2$, $n \in \mathbb{N}^+$, then*

$$\int_0^{2\pi} \frac{1}{\sqrt[n]{\kappa}} d\theta \leqslant \left(\frac{L}{2A}\right)^{1-\frac{1}{n}} \left(L + \sqrt{L^2-4\pi A}\right), \tag{3.4}$$

and the equality in (3.4) holds if and only if γ is a circle.

Proof. Set $F(x) = -\sqrt[n]{x}$ and by (2.2),

$$\begin{aligned}\int_0^{2\pi} \frac{1}{\sqrt[n]{\kappa}} d\theta &\leqslant \pi \left[(-t_1)^{\frac{1}{n}} + (-t_2)^{\frac{1}{n}}\right] \\ &= \pi \cdot \left(\frac{1}{2\pi}\right)^{\frac{1}{n}} \left[\left(L - \sqrt{L^2-4\pi A}\right)^{\frac{1}{n}} + \left(L + \sqrt{L^2-4\pi A}\right)^{\frac{1}{n}}\right].\end{aligned}$$

Since

$$\begin{aligned}&\left(L - \sqrt{L^2-4\pi A}\right)^{\frac{1}{n}} + \left(L + \sqrt{L^2-4\pi A}\right)^{\frac{1}{n}} \\ &= (4\pi A)^{\frac{1}{n}} \left[\frac{1}{(L+\sqrt{L^2-4\pi A})^{\frac{1}{n}}} + \frac{1}{(L-\sqrt{L^2-4\pi A})^{\frac{1}{n}}}\right] \\ &\leqslant 2 \cdot (4\pi A)^{\frac{1}{n}} \cdot \frac{1}{\left(L-\sqrt{L^2-4\pi A}\right)^{\frac{1}{n}}} \leqslant 2 \cdot \left(\frac{4\pi A}{L}\right)^{\frac{1}{n}} \cdot \frac{1}{1-\left(1-\frac{4\pi A}{L^2}\right)^{\frac{1}{2}}} \\ &= 2 \cdot \left(\frac{L}{4\pi A}\right)^{1-\frac{1}{n}} \left(L + \sqrt{L^2-4\pi A}\right),\end{aligned}$$

where the second inequality holds since $0 < 1-(1-\frac{4\pi A}{L^2})^{\frac{1}{2}} \leqslant 1$, we conclude that

$$\int_0^{2\pi} \frac{1}{\sqrt[n]{\kappa}} d\theta \leqslant \left(\frac{L}{2A}\right)^{1-\frac{1}{n}} \left(L + \sqrt{L^2-4\pi A}\right).$$

It is clear that the equality in (3.4) holds when γ is a circle. The above expression is strict when γ is not a circle via the choice of $F(x) = -\sqrt[n]{x}$ and the estimate of $(L-\sqrt{L^2-4\pi A})^{\frac{1}{n}} + (L+\sqrt{L^2-4\pi A})^{\frac{1}{n}}$. $\square$

Another significant application of Theorem 2.1 is the following geometric Jensen inequality.

THEOREM 3.4. (Geometric Jensen inequality) *Let γ be a closed, strictly convex C^2 curve with length L and radius of curvature $\rho(\theta)$, and $F(x)$ be a strictly convex function on $(0,+\infty)$, then*

$$\frac{1}{2\pi}\int_{S^1} F(\rho(\theta))d\theta \geqslant F\left(\frac{L}{2\pi}\right). \tag{3.5}$$

Moreover, the equality in (3.5) holds if and only if γ is a circle.

Proof. Since γ is a closed and strictly convex C^2 curve, $\rho(\theta)$ is continuous on $[0,2\pi]$, by Jensen's theorem,

$$\frac{1}{2\pi}\int_0^{2\pi} F(\rho(\theta))d\theta \geqslant F\left[\frac{1}{2\pi}\int_0^{2\pi}\rho(\theta)d\theta\right] = F\left(\frac{L}{2\pi}\right).$$

The equality in (3.5) holds when γ is a circle ($\rho(\theta) \equiv \frac{L}{2\pi}$). From Theorem 2.2 and Lemma 2.7(3), (3.5) is strict when γ is not a circle, the desired result is obtained. $\square$

REMARK 2. From Lemma 2.7(2),

$$F(-t_1)+F(-t_2) = F\left(\frac{L}{2\pi}-\frac{u}{2\pi}\right)+F\left(\frac{L}{2\pi}+\frac{u}{2\pi}\right) \geqslant 2F\left(\frac{L}{2\pi}\right),$$

it is clear that (2.2) is stronger than (3.5).

Moreover, choosing $F(x) = \frac{1}{x^\alpha}$, $F(x) = x^\beta$ and $F(x) = -x^\gamma$, respectively, where $\alpha > 0$, $\beta > 1$ and $0 < \gamma < 1$, we conclude the following corollary.

COROLLARY 3.5. *If γ is a closed, strictly convex C^2 curve with length L and curvature κ, then*

$$\int_0^{2\pi} \kappa^\alpha d\theta \geqslant \frac{(2\pi)^{\alpha+1}}{L^\alpha}, \tag{3.6}$$

$$\int_0^{2\pi} \frac{1}{\kappa^\beta} d\theta \geqslant \frac{L^\beta}{(2\pi)^{\beta-1}}, \tag{3.7}$$

$$\int_0^{2\pi} \frac{1}{\kappa^\gamma} d\theta \leqslant \frac{L^\gamma}{(2\pi)^{\gamma-1}}, \tag{3.8}$$

where $\alpha > 0$, $\beta > 1$ and $0 < \gamma < 1$. Moreover, the equality signs in (3.6)–(3.8) hold if and only if γ is a circle.

As a special case of Theorem 3.4, one can derive Ros's inequality in $\mathbb{R}^2$.

COROLLARY 3.6. (Ros's inequality in $\mathbb{R}^2$ ([15], [19] and [23])) *If γ is a closed, strictly convex C^2 curve with length L, area A and curvature κ, then*

$$\int_\gamma \frac{1}{\kappa} ds \geqslant \frac{L^2}{2\pi} \geqslant 2A, \tag{3.9}$$

and the equality in (3.9) holds if and only if γ is a circle.

4. An upper bound estimate of Green-Osher's difference

Motivated by the usage of unit-speed outward normal flow in the proof of Green-Osher's inequality, we introduce a new flow (4.1) below for convex curves to investigate new geometric inequalities. Let $X_0(\varphi)$ be a strictly convex, C^2 curve in the plane $\mathbb{R}^2$. Let the origin of $\mathbb{R}^2$ be in the domain enclosed by the curve X_0. Deform X_0 according to

$$\begin{cases} \frac{\partial X}{\partial t}(\varphi,t) = (p(\varphi,t) - \rho(\varphi,t))N(\varphi,t), & (\varphi,t) \in S^1 \times (0,T), \\ X(\varphi,0) = X_0(\varphi), & \varphi \in S^1, \end{cases} \tag{4.1}$$

where $N(\varphi,t)$ is the inward pointing unit normal vector field to the evolving curve, $p(\varphi,t) = -\langle X(\varphi,t), N(\varphi,t)\rangle$ and $\rho(\varphi,t)$ denote its support function and radius of curvature at (φ,t), respectively. By considering the behavior of the evolving curve in the flow (4.1), we can give an upper bound estimate of Green-Osher's difference.

THEOREM 4.1. *Let $\gamma(\theta)$ be a closed, strictly convex C^2 curve in the plane with radius of curvature $\rho(\theta)$, where θ is the tangential angle, and $F(x)$ be a convex function on $(0,+\infty)$, then*

$$\frac{1}{2\pi}\int_{S^1} F(\rho(\theta))d\theta - \frac{1}{2}(F(-t_1)+F(-t_2)) \leqslant c\int_{S^1}\left(\frac{\partial \rho}{\partial \theta}\right)^2 d\theta, \tag{4.2}$$

where the constant $c = \frac{1}{16\pi}\max\{F''(x) | m \leqslant x \leqslant M\}$ and m, M are the minimum and maximum of the function $\rho(\theta)$ for $\theta \in S^1$, respectively. Moreover, if $F(x)$ is strictly convex, the equality in (4.2) holds if and only if $\gamma(\theta)$ is a circle.

Proof. Since the initial curve is strictly convex, the tangential angle θ can be used as its parameter. To simplify the geometric analysis of this flow, by Proposition 1.1 of [3], we can add a tangential component $\alpha = -\frac{\partial p}{\partial \theta} + \frac{\partial \rho}{\partial \theta}$ to the flow (4.1) to guarantee that $\frac{\partial \theta}{\partial t} = 0$, that is to say, we need only to consider the following flow

$$\begin{cases} \frac{\partial X}{\partial t}(\theta,t) = \alpha(\theta,t)T(\theta,t) + (p(\theta,t) - \rho(\theta,t))N(\theta,t), & (\theta,t) \in S^1 \times (0,T), \\ X(\theta,0) = X_0(\theta), & \theta \in S^1. \end{cases} \tag{4.3}$$

It is easy to show that this flow preserves the perimeter of the evolving curve (see Gage [8], Jiang-Pan [11], Pan-Yang [18]). The evolution equation of the radius of curvature is

$$\frac{\partial \rho}{\partial t}(\theta,t)=\frac{\partial^2 \rho}{\partial \theta^2}(\theta,t).$$

By the maximum principle for heat equations, $m \leqslant \rho(\theta,t) \leqslant M$, where m, M are the minimum and maximum of the function $\rho(\theta,0)$, respectively.

Set $w=\rho-\frac{L_0}{2\pi}$, L_0 is the perimeter of initial curve, then $w_t=w_{\theta\theta}$. By Wirtinger's inequality, we have

$$\frac{d}{dt}\int_0^{2\pi} w^2 d\theta=\int_0^{2\pi} 2ww_{\theta\theta}d\theta=-2\int_0^{2\pi} w_\theta^2 d\theta \leqslant -2\int_0^{2\pi} w^2 d\theta.$$

Hence,

$$\int_0^{2\pi} w(\theta,t)^2 d\theta \leqslant \int_0^{2\pi} w(\theta,0)^2 d\theta \cdot e^{-2t},$$

as the time t goes to infinity, $w(\theta,t)$ converges to 0, it means that $\rho(\theta,t)$ converges to a constant $\frac{L_0}{2\pi}$, which implies that the evolving curve converges to a circle with perimeter equal to L_0. Since the function $F(x)$ is convex on $(0,+\infty)$, from Lemma 2.7(2) it follows that

$$F(-t_1)+F(-t_2)=F\left(\frac{L_0}{2\pi}+\frac{u}{2\pi}\right)+F\left(\frac{L_0}{2\pi}-\frac{u}{2\pi}\right) \geqslant 2F\left(\frac{L_0}{2\pi}\right),$$

where $u=\sqrt{L_0-4\pi A_0} \geqslant 0$, L_0, A_0 are the length and area of the initial curve, respectively. Moreover,

$$\frac{1}{2\pi}\int_{S^1} F(\rho_0(\theta))d\theta-\frac{1}{2}(F(-t_1)+F(-t_2)) \leqslant \frac{1}{2\pi}\int_{S_1} F(\rho_0(\theta))d\theta-F\left(\frac{L_0}{2\pi}\right).$$

Therefore, we need only to show that

$$\frac{1}{2\pi}\int_{S^1} F(\rho_0(\theta))d\theta-F\left(\frac{L_0}{2\pi}\right) \leqslant c\int_{S^1}\left(\frac{\partial \rho_0}{\partial \theta}\right)^2 d\theta, \tag{4.4}$$

for $c=\frac{1}{16\pi}\max\{F''(x) | m \leqslant x \leqslant M\}$. Under the flow (4.3), from Wirtinger's inequality, we get

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2\pi}\int_{S^1} F(\rho(\theta,t))d\theta-F\left(\frac{L}{2\pi}\right)-c\int_{S^1}\left(\frac{\partial \rho}{\partial \theta}\right)^2 d\theta\right]\\
&=\frac{1}{2\pi}\int_{S^1} F'(\rho)\frac{\partial^2 \rho}{\partial \theta^2}d\theta-2c\int_{S^1}\frac{\partial \rho}{\partial \theta}\frac{\partial^3 \rho}{\partial \theta^3}d\theta\\
&=-\frac{1}{2\pi}\int_{S^1} F''(\rho)\left(\frac{\partial \rho}{\partial \theta}\right)^2 d\theta+2c\int_{S^1}\left(\frac{\partial^2 \rho}{\partial \theta^2}\right)^2 d\theta\\
&\geqslant -\frac{1}{2\pi}\int_{S^1} F''(\rho)\left(\frac{\partial \rho}{\partial \theta}\right)^2 d\theta+8c\int_{S^1}\left(\frac{\partial \rho}{\partial \theta}\right)^2 d\theta \geqslant 0,
\end{aligned}$$

which means that the quantity $Q \triangleq \frac{1}{2\pi}\int_{S^1} F(\rho(\theta,t))d\theta - F\left(\frac{L}{2\pi}\right) - c\int_{S^1}\left(\frac{\partial\rho}{\partial\theta}\right)^2 d\theta$ is increasing in the time t. Since the evolving curve converges to a circle, the quantity Q tends to 0 as t goes to infinity, and thus (4.4) is proved.

Moreover, if $F(x)$ is strictly convex on $(0,+\infty)$, the equality in (4.2) holds when the curve is a circle. On the other hand, to prove the curve is a circle under the condition that the equality in (4.2) holds, we have to show that the inequality (4.2) is strict when the curve is not a circle and $F(x)$ is strictly convex. By Lemma 2.7(3) and (4.4), one can obtain the desired result. □

Set $F(x) = x^2$ in (4.4), we obtain a Wirtinger-type inequality,

$$\int_{S^1} \rho^2 d\theta \leqslant \frac{L^2}{2\pi} + \frac{1}{4}\int_{S^1}\left(\frac{\partial\rho}{\partial\theta}\right)^2 d\theta. \tag{4.5}$$

The equality in (4.5) holds when γ is a circle, while there exist non-circular curves such that the equality in (4.5) holds, e.g. the curve γ with radius of curvature $\rho(\theta) = 4 - 3\sin(2\theta)$.

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