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MULTIPLE OPIAL–TYPE INEQUALITIES FOR GENERAL KERNELS WITH APPLICATIONS

SAJID IQBAL, JOSIP PEČARIĆ AND MUHAMMAD SAMRAIZ

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Abstract. The main purpose of this paper is to give the general multiple Opial-type inequalities for general kernels. We consider the monotocity and boundedness of the weight functions to prove new inequalities. As applications of our general results we establish new inequalities for Widder's derivatives and linear differential operator. Results from [11] are obtain by applying the Canavati fractional derivatives to our main results.

1. Introduction

Mathematical inequalities which involve derivatives and integrals of functions is of great interest. Opial's inequality [14] is of great importance in mathematics with respect to the applications in theory of differential equations and difference equations. Many mathematicians gave the improvements and generalizations in last few decades to add the considerable contribution in the literature and it has attracted a great deal of attention in the recent literature (see, for instance, [1], [3], [4], [7], [9], [12], [15]).

Let us recall that the original Opial's inequality [14] (see also [13, p. 114]) states the following:

THEOREM 1.1. *Let $a>0$. If $f\in C^1[0,a]$ with $f(0)=f(a)=0$ and $f(t)>0$ on $(0,a)$, then*

$$\int_0^a |f(t)f'(t)|dt \leqslant \frac{a}{4}\int_0^a (f'(t))^2 dt.$$

The constant $\frac{a}{4}$ is the best possible.

Agarwal, Pang and Alzer [2, 3, 6] study the Opial-type inequalities involving ordinary derivatives and their applications in differential equations and difference equations. Here our main purpose is to give the Opial-type inequalities for general kernels. We also provide connection between our results in this paper with [11].

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By $C^n[a,b]$ we denote the space of all functions on $[a,b]$ which have continuous derivatives up to order n, and $AC[a,b]$ is the space of all absolutely continuous functions on $[a,b]$. By $AC^n[a,b]$ we denote the space of all functions $f \in C^{n-1}[a,b]$ with $f^{(n-1)} \in AC[a,b]$.

By $L_p[a,b]$, $1 \leqslant p < \infty$, we denote the space of all Lebesgue measurable functions f for which $|f^p|$ is Lebesgue integrable on $[a,b]$, and by $L_\infty[a,b]$ the set of all functions measurable and essentially bounded on $[a,b]$. Clearly, $L_\infty[a,b] \subset L_p[a,b]$ for all $p \geqslant 1$.

We say that a function $y : [a,b] \to \mathbb{R}$ belongs to the class $U(k,f)$ if it admits the representation

$$y(t) = \int_a^t k(t,\tau) f(\tau)\, d\tau, \tag{1.1}$$

where f is a continuous function on $[a,b]$ and k is an arbitrary continuous kernel.

In (1.1) for nonnegative measurable kernel k, the positivity of f implies the positivity of y.

The paper is organized in the following way: After introduction in Section 2, we prove the Opial-type inequality for general kernels. We consider the monotonocity and boundedness of weight functions to prove new inequalities. In Section 3, we give results for one weighted and non-weighted case. In Section 4, we give application of our main results for linear differential operator. In Section 5, we give results for Widder's derivatives. We conclude this paper by providing applications for the Canavati fractional derivative which is in fact shows that results in this paper generalizes results from [11].

2. Main Results

Theorems and proofs in this section are based on a technique from [11] which resulted with new inequalities for the general kernels.

Our first main result is given in the following theorem.

THEOREM 2.1. *Let* $y_i \in U(k_i, f)$, $i = 1,\dots,N$, $N \in \mathbb{N}$. *Let* w_1 *and* w_2 *be continuous weight functions on* $[a,x]$ *with* $w_1 \geqslant 0$ *and* $w_2 > 0$. *Let* $r_i \geqslant 0$, $r = \sum_{i=1}^N r_i > 0$, $p > 0$, $q \geqslant 0$, $\sigma = \frac{1}{p+q} < 1$, *and* $f \in L_{p+q}[a,b]$. *Then*

$$\int_a^x w_1(t) \prod_{i=1}^N |y_i(t)|^{r_i p} |f(t)|^q dt$$

$$\leqslant \left(\frac{q}{rp+q}\right)^{\sigma q} \left(\int_a^x [w_1(t)]^{\frac{1}{\sigma p}} [w_2(t)]^{-\frac{q}{p}} \prod_{i=1}^N \left(\int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}} |k_i(t,\tau)|^{\frac{1}{1-\sigma}} d\tau \right)^{\frac{(1-\sigma) r_i}{\sigma}} dt \right)^{\sigma p}$$

$$\times \left(\int_a^x w_2(t)|f(t)|^{\frac{1}{\sigma}} dt \right)^{\sigma(rp+q)}. \tag{2.1}$$

Proof. Let $q \neq 0$. Since $w_2 > 0$ we have

$$|y_i(t)| \leqslant \int_a^t [w_2(\tau)]^{-\sigma} [w_2(\tau)]^{\sigma} |k_i(t,\tau)| \, |f(\tau)| d\tau.$$

Using Hölder's inequality for $\frac{1}{1-\sigma}$ and $\frac{1}{\sigma}$, for $t \in [a,x]$ we obtain

$$|y_i(t)| \leqslant \left(\int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}} |k_i(t,\tau)|^{\frac{1}{1-\sigma}} d\tau \right)^{1-\sigma} \left(\int_a^t w_2(\tau)|f(\tau)|^{\frac{1}{\sigma}} d\tau \right)^{\sigma}.$$

Since $w_2 > 0$, and $r = \sum_{i=1}^N r_i > 0$, we have

$$\begin{aligned}
&\int_a^x w_1(t) \prod_{i=1}^N |y_i(t)|^{r_i p} |f(t)|^q dt \\
&\leqslant \int_a^x w_1(t)[w_2(t)]^{-\sigma q}[w_2(t)]^{\sigma q}|f(t)|^q \\
&\quad \times \prod_{i=1}^N \left(\int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}} |k_i(t,\tau)|^{\frac{1}{1-\sigma}} d\tau \right)^{(1-\sigma)r_i p} \left(\int_a^t w_2(\tau)|f(\tau)|^{\frac{1}{\sigma}} d\tau \right)^{\sigma r_i p} dt \\
&= \int_a^x w_1(t)[w_2(t)]^{-\sigma q}[w_2(t)]^{\sigma q}|f(t)|^q \left(\int_a^t w_2(\tau)|f(\tau)|^{\frac{1}{\sigma}} d\tau \right)^{\sigma r p} \\
&\quad \times \prod_{i=1}^N \left(\int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}} |k_i(t,\tau)|^{\frac{1}{1-\sigma}} d\tau \right)^{(1-\sigma)r_i p} dt.
\end{aligned} \tag{2.2}$$

Applying Hölder's inequality for $\{\frac{1}{\sigma p}, \frac{1}{\sigma q}\}$ and simple integration, we get

$$\begin{aligned}
&\int_a^x w_1(t) \prod_{i=1}^N |y_i(t)|^{r_i p} |f(t)|^q dt \\
&\leqslant \left(\int_a^x w_1(t)^{\frac{1}{\sigma p}} [w_2(t)]^{-\frac{q}{p}} \prod_{i=1}^N \left(\int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}} |k_i(t,\tau)|^{\frac{1}{1-\sigma}} d\tau \right)^{\frac{(1-\sigma)r_i}{\sigma}} dt \right)^{\sigma p}
\end{aligned}$$

$$\times\left(\int_a^x w_2(t)|f(t)|^{\frac{1}{\sigma}}\left(\int_a^t w_2(\tau)|f(\tau)|^{\frac{1}{\sigma}}d\tau\right)^{\frac{rp}{q}}dt\right)^{\sigma q}$$

$$=\left(\int_a^x w_1(t)^{\frac{1}{\sigma p}}[w_2(t)]^{-\frac{q}{p}}\prod_{i=1}^N\left(\int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}}|k_i(t,\tau)|^{\frac{1}{1-\sigma}}d\tau\right)^{\frac{(1-\sigma)r_i}{\sigma}}dt\right)^{\sigma p}$$

$$\times\left(\frac{q}{rp+q}\right)^{\sigma q}\left(\int_a^x w_2(t)|f(t)|^{\frac{1}{\sigma}}dt\right)^{\sigma(rp+q)},$$

which gives us the inequality (2.1).

If we take $q=0$ $\left(\sigma=\frac{1}{p}\right)$ in inequality (2.2), we get,

$$\int_a^x w_1(t)\prod_{i=1}^N|y_i(t)|^{r_ip}dt$$

$$\leqslant\left(\int_a^x w_1(t)\prod_{i=1}^N\left(\int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}}|k_i(t,\tau)|^{\frac{1}{1-\sigma}}d\tau\right)^{(1-\sigma)r_ip}dt\right)\left(\int_a^x w_2(\tau)|f(\tau)|^{\frac{1}{\sigma}}d\tau\right)^{r},$$

from which we get inequality (2.1) for $q=0$. This complete the proof. □

We use monotonicity of w_1 and w_2 to prove our next result.

THEOREM 2.2. *Suppose that the assumptions of the Theorem 2.1 hold. Suppose also that w_1 is an increasing and w_2 is decreasing functions. Then*

$$\int_a^x w_1(t)\prod_{i=1}^N|y_i(t)|^{r_ip}|f(t)|^q dt$$

$$\leqslant\left(\frac{q}{rp+q}\right)^{\sigma q}w_1(x)[w_2(x)]^{-\sigma(rp+q)}\left[\int_a^x\prod_{i=1}^N\left(\int_a^t|k_i(t,\tau)|^{\frac{1}{1-\sigma}}d\tau\right)^{\frac{(1-\sigma)r_i}{\sigma}}dt\right]^{\sigma p}$$

$$\times\left(\int_a^x w_2(t)|f(t)|^{\frac{1}{\sigma}}dt\right)^{\sigma(rp+q)}. \quad (2.3)$$

Proof. We start the proof with inequality (2.1) proved in Theorem 2.1. By monotonicity of w_1 and w_2 we have

$$\left[\int_a^x [w_1(t)]^{\frac{1}{p\sigma}} [w_2(t)]^{-\frac{q}{p}} \prod_{i=1}^N \left(\int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}} |k_i(t,\tau)|^{\frac{1}{1-\sigma}} d\tau\right)^{\frac{(1-\sigma)r_i}{\sigma}} dt\right]^{\sigma p}$$

$$\leqslant w_1(x)[w_2(x)]^{-\sigma(rp+q)} \left[\int_a^x \prod_{i=1}^N \left(\int_a^t |k_i(t,\tau)|^{\frac{1}{1-\sigma}} d\tau\right)^{\frac{(1-\sigma)r_i}{\sigma}} dt\right]^{\sigma p}. \quad (2.4)$$

Use of inequality (2.4) in inequality (2.1) give us the inequality (2.3).
For $q = 0$, we proceed same as in Theorem 2.1. □

To prove the next theorem we suppose that the weight functions are bounded.

THEOREM 2.3. *Suppose that the assumptions of the Theorem 2.1 hold. Suppose also $w_1(t) \leqslant B$ and $A \leqslant w_2(t)$ for $t \in [a,x]$. Then*

$$\int_a^x w_1(t) \prod_{i=1}^N |y_i(t)|^{r_i p} |f(t)|^q dt$$

$$\leqslant \left(\frac{q}{\sigma p+q}\right)^{\sigma q} BA^{-\sigma(rp+q)} \left(\int_a^x \prod_{i=1}^N \left(\int_a^t |k_i(t,\tau)|^{\frac{1}{1-\sigma}} d\tau\right)^{\frac{(1-\sigma)r_i}{\sigma}} dt\right)^{\sigma p}$$

$$\times \left(\int_a^x w_2(t)|f(t)|^{\frac{1}{\sigma}} dt\right)^{\sigma(rp+q)}. \quad (2.5)$$

Proof. Applying Theorem 2.2 with conditions $w_1(t) \leqslant B$, and $w_2(t) \geqslant A$ to get inequality (2.5). □

With extra parameters $s_1, s_2,$ and s_3, we can establish some new inequalities.

THEOREM 2.4. *Suppose that the assumptions of the Theorem 2.1 hold. Suppose also that $s_k > 1$ and $\frac{1}{s_k} + \frac{1}{s'_k} = 1$ for $k = 1,2,3$. Then*

$$\int_a^x w_1(t) \prod_{i=1}^N |y_i(t)|^{r_i p} |f(t)|^q dt$$

$$\leqslant \left(\frac{q}{rp+q}\right)^{\sigma q} P(x)Q(x)R(x) \left[\int_a^x \prod_{i=1}^N \left(\int_a^t |k_i(t,\tau)|^{\frac{1}{1-\sigma}s_1} d\tau\right)^{\frac{(1-\sigma)}{s_1 p} r_i s_2 s_3} dt\right]^{\frac{\sigma p}{s_2 s_3}}$$

$$\times \left(\int_a^x w_2(t)|f(t)|^{\frac{1}{\sigma}} dt\right)^{\sigma(rp+q)}, \quad (2.6)$$

where

$$P(x)=\left(\int_a^x [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}s_1'}d\tau\right)^{\frac{(1-\sigma)rp}{s_1'}},$$

$$Q(x)=\left(\int_a^x [w_1(t)]^{\frac{s_2'}{\sigma p}}dt\right)^{\frac{\sigma p}{s_2'}},$$

and

$$R(x)=\left(\int_a^x [w_2(t)]^{-\frac{q}{p}s_2s_3'}dt\right)^{\frac{\sigma p}{s_2 s_3'}}.$$

Proof. We start the proof with the inequality (2.1) proved in Theorem 2.1 and then applying Hölder's inequality for the parameters s_1 and s_1', we get

$$\int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}}|k_i(t,\tau)|^{\frac{1}{1-\sigma}}d\tau \leqslant \left(\int_a^t w_2(\tau)^{-\frac{\sigma}{1-\sigma}s_1'}d\tau\right)^{\frac{1}{s_1'}}\cdot\left(\int_a^t |k_i(t,\tau)|^{\frac{1}{1-\sigma}s_1}d\tau\right)^{\frac{1}{s_1}}.$$

Now follows

$$\prod_{i=1}^N\left(\int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}}|k_i(t,\tau)|^{\frac{1}{1-\sigma}}d\tau\right)^{\frac{(1-\sigma)r_i}{\sigma}} \leqslant \prod_{i=1}^N\left(\int_a^t w_2(\tau)^{-\frac{\sigma}{1-\sigma}s_1'}d\tau\right)^{\frac{1}{s_1'}\frac{(1-\sigma)r_i}{\sigma}}\cdot\left(\int_a^t |k_i(t,\tau)|^{\frac{1}{1-\sigma}s_1}d\tau\right)^{\frac{1}{s_1}\frac{(1-\sigma)r_i}{\sigma}},$$

we get

$$\prod_{i=1}^N\left(\int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}}|k_i(t,\tau)|^{\frac{1}{1-\sigma}}d\tau\right)^{\frac{(1-\sigma)r_i}{\sigma}} \leqslant \left(\int_a^t w_2(\tau)^{-\frac{\sigma}{1-\sigma}s_1'}d\tau\right)^{\frac{1}{s_1'}\frac{(1-\sigma)r}{\sigma}}\cdot\prod_{i=1}^N\left(\int_a^t |k_i(t,\tau)|^{\frac{1}{1-\sigma}s_1}d\tau\right)^{\frac{1}{s_1}\frac{(1-\sigma)r_i}{\sigma}}.$$

Applying Hölder's inequality for the s_2, s_2' and s_3, s_3' we obtain

$$
\begin{aligned}
&\left[\int_a^x [w_1(t)]^{\frac{1}{\sigma p}}[w_2(t)]^{-\frac{q}{p}} \prod_{i=1}^N \left(\int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}} |k_i(t,\tau)|^{\frac{1}{1-\sigma}} d\tau\right)^{\frac{(1-\sigma)r_i}{\sigma}} dt\right]^{\sigma p} \\
&\leqslant \left[\int_a^x [w_1(t)]^{\frac{1}{\sigma p}}[w_2(t)]^{-\frac{q}{p}} \left(\int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}s_1'} d\tau\right)^{\frac{(1-\sigma)r}{s_1'\sigma}}\right. \\
&\quad \left.\times \prod_{i=1}^N \left(\int_a^t |k_i(t,\tau)|^{\frac{1}{1-\sigma}s_1} d\tau\right)^{\frac{(1-\sigma)r_i}{\sigma s_1}} dt\right]^{\sigma p} \\
&\leqslant \left(\int_a^x [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}s_1'} d\tau\right)^{\frac{(1-\sigma)rp}{s_1'}} \left[\int_a^x [w_1(t)]^{\frac{1}{\sigma p}}[w_2(t)]^{-\frac{q}{p}}\right. \\
&\quad \left.\times \prod_{i=1}^N \left(\int_a^t |k_i(t,\tau)|^{\frac{1}{1-\sigma}s_1} d\tau\right)^{\frac{(1-\sigma)r_i}{s_1 p}} dt\right]^{\sigma p} \\
&= P(x)\left[\int_a^x [w_1(t)]^{\frac{1}{\sigma p}}[w_2(t)]^{-\frac{q}{p}} \prod_{i=1}^N \left(\int_a^t |k_i(t,\tau)|^{\frac{1}{1-\sigma}s_1} d\tau\right)^{\frac{(1-\sigma)r_i}{s_1 p}} dt\right]^{\sigma p} \\
&\leqslant P(x)\left(\int_a^x [w_1(t)]^{\frac{s_2'}{\sigma p}} dt\right)^{\frac{\sigma p}{s_2'}} \left(\int_a^x w_2(t)^{-\frac{q s_2}{p}} \prod_{i=1}^N \left(\int_a^t |k_i(t,\tau)|^{\frac{1}{1-\sigma}s_1} d\tau\right)^{\frac{(1-\sigma)r_i s_2}{s_1 p}} dt\right)^{\frac{\sigma p}{s_2}} \\
&\leqslant P(x)Q(x)\left(\int_a^x [w_2(t)]^{-\frac{q}{p}s_2 s_3'} dt\right)^{\frac{\sigma p}{s_2 s_3'}} \left[\int_a^x \prod_{i=1}^N \left(\int_a^t |k_i(t,\tau)|^{\frac{1}{1-\sigma}s_1} d\tau\right)^{\frac{(1-\sigma)}{s_1 p} r_i s_2 s_3} dt\right]^{\frac{\sigma p}{s_2 s_3}} \\
&= P(x)Q(x)R(x)\left[\int_a^x \prod_{i=1}^N \left(\int_a^t |k_i(t,\tau)|^{\frac{1}{1-\sigma}s_1} d\tau\right)^{\frac{(1-\sigma)}{s_1 p} r_i s_2 s_3} dt\right]^{\frac{\sigma p}{s_2 s_3}}.
\end{aligned}
$$

Then from inequality (2.1) we can get inequality (2.6). □

3. One weighted and non-weighted cases

Our first result of this section is a direct consequence of Theorem 2.1.

THEOREM 3.1. *Let $N \in \mathbb{N}$ and w be continuous positive weight function on $[a,x]$. Let $r_i \geqslant 0,\ r = \sum_{i=1}^{N} r_i > 0,\ p > 0,\ q \geqslant 0,\ \sigma = \frac{1}{p+q} < 1$, and $f \in L_{p+q}[a,b]$. Then*

$$\int_a^x w(t) \prod_{i=1}^{N} |y_i(t)|^{r_i p} |f(t)|^q dt$$
$$\leqslant \left(\frac{q}{rp+q}\right)^{\sigma q} \left(\int_a^x w(t) \prod_{i=1}^{N} \left(\int_a^t [w(\tau)]^{-\frac{\sigma}{1-\sigma}} |k_i(t,\tau)|^{\frac{1}{1-\sigma}} d\tau \right)^{\frac{(1-\sigma) r_i}{\sigma}} dt \right)^{\sigma p}$$
$$\times \left(\int_a^x w(t) |f(t)|^{\frac{1}{\sigma}} dt \right)^{\sigma(rp+q)}.$$

Now if we have decreasing weight functions, then we need the assumption $r \geqslant 1$.

THEOREM 3.2. *Suppose that the assumptions of the Theorem 3.1 hold. Suppose also that $r \geqslant 1$ and w is a decreasing function. Then*

$$\int_a^x w(t) \prod_{i=1}^{N} |y_i(t)|^{r_i p} |f(t)|^q dt$$
$$\leqslant \left(\frac{q}{rp+q}\right)^{\sigma q} [w(x)]^{\frac{p(1-r)}{p+q}} \left(\int_a^x \prod_{i=1}^{N} \left(\int_a^t |k_i(t,\tau)|^{\frac{1}{1-\sigma}} d\tau \right)^{\frac{(1-\sigma) r_i}{\sigma}} dt \right)^{\sigma p}$$
$$\times \left(\int_a^x w(t) |f(t)|^{\frac{1}{\sigma}} dt \right)^{\sigma(rp+q)}.$$

Proof. Let $q \neq 0$. Since w is a decreasing function, we have

$$1 \leqslant \left(\frac{w(\tau)}{w(t)}\right)^{\sigma}, \quad \tau \leqslant t.$$

Also

$$\begin{aligned}\prod_{i=1}^{N}|y_i(t)|^{r_ip} &\leqslant \prod_{i=1}^{N}\left(\int_a^t |k_i(t,\tau)||f(\tau)|d\tau\right)^{r_ip}\\ &= \prod_{i=1}^{N}\left(\int_a^t [w(\tau)]^{\sigma}[w(\tau)]^{-\sigma}|k_i(t,\tau)||f(\tau)|d\tau\right)^{r_ip}\\ &\leqslant [w(t)]^{-\sigma rp}\prod_{i=1}^{N}\left(\int_a^t [w(\tau)]^{\sigma}|k_i(t,\tau)||f(\tau)|d\tau\right)^{r_ip}.\end{aligned}$$

Using Hölder's inequality for $\frac{1}{1-\sigma}$ and $\frac{1}{\sigma}$ for $t\in[a,b]$, we have

$$\prod_{i=1}^{N}|y_i(t)|^{r_ip} \leqslant [w(t)]^{-\sigma rp}\prod_{i=1}^{N}\left(\int_a^t |k_i(t,\tau)|^{\frac{1}{1-\sigma}}d\tau\right)^{(1-\sigma)r_ip}\left(\int_a^t w(\tau)|f(\tau)|^{\frac{1}{\sigma}}d\tau\right)^{\sigma rp}.$$

Therefore

$$\begin{aligned}&\int_a^x w(t)\prod_{i=1}^{N}|y_i(t)|^{r_ip}|f(t)|^q dt\\ &\leqslant \int_a^x [w(t)]^{1-\sigma rp}\prod_{i=1}^{N}\left(\int_a^t |k_i(t,\tau)|^{\frac{1}{1-\sigma}}d\tau\right)^{(1-\sigma)r_ip}\left(\int_a^t w(\tau)|f(\tau)|^{\frac{1}{\sigma}}d\tau\right)^{\sigma rp}|f(t)|^q dt.\end{aligned}$$

Now applying Hölder's inequality for $\frac{1}{\sigma p}$ and $\frac{1}{\sigma q}$, we get

$$\begin{aligned}&\int_a^x w(t)\prod_{i=1}^{N}|y_i(t)|^{r_ip}|f(t)|^q dt\\ &\leqslant \left(\int_a^x \prod_{i=1}^{N}\left(\int_a^t |k_i(t,\tau)|^{\frac{1}{1-\sigma}}d\tau\right)^{\frac{(1-\sigma)r_i}{\sigma}}dt\right)^{\sigma p}\\ &\quad\times\left(\int_a^x [w(t)]^{\frac{1-\sigma rp}{\sigma q}}\left(\int_a^t w(\tau)|f(\tau)|^{\frac{1}{q}}d\tau\right)^{\frac{rp}{q}}|f(t)|^{\frac{1}{\sigma}}dt\right)^{\sigma q}\\ &= \left(\int_a^x \prod_{i=1}^{N}\left(\int_a^t |k_i(t,\tau)|^{\frac{1}{1-\sigma}}d\tau\right)^{\frac{(1-\sigma)r_i}{\sigma}}dt\right)^{\sigma p}\end{aligned}$$

$$\times\left(\int_a^x [w(t)]^{\frac{1-\sigma rp-\sigma q}{\sigma q}} w(t)\left(\int_a^t w(\tau)|f(\tau)|^{\frac{1}{\sigma}}d\tau\right)^{\frac{rp}{q}}|f(t)|^{\frac{1}{\sigma}}dt\right)^{\sigma q}$$
$$\leqslant\left(\frac{q}{rp+q}\right)^{\sigma q}[w(x)]^{\frac{p(1-r)}{p+q}}\left(\int_a^x\prod_{i=1}^N\left(\int_a^t|k_i(t,\tau)|^{\frac{1}{1-\sigma}}d\tau\right)^{\frac{(1-\sigma)r_i}{\sigma}}dt\right)^{\sigma p}$$
$$\times\left(\int_a^x w(t)|f(t)|^{\frac{1}{\sigma}}dt\right)^{\sigma(rp+q)}. \tag{3.1}$$

For $q=0$ we proceed as in Theorem 2.1. This complete the proof. □

If $r=1$ we have Alzer's inequality [5, Theorem 2.1] for general kernel.

COROLLARY 3.3. *Suppose that the assumptions of the Theorem 3.1 are satisfied and let $r=1$. Then*

$$\int_a^x w(t)\prod_{i=1}^N|y_i(t)|^{r_ip}|f(t)|^q dt$$
$$\leqslant\left(\frac{q}{p+q}\right)^{\sigma q}\left(\int_a^x\prod_{i=1}^N\left(\int_a^t|k_i(t,\tau)|^{\frac{1}{1-\sigma}}d\tau\right)^{\frac{(1-\sigma)r_i}{\sigma}}dt\right)^{\sigma p}\left(\int_a^x w(t)|f(t)|^{\frac{1}{\sigma}}dt\right).$$

Now we suppose that the weight function w is bounded to prove the next result.

THEOREM 3.4. *Suppose that the assumptions of the Theorem 3.1 are satisfied. Suppose also that $r\geqslant 1$, and $A\leqslant w(t)\leqslant B$ for $t\in[a,x]$. Then*

$$\int_a^x w(t)\prod_{i=1}^N|y_i(t)|^{r_ip}|f(t)|^q dt$$
$$\leqslant\left(\frac{q}{rp+q}\right)^{\sigma q}\left(\frac{B}{A^r}\right)^{\sigma p}\left(\int_a^x\prod_{i=1}^N\left(\int_a^t|k_i(t,\tau)|^{\frac{1}{1-\sigma}}d\tau\right)^{\frac{(1-\sigma)r_i}{\sigma}}dt\right)^{\sigma p}$$
$$\times\left(\int_a^x w(t)|f(t)|^{\frac{1}{\sigma}}dt\right)^{\sigma(rp+q)}.$$

Proof. The proof follows from inequality (3.1) by using $A\leqslant w(t)\leqslant B$, $t\in[a,b]$. □

Here we give the corresponding non-weighted case of our weighted result.

THEOREM 3.5. *Let* $y_i \in U(k_i, f)$, $i = 1, ..., N$, $N \in \mathbb{N}$. *Let* $r_i \geqslant 0$, $r = \sum_{i=1}^{N} r_i > 0$, $p > 0$, $q \geqslant 0$, $\sigma = \frac{1}{p+q} < 1$, *and* $f \in L_{p+q}[a,b]$. *Then*

$$\int_a^x \prod_{i=1}^{N} |y_i(t)|^{r_i p} |f(t)|^q dt \leqslant \left(\frac{q}{rp+q}\right)^{\sigma q} \left(\int_a^x \prod_{i=1}^{N} \left(\int_a^t |k_i(t,\tau)|^{\frac{1}{1-\sigma}} d\tau \right)^{\frac{(1-\sigma) r_i p}{\sigma}} dt \right)^{\sigma p} \left(\int_a^x |f(t)|^{\frac{1}{\sigma}} dt \right)^{\sigma(rp+q)}.$$

Proof. Similar to the proof of Theorem 2.1. □

4. Results for linear differential operator

Let $[a,b] \subset \mathbb{R}$, and $h, a_i \in [a,b]$ for $i = 0, ..., n-1 (n \in \mathbb{N})$. Let

$$L = D^n + a_{n-1}(x) D^{n-1} + ... + a_0(x),$$

be a fixed linear differential operator on $C^n[a,b]$. Let $y_1(x), ... y_n(x)$ be a set of linearly independent solution to $Ly = 0$ and the associated Green's function for L is

$$H(x,t) := \frac{\begin{vmatrix} y_1(t) & \cdots & y_n(t) \\ y_1'(t) & \cdots & y_n'(t) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ y_1^{(n-2)}(t) & \cdot & y_n^{(n-2)}(t) \\ y_1(x) & \cdots & y_n(x) \end{vmatrix}}{\begin{vmatrix} y_1(t) & \cdots & y_n(t) \\ y_1'(t) & \cdots & y_n'(t) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ y_1^{(n-2)}(t) & \cdot & y_n^{(n-2)}(t) \\ y_1^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t) \end{vmatrix}},$$

which is continuous function on $[a,b]^2$, then

$$y(x) = \int_a^x H(x,t) h(t) dt, \quad \text{for all} \quad x \in [a,b]$$

is the unique solution to the initial value problem

$$Ly = h, \quad y^{(i)}(a) = 0, \quad i = 0, 1, ..., n-1.$$

Results given in Section 2 and Section 3 can be analogously done for the linear differential operator. Here as an example inequality we give next result based on Theorem 2.1.

THEOREM 4.1. *Let* $y_i \in U(H_i,h)$, $i=1,\dots,N$, $N\in\mathbb{N}$. *Let* w_1 *and* w_2 *be continuous weight functions on* $[a,x]$ *with* $w_1 \geqslant 0$ *and* $w_2>0$. *Let* $r_i \geqslant 0$, $r=\sum_{i=1}^{N} r_i>0$, $p>0$, $q\geqslant 0$, $\sigma=\frac{1}{p+q}<1$, *and* $h\in L_{p+q}[a,b]$. *Then*

$$\int_a^x w_1(t)\prod_{i=1}^{N}|y_i(t)|^{r_ip}|h(t)|^q dt$$
$$\leqslant \left(\frac{q}{rp+q}\right)^{\sigma q}\left(\int_a^x [w_1(t)]^{\frac{1}{\sigma p}}[w_2(t)]^{-\frac{q}{p}}\prod_{i=1}^{N}\left(\int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}}|H_i(t,\tau)|^{\frac{1}{1-\sigma}}d\tau\right)^{\frac{(1-\sigma)r_i}{\sigma}}dt\right)^{\sigma p}$$
$$\times\left(\int_a^x w_2(t)|h(t)|^{\frac{1}{\sigma}}dt\right)^{\sigma(rp+q)}. \quad (4.1)$$

Proof. Applying Theorem 2.1 with $k_i(t,\tau)=H_i(t,\tau)$ and $f=h$, we obtain the inequality (4.1). □

5. Results for Widder's derivatives

We continue with the process of application for fractional derivative and we give results for Widder's derivatives to produce new inequalities. First it is necessary to give some important details about Widder's derivatives (see [16]).

Let $f,u_0,u_1,\dots,u_n\in C^{n+1}([a,b]), n\geqslant 0$, and the Wronskians

$$W_i(x):=W[u_0(x),u_1(x),\dots,u_i(x)]=\begin{vmatrix} u_0(x) & \cdots & u_i(x)\\ u_0'(x) & \cdots & u_i'(x)\\ \vdots & & \vdots\\ u_0^{(i)}(x) & \cdots & u_i^{(i)}(x)\end{vmatrix},$$

$i=0,1,\dots,n$. Here $W_0(x)=u_0(x)$. Assume $W_i(x)>0$ over $[a,b]$, $i=0,1,\dots,n$. For $i\geqslant 0$, the differential operator of order i (Widder's derivative):

$$L_if(x):=\frac{W[u_0(x),u_1(x),\dots,u_{i-1}(x),f(x)]}{W_{i-1}(x)},$$

$i=1,\dots,n+1$; $L_0f(x):=f(x)$ for all $x\in[a,b]$.

Consider also

$$g_i(x,t) := \frac{1}{W_i(t)} \begin{vmatrix} u_0(t) & \cdots & u_i(t) \\ u_0'(t) & \cdots & u_i'(t) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ u_0(x) & \cdots & u_i(x) \end{vmatrix},$$

$i = 1,2,\dots,n;\ g_0(x,t) := \frac{u_0(x)}{u_0(t)}$ for all $x,t \in [a,b]$.

EXAMPLE 5.1. [16]. Sets of the form $\{u_0,u_1,u_2,\dots,u_n\}$ are $\{1,x,x^2,\dots,x^n\}$, $\{1,\sin x,\cos x,-\sin 2x,\cos 2x,\dots,(-1)^{n-1}\sin nx,(-1)^{n-1}\cos nx\}$, etc.

We also mention the generalized Widder-Talylor's formula, see [16] (see also [8]).

THEOREM 5.2. *Let the functions* $f,u_0,u_1,\dots,u_n \in C^{n+1}([a,b])$, *and the Wronkians* $W_0(x),W_1(x),\dots,W_n(x) > 0$ *on* $[a,b], x \in [a,b]$. *Then for* $t \in [a,b]$ *we have*

$$f(x) = f(t)\frac{u_0(x)}{u_0(t)} + L_1 f(t) g_1(x,t) + \dots + L_n f(t) g_n(x,t) + R_n(x),$$

where

$$R_n(x) := \int_t^x g_n(x,t) L_{n+1} f(t)\,dt.$$

For example (see [16]) one could take $u_0(x) = c > 0$. If $u_i(x) = x^i$, $i = 0,1,\dots,n$, defined on [a,b], then

$$L_i f(t) = f^{(i)}(t) \text{ and } g_i(x,t) = \frac{(x-t)^i}{i!}, \quad t \in [a,b].$$

We need

COROLLARY 5.3. *By additionally assuming for fixed* $a \in [a,b]$ *that* $L_i f(a) = 0$, $i = 0,1,\dots,n$, *we get that*

$$f(x) := \int_a^x g_n(x,t) L_{n+1} f(t)\,dt \quad for\ all\ x \in [a,b].$$

Results given in Section 2 and Section 3 can be analogously done for Widder's derivative. Here as an example inequality we give next result based on Theorem 2.1.

THEOREM 5.4. *Let* $f_i \in U(g_i, L_{n+1} f)$, $i = 1,\dots,N$, $N \in \mathbb{N}$. *Let* w_1 *and* w_2 *be continuous weight functions on* $[a,x]$ *with* $w_1 \geqslant 0$ *and* $w_2 > 0$. *Let* $r_i \geqslant 0$, $r = \sum_{i=1}^{N} r_i > 0$, $p > 0$, $q \geqslant 0$, $\sigma = \frac{1}{p+q} < 1$, *and* $L_{n+1} f \in L_{p+q}[a,b]$. *Then*

$$\int_a^x w_1(t)\prod_{i=1}^N |f_i(t)|^{r_ip}|L_{n+1}f(t)|^q dt$$
$$\leqslant \left(\frac{q}{rp+q}\right)^{\sigma q}\left(\int_a^x [w_1(t)]^{\frac{1}{\sigma p}}[w_2(t)]^{-\frac{q}{p}}\prod_{i=1}^N\left(\int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}}|g_i(t,\tau)|^{\frac{1}{1-\sigma}}d\tau\right)^{\frac{(1-\sigma)r_i}{\sigma}}dt\right)^{\sigma p}$$
$$\times\left(\int_a^x w_2(t)|L_{n+1}f(t)|^{\frac{1}{\sigma}}dt\right)^{\sigma(rp+q)}. \quad (5.1)$$

Proof. Applying Theorem 2.1 with $y_i = f_i,\ f = L_{n+1}f$ and $k_i(t,\tau) = g_i(t,\tau)$, we obtain the inequality (5.1). □

6. Concluding Remarks

Let $x \in [a,b],\ \alpha > 0,\ n = [\alpha]+1,\ [\alpha]$ denotes the integral part of α and Γ is the gamma function $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt$. For $f \in L_1[a,b]$ the *Riemann-Liouville fractional integrals* $J^\alpha_{a+}f$ (left-sided) of order α are defined by

$$J^\alpha_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)\,dt.$$

The subspace $C^\alpha_{a+}[a,b]$ of $C^{n-1}[a,b]$ is defined by

$$C^\alpha_{a+}[a,b] = \left\{f \in C^{n-1}[a,b] : J^{n-\alpha}_{a+}f^{(n-1)} \in C^1[a,b]\right\}.$$

For $f \in C^\alpha_{a+}[a,b]$ the *Canavati fractional derivatives* $D^\alpha_{a+}f$ (left-sided) of order α are defined by

$$D^\alpha_{a+}f(x) = \frac{d}{dx}J^{n-\alpha}_{a+}f^{(n-1)}(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d}{dx}\int_a^x (x-t)^{n-\alpha-1}f^{(n-1)}(t)\,dt.$$

In addition, we stipulate

$$D^0_{a+}f := f =: J^0_{a+}f.$$

If $\alpha \in \mathbb{N}$ then $D^\alpha_{a+}f = f^{(\alpha)}$, the ordinary α-order derivatives.

The composition identity for the Canavati left-sided fractional derivatives comes from [10].

THEOREM 6.1. [10, Theorem 2.1] *Let $\alpha > \beta \geqslant 0,\ n = [\alpha]+1,\ m = [\beta]+1$. Let $f \in C^\alpha_{a+}[a,b]$ be such that $f^{(i)}(a) = 0$ for $i = m-1, m, \ldots, n-2$. Then $f \in C^\beta_{a+}[a,b]$ and*

$$D^\beta_{a+}f(x) = \frac{1}{\Gamma(\alpha-\beta)}\int_a^x (x-t)^{\alpha-\beta-1}D^\alpha_{a+}f(t)\,dt, \quad x \in [a,b].$$

REMARK 6.2. Let $N \in \mathbb{N}$, $\alpha > \beta_i \geqslant 0$, $m = \min\{[\beta_i] + 1 \colon i = 1, \ldots, N\}$ and $n = [\alpha] + 1$. Let $f \in C^{\alpha}_{a+}[a,b]$ be such that $f^{(i)}(a) = 0$ for $i = m-1, \ldots, n-2$. Let w_1 and w_2 be continuous weight functions on $[a,x]$ with $w_1 \geqslant 0$ and $w_2 > 0$. Let $r_i \geqslant 0$, $r = \sum_{i=1}^{N} r_i > 0$. Let $p > 0$, $q \geqslant 0$, $\sigma = \frac{1}{p+q} < 1$, $\rho = \sum_{i=1}^{N} r_i(\alpha - \beta_i) - r\sigma$ and let $\alpha > \beta_i + \sigma$ for $i = 1, \ldots, N$. Let also $D^{\alpha}_{a+} f \in L_{p+q}[a,b]$. Then by replacing y_i by $D^{\beta_i}_{a+} f$, f by $D^{\alpha}_{a+} f$ and taking particular kernel

$$k_i(t,\tau) = \begin{cases} \frac{(t-\tau)^{\alpha-\beta_i-1}}{\Gamma(\alpha-\beta_i)}, & a \leqslant \tau \leqslant t; \\ 0, & t < \tau \leqslant b, \end{cases} \tag{6.1}$$

in Theorem 2.1, we get [11, Theorem 2.1]. So Theorem 2.1 is generalized form of the [11, Theorem 2.1]. Similarly we obtain all results from [11].

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Sajid Iqbal
Department of Mathematics
University of Sargodha (Sub-Campus Bhakkar)
Bhakkar, Pakistan
e-mail: sajid_uos2000@yahoo.com

Josip Pečarić
University of Zagreb, Faculty of Textile Technology
Prilaz baruna Filipovića 28a
10000 Zagreb, Croatia
e-mail: pecaric@element.hr

Muhammad Samraiz
Department of Mathematics
University of Sargodha
Sargodha, Pakistan
e-mail: msamraiz@uos.edu.pk