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ON QUASI–$*$–n–PARANORMAL OPERATORS

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Abstract. For a positive integer n, an operator $T \in B(H)$ is called quasi-$*$-n-paranormal if $||T^{2+n}x||^{\frac{1}{1+n}}||Tx||^{\frac{n}{1+n}} \geqslant ||T^*Tx||$ for every $x \in H$, which is a further generalization of hyponormal and a subclass of normaloid. In this paper, we give necessary and sufficient conditions for T to be a quasi-$*$-n-paranormal operator. And prove that the spectrum is continuous on the class of all quasi-$*$-n-paranormal operators.

1. Introduction

Let $B(H)$ and $K(H)$ denote, respectively, the C^*-algebra of all bounded linear operators and the ideal of compact operators on an infinite dimensional separable Hilbert space H.

In paper [15] authors introduced the class of quasi-$*$-n-paranormal operators defined as follows:

DEFINITION 1.1. For a positive integer n, T is a quasi-$*$-n-paranormal operator if

$$||T^{2+n}x||^{\frac{1}{1+n}}||Tx||^{\frac{n}{1+n}} \geqslant ||T^*Tx|| \text{ for every } x \in H.$$

A quasi-$*$-n-paranormal operator for a positive integer n is an extension of hyponormal operator, i.e., $T^*T \geqslant TT^*$ (equivalently, if $||T^*x|| \leqslant ||Tx||$ for all x in H) and $*$-n-paranormal operator, i.e., $||T^{1+n}x||^{\frac{1}{1+n}} \geqslant ||T^*x||$ for unit vector x. By definition a $*$-n-paranormal operator is a quasi-$*$-n-paranormal operator. A quasi-$*$-n-paranormal operator is normaloid [15], i.e., $||T^n|| = ||T||^n$, for $n \in \mathbb{N}$ (equivalently, $||T|| = r(T)$, the spectral radius of T). A $*$-1-paranormal operator is called a $*$-paranormal operator, i.e., $||T^2x|| \geqslant ||T^*x||^2$ for unit vector x, which has been studied by many authors and it is known that $*$-paranormal operators have many interesting properties similar to those of hyponormal operators (see [5, 7, 10, 13]).

THEOREM 1.2. *Each hyponormal operator is a quasi-$*$-n-paranormal operator.*

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Proof. Recall from [1, Theorem 1] that if T is a hyponormal operator, then

$$|T^{1+n}|^{\frac{2}{1+n}} \geqslant T^*T \geqslant TT^*,$$

using the Hölder-McCarthy inequality, then for every $x \in H$,

$$\begin{aligned}
||T^*Tx||^2 &= (TT^*Tx, Tx) \\
&\leqslant (|T^{1+n}|^{\frac{2}{1+n}}Tx, Tx) \\
&\leqslant (|T^{1+n}|^2 Tx, Tx)^{\frac{1}{1+n}} ||Tx||^{2(1-\frac{1}{1+n})} \\
&= ||T^{2+n}x||^{\frac{2}{1+n}} ||Tx||^{\frac{2n}{1+n}},
\end{aligned}$$

therefore, we have

$$||T^{2+n}x||^{\frac{1}{1+n}} ||Tx||^{\frac{n}{1+n}} \geqslant ||T^*Tx|| \text{ for every } x \in H. \quad \square$$

EXAMPLE 1.3. Let T be the unilateral weighted shift operator with weights $\alpha := \{\alpha_n\}_{n \geqslant 1}$ of positive real numbers, that is,

$$T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ \alpha_1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \alpha_2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \alpha_3 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \alpha_4 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is well known that T is hyponormal if and only if α is monotonically increasing. Simple calculations show that T is a quasi-$*$-n-paranormal operator if and only if

$$(\alpha_{i+1+n}\alpha_{i+n}\ldots\alpha_{i+2}\alpha_{i+1})^{\frac{1}{1+n}} \geqslant \alpha_i \ \ (i = 1, 2, 3, \ldots).$$

2. quasi-$*$-n-paranormal operators

In the sequel, we shall write $N(T)$ and $R(T)$ for the null space and range space of T, respectively.

LEMMA 2.1. [14] *T is a $*$-n-paranormal operator if and only if*

$$T^{*1+n}T^{1+n} - (1+n)\mu^n TT^* + n\mu^{1+n} \geqslant 0 \ \ \textit{for all } \mu > 0.$$

It is well known that for any operators A, B and C, $||Ax||||Cx|| \geqslant ||Bx||^2$ for all $x \in H \Longleftrightarrow A^*A - 2\lambda B^*B + \lambda^2 C^*C \geqslant 0$ for all $\lambda > 0$. Thus we have the following Lemma 2.2.

LEMMA 2.2. *T is a quasi-$*$-n-paranormal operator if and only if*

$$T^*(T^{*1+n}T^{1+n} - (1+n)\mu^n TT^* + n\mu^{1+n})T \geqslant 0 \ \ \textit{for all } \mu > 0.$$

THEOREM 2.3. *If T does not have a dense range, then the following statements are equivalent:*

(1) *T is a quasi-$*$-n-paranormal operator;*

(2) *$T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$ on $H = \overline{R(T)} \oplus N(T^*)$, where $A^{*1+n}A^{1+n} - (1+n)\mu^n(AA^* + BB^*) + n\mu^{1+n} \geqslant 0$ for all $\mu > 0$.*

Proof. (1) $\Rightarrow$ (2) Consider the matrix representation of T with respect to the decomposition $H = \overline{R(T)} \oplus N(T^*)$:

$$T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}.$$

Let P be the projection onto $\overline{R(T)}$. Since T is a quasi-$*$-n-paranormal operator, we have

$$P(T^{*1+n}T^{1+n} - (1+n)\mu^n TT^* + n\mu^{1+n})P \geqslant 0.$$

Therefore $A^{*1+n}A^{1+n} - (1+n)\mu^n(AA^* + BB^*) + n\mu^{1+n} \geqslant 0$ for all $\mu > 0$.

(2) $\Rightarrow$ (1) Suppose that $T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$ on $H = \overline{R(T)} \oplus N(T^*)$. Then we have

$$\begin{aligned}
&T^*(T^{*1+n}T^{1+n} - (1+n)\mu^n TT^* + n\mu^{1+n})T \\
&= \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}^* \\
&\quad \times \left(\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}^{*1+n} \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}^{1+n} - (1+n)\mu^n \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}^* + n\mu^{1+n} \right) \\
&\quad \times \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} D & A^{*1+n}A^nB \\ B^*A^{*n}A^{1+n} & B^*A^{*n}A^nB + n\mu^{1+n} \end{pmatrix} \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} A^*DA & A^*DB \\ B^*DA & B^*DB \end{pmatrix}
\end{aligned}$$

where $D = A^{*1+n}A^{1+n} - (1+n)\mu^n(AA^* + BB^*) + n\mu^{1+n}$. Let $\mu > 0$ be arbitrary and $v = x \oplus y$ be a vector in $H = \overline{R(T)} \oplus N(T^*)$, where $x \in \overline{R(T)}$ and $y \in N(T^*)$. Then

$$\begin{aligned}
&(T^{*2+n}T^{2+n} - (1+n)\mu^n T^*TT^*T + n\mu^{1+n}T^*Tv, v) \\
&= (A^*DAx, x) + (A^*DBy, x) + (B^*DAx, y) + (B^*DBy, y) \\
&= (D(Ax + By), (Ax + By)) \\
&= ((A^{*1+n}A^{1+n} - (1+n)\mu^n(AA^* + BB^*) + n\mu^{1+n})(Ax + By), (Ax + By)).
\end{aligned}$$

Since

$$A^{*1+n}A^{1+n} - (1+n)\mu^n(AA^* + BB^*) + n\mu^{1+n} \geqslant 0 \text{ for all } \mu > 0,$$

$(T^{*2+n}T^{2+n} - (1+n)\mu^n T^*T^*TT + n\mu^{1+n}T^*Tv, v) \geqslant 0$ for all $v \in H$ and for all $\mu > 0$,

hence

$$T^{*2+n}T^{2+n} - (1+n)\mu^n T^*TT^*T + n\mu^{1+n}T^*T \geqslant 0 \ \text{ for all } \mu > 0.$$

Thus T is a quasi-$*$-n-paranormal operator. □

COROLLARY 2.4. *If T is a quasi-$*$-n-paranormal operator and $R(T)$ is not dense, then T has the following matrix representation:*

$$T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \text{ on } H = \overline{R(T)} \oplus N(T^*)$$

where A is a $*$-n-paranormal operator on $\overline{R(T)}$.

THEOREM 2.5. *Let T be a quasi-$*$-n-paranormal operator and M be an invariant subspace for T. Then the restriction $T|_M$ is also a quasi-$*$-n-paranormal operator.*

Proof. Let P be the projection onto M. Then $TP = PTP$, so that

$$(T|_M)^* = PT^*P.$$

Hence, for $x \in M$ we have

$$\begin{aligned} ||(T|_M)^*T|_M x|| &= ||PT^*PPTPx|| = ||PT^*Tx|| \leqslant ||T^*Tx|| \\ &\leqslant ||T^{2+n}x||^{\frac{1}{1+n}} ||Tx||^{\frac{n}{1+n}} = ||(T|_M)^{2+n}x||^{\frac{1}{1+n}} ||T|_M x||^{\frac{n}{1+n}}. \end{aligned}$$

Thus $T|_M$ is a quasi-$*$-n-paranormal operator. □

COROLLARY 2.6. *Let T be a quasi-$*$-paranormal operator and M be an invariant subspace for T. Then the restriction $T|_M$ is also a quasi-$*$-paranormal operator.*

3. Spectral properties of quasi-$*$-n-paranormal operators

For every $T \in B(H)$, $\sigma(T)$ is a compact subset of $\mathbb{C}$. The function σ viewed as a function from $B(H)$ into the set of all compact subsets of $\mathbb{C}$, equipped with the Hausdorff metric, is well known to be upper semi-continuous, but fails to be continuous in general. Conway and Morrel [3] have carried out a detailed study of spectral continuity in $B(H)$. Recently, the continuity of spectrum was considered when restricted to certain subsets of the entire manifold of Toeplitz operators in [8, 11]. It has been proved that is continuous in the set of normal operators and hyponormal operators in [9]. And this result has been extended to quasihyponormal operators by Djordjević in [4], to p-hyponormal operators by Hwang and Lee in [12], and to (p,k)-quasihyponormal, M-hyponormal, $*$-paranormal and paranormal operators by Duggal, Jeon and Kim in [6]. In this section we extend this result to quasi-$*$-n-paranormal operators.

LEMMA 3.1. *Let T be a quasi-$*$-n-paranormal operator. Then the following assertions hold:*

(1) *If T is quasinilpotent, then $T=0$.*

(2) *For every non-zero $\lambda \in \sigma_p(T)$, the matrix representation of T with respect to the decomposition $H = N(T-\lambda) \oplus (N(T-\lambda))^{\perp}$ is: $T = \begin{pmatrix} \lambda & 0 \\ 0 & B \end{pmatrix}$ for some operator B satisfying $\lambda \notin \sigma_p(B)$ and $\sigma(T) = \{\lambda\} \cup \sigma(B)$.*

Proof. (1) Suppose T is a quasi-$*$-n-paranormal operator. T is normaloid by [15], thus $T=0$.

(2) We may assume $Tx = \lambda x$, where $||x|| = 1$ and $\lambda \neq 0$. Then

$$|\lambda| ||T^*x|| = ||T^*Tx|| \leqslant ||T^{2+n}x||^{\frac{1}{1+n}} ||Tx||^{\frac{n}{1+n}} = |\lambda|^2.$$

Hence $||T^*x|| \leqslant |\lambda|$ and

$$0 \leqslant ||(T-\lambda)^*x||^2 = ||T^*x||^2 - 2Re(T^*x, \bar{\lambda}x) + |\lambda|^2 \leqslant 2|\lambda|^2 - 2|\lambda|^2 = 0.$$

We have $T^*x = \bar{\lambda}x$, i.e., $N(T-\lambda)$ reduces T. So we have that $T = \begin{pmatrix} \lambda & 0 \\ 0 & B \end{pmatrix}$ on $H = N(T-\lambda) \oplus (N(T-\lambda))^{\perp}$ for some operator B satisfying $\lambda \notin \sigma_p(B)$ and $\sigma(T) = \{\lambda\} \cup \sigma(B)$. □

LEMMA 3.2. [2] *Let H be a complex Hilbert space. Then there exists a Hilbert space K such that $H \subset K$ and a map $\varphi : B(H) \to B(K)$ such that*

(1) *φ is a faithful $*$-representation of the algebra $B(H)$ on K;*

(2) *$\varphi(A) \geqslant 0$ for any $A \geqslant 0$ in $B(H)$;*

(3) *$\sigma_a(T) = \sigma_a(\varphi(T)) = \sigma_p(\varphi(T))$ for any $T \in B(H)$.*

THEOREM 3.3. *The spectrum σ is continuous on the set of quasi-$*$-n-paranormal operators.*

Proof. Suppose T is a quasi-$*$-n-paranormal operator. Let φ: $B(H) \to B(K)$ be Berberian's faithful $*$-representation of Lemma 3.2. In the following, we shall show that $\varphi(T)$ is also a quasi-$*$-n-paranormal operator. In fact, since T is a quasi-$*$-n-paranormal operator, we have

$$T^*(T^{*1+n}T^{1+n} - (1+n)\mu^n TT^* + n\mu^{1+n})T \geqslant 0 \text{ for all } \mu > 0.$$

Hence we have

$$\begin{aligned} &(\varphi(T))^*(\varphi(T)^{*1+n}\varphi(T)^{1+n} - (1+n)\mu^n \varphi(T)\varphi(T)^* + n\mu^{1+n})\varphi(T) \\ &= \varphi(T^*(T^{*1+n}T^{1+n} - (1+n)\mu^n TT^* + n\mu^{1+n})T) \text{ by Lemma 3.2} \\ &\geqslant 0 \text{ by Lemma 3.2,} \end{aligned}$$

so $\varphi(T)$ is also a quasi-$*$-n-paranormal operator. By Lemma 3.1, we have T belongs to the set $C(i)$ (see definition in [6]). Therefore, we have that the spectrum σ is continuous on the set of quasi-$*$-n-paranormal operators by [6, Theorem 1.1]. $\square$

The following example provides an operator T which is a quasi-$*$-n-paranormal operator, however, the relation $N(T) \subseteq N(T^*)$ does not hold.

EXAMPLE 3.4. [15] Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ be operators on $\mathbb{R}^2$, and let $H_n = \mathbb{R}^2$ for all positive integers n. Consider the operator $T_{A,B}$ on $\oplus_{n=1}^{+\infty} H_n$ defined by

$$T_{A,B} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ A & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & B & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & B & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & B & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & B & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then $T_{A,B}$ is a quasi-$*$-n-paranormal operator, $T_{A,B}(x) = 0$ while $T^*_{A,B}(x) \neq 0$ for the vector $x = (0,0,1,-1,0,0,\cdots)$. Therefore, the relation $N(T_{A,B}) \subseteq N(T^*_{A,B})$ does not always hold.

In the following, we shall give a sufficient condition for quasi-$*$-n-paranormal operators to be normal and compact.

THEOREM 3.5. *Let T be a quasi-$*$-n-paranormal operator and Riesz (i.e., $\sigma_e(T) = \{0\}$). Then T is normal and compact.*

Proof. Decompose T into $T = T_1 \oplus T_2$, where T_1 is normal part and T_2 is pure. Then T_2 is quasi-$*$-n-paranormal and Riesz. Since non-zero eigenvalues of a quasi-$*$-n-paranormal operator are normal, $\sigma(T_2) = 0$; hence T_2 is quasinilpotent, and therefore $T_2 = 0$. The proof now follows since a normal Riesz operator is compact. $\square$

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REFERENCES

[1] A. ALUTHGE AND D. WANG, *Powers of p-hyponormal operators*, J. Ineq. Appl. **3** (1999), 279–284.
[2] S. K. BERBERIAN, *Approximate proper vectors*, Proc. Amer. Math. Soc. **13** (1962), 111–114.
[3] J. B. CONWAY AND B. B. MORREL, *Operators that are points of spectral continuity*, Integr. Equ. Oper. Theory **2** (1979), 174–198.
[4] S. V. DJORDJEVIĆ, *Continuity of the essential spectrum in the class of quasihyponormal operators*, Vesnik Math. **50** (1998), 71–74.
[5] B. P. DUGGAL, *Upper triangular matrix operators with diagonal (T_1,T_2), T_2 k-nilpotent*, Rend. Circ. Mat. Palermo **62**, 2 (2013), 215–226.
[6] B. P. DUGGAL, I. H. JEON AND I. H. KIM, *Continuity of the spectrum on a class of upper triangular operator matrices*, J. Math. Anal. Appl. **370** (2010), 584–587.
[7] B. P. DUGGAL, I. H. JEON AND I. H. KIM, *On $*$-paranormal contractions and properties for $*$-class A operators*, Linear Algebra Appl. **436** (2012), 954–962.
[8] D. R. FARENICK AND W. Y. LEE, *Hyponormality and spectra of Toeplitz operators*, Trans. Amer. Math. Soc. **348** (1996), 4153–4174.
[9] P. R. HALMOS, *A Hilbert Space Problem Book*, Springer-Verlag, New York, 1982.
[10] Y. M. HAN AND A. H. KIM, *A note on $*$-paranormal operators*, Integr. Equ. Oper. Theory **49**, 4 (2004), 435–444.
[11] I. S. HWANG AND W. Y. LEE, *On the continuity of spectra of Toeplitz operators*, Arch. Math. **70** (1998), 66–73.
[12] I. S. HWANG AND W. Y. LEE, *The spectrum is continuous on the set of p-hyponormal operators*, Math. Z. **235** (2000), 151–157.
[13] M. Y. LEE, S. H. LEE AND C. S. RHOO, *Some remarks on the structure of k-$*$-paranormal operators*, Kyungpook Math. J. **35** (1995), 205–211.
[14] F. ZUO AND J. L. SHEN, *A note on $*$-n-paranormal operators*, Adv. Math.(China) **42**, 2 (2013), 153–158.
[15] F. ZUO AND W. YAN, *Finite operators and Weyl type theorems for quasi-$*$-n-paranormal operators*, Filomat, preprint.

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