Abstract. In this note, using the method of Farid et al. (2010) [G. Farid, J. Pečarić, Atiq Ur Rehman, On Refinements of Aczél’s, Popoviciu, Bellman’s Inequalities and Related Results, J. Inequal. Appl. 2010 (2010), Article ID 579567], some reversed versions of Aczél-type inequality and Bellman-type inequality proposed by Farid, Pečarić and Ur Rehman are established. The obtained results are the refinements of reversed Aczél-Popoviciu inequality, Aczél-Bjelica inequality and Bellman’s inequality.

1. Introduction

The famous Aczél’s inequality, which is of wide application in the theory of functional equations in non-Euclidean geometry, was presented by Aczél [1] as follows.

THEOREM A. Let n be a fixed positive integer, and let A, B, a_i, b_i (i = 1, 2, ···, n) be real numbers such that \( A^2 - \sum_{i=1}^{n} a_i^2 > 0 \) and \( B^2 - \sum_{i=1}^{n} b_i^2 > 0 \). Then

\[
 \left( A^2 - \sum_{i=1}^{n} a_i^2 \right) \left( B^2 - \sum_{i=1}^{n} b_i^2 \right) \leq \left( a_1 b_1 - \sum_{i=2}^{n} a_i b_i \right)^2.
\]

Later in 1959 Popoviciu [6] gave a generalization of the above inequality in the following theorem.

THEOREM B. Let n be a fixed positive integer, let \( p > 1, q > 1, \frac{1}{p} + \frac{1}{q} = 1 \), and let A, B, a_i, b_i (i = 1, 2, ···, n) be nonnegative real numbers such that \( A^p - \sum_{i=1}^{n} a_i^p > 0 \) and \( B^q - \sum_{i=1}^{n} b_i^q > 0 \). Then

\[
 \left( A^p - \sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} \left( B^q - \sum_{i=1}^{n} b_i^q \right)^{\frac{1}{q}} \leq AB - \sum_{i=1}^{n} a_i b_i,
\]

which is called as Aczél-Popoviciu inequality.

In 1982, Vasić and Pečarić [9] presented the following reversed version of Aczél-Popoviciu inequality (2).
Theorem C. Let $n$ be a fixed positive integer, let $p < 1$ $(p \neq 0)$, $\frac{1}{p} + \frac{1}{q} = 1$, and let $A$, $B$, $a_i$, $b_i$ $(i = 1, 2, \ldots, n)$ be positive numbers such that $A^p - \sum_{i=1}^{n} a_i^p > 0$ and $B^q - \sum_{i=1}^{n} b_i^q > 0$. Then

$$\left( A^p - \sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} \left( B^q - \sum_{i=1}^{n} b_i^q \right)^{\frac{1}{q}} \geq AB - \sum_{i=1}^{n} a_ib_i,$$  \hspace{1cm} (3)

which is called as Aczél-Vasič-Pečarić inequality.

In 1990, Bjelica [3] obtained an interesting Aczél-type inequality as follows.

Theorem D. Let $n$ be a fixed positive integer, let $0 < p \leq 2$, and let $A$, $B$, $a_i$, $b_i$ $(i = 1, 2, \ldots, n)$ be nonnegative real numbers such that $A^p - \sum_{i=1}^{n} a_i^p > 0$ and $B^p - \sum_{i=1}^{n} b_i^p > 0$. Then

$$\left( A^p - \sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} \left( B^p - \sum_{i=1}^{n} b_i^p \right)^{\frac{1}{p}} \leq AB - \sum_{i=1}^{n} a_ib_i,$$  \hspace{1cm} (4)

which is called as Aczél-Bjelica inequality.

Bellman inequality [2] related with Aczél’s inequality is stated as follows.

Theorem E. Let $n$ is a fixed positive integer, and let $a_i$, $b_i$ $(i = 1, 2, \ldots, n)$ be positive numbers such that $a_i^p - \sum_{i=2}^{n} a_i^p > 0$ and $b_i^p - \sum_{i=2}^{n} b_i^p > 0$. If $p \geq 1$ (or $p < 0$), then

$$\left[ \left( a_1^p - \sum_{i=2}^{n} a_i^p \right)^{\frac{1}{p}} + \left( b_1^p - \sum_{i=2}^{n} b_i^p \right)^{\frac{1}{p}} \right]^p \leq (a_1 + b_1)^p - \sum_{i=2}^{n} (a_i + b_i)^p;$$  \hspace{1cm} (5)

If $0 < p < 1$, then the reverse inequality is valid.

Remark 1.1. The case $p \geq 1$ of Theorem E was given by Bellman [2]. The cases $0 < p < 1$ and $p < 0$ were proved in [9] by Vasić and Pečarić.

In 2010, Farid, Pečarić and Ur Rehman [5] established some interesting refinements of (2), (3), (4), and (5) as follows.

Theorem F. Let $n$ and $j$ be fixed positive integers with $1 \leq j < n$, let $1 < p \leq 2$, and let $A$, $B$, $a_i$, $b_i$ $(i = 1, 2, \ldots, n)$ be nonnegative real numbers such that $A^p - \sum_{i=1}^{n} a_i^p > 0$ and $B^p - \sum_{i=1}^{n} b_i^p > 0$. Then

$$\left( A^p - \sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} \left( B^p - \sum_{i=1}^{n} b_i^p \right)^{\frac{1}{p}} \leq \alpha \beta - \left( \sum_{i=k+1}^{n} a_i^p \right)^{\frac{1}{p}} \left( \sum_{i=k+1}^{n} b_i^p \right)^{\frac{1}{p}} \leq \alpha \beta - \sum_{i=k+1}^{n} a_ib_i \leq AB - \sum_{i=1}^{n} a_ib_i,$$  \hspace{1cm} (6)
where $\alpha = \left( A^p - \sum_{i=1}^{k} a_i^p \right)^{\frac{1}{p}}$, $\beta = \left( B^q - \sum_{i=1}^{k} b_i^q \right)^{\frac{1}{q}}$.

**Theorem G.** Let $n$ and $j$ be fixed positive integers with $1 \leq j < n$, let $p > 1$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and let $A$, $B$, $a_i$, $b_i$ ($i = 1, 2, \cdots, n$) be nonnegative real numbers such that $A^p - \sum_{i=1}^{n} a_i^p > 0$ and $B^q - \sum_{i=1}^{n} b_i^q > 0$. Then

\[
\left( A^p - \sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} \left( B^q - \sum_{i=1}^{n} b_i^q \right)^{\frac{1}{q}} \\
\leq \alpha \beta - \left( \sum_{i=k+1}^{n} a_i^p \right)^{\frac{1}{p}} \left( \sum_{i=k+1}^{n} b_i^q \right)^{\frac{1}{q}} \\
\leq \alpha \beta - \sum_{i=k+1}^{n} a_i b_i \\
\leq AB - \sum_{i=1}^{n} a_i b_i, \tag{7}
\]

where $\alpha = \left( A^p - \sum_{i=1}^{k} a_i^p \right)^{\frac{1}{p}}$, $\beta = \left( B^q - \sum_{i=1}^{k} b_i^q \right)^{\frac{1}{q}}$.

**Theorem H.** Let $n$ and $j$ be fixed positive integers with $1 \leq j < n$, let $p \geq 1$, and let $A$, $B$, $a_i$, $b_i$ ($i = 1, 2, \cdots, n$) be nonnegative real numbers such that $A^p - \sum_{i=1}^{n} a_i^p > 0$ and $B^q - \sum_{i=1}^{n} b_i^q > 0$. Then

\[
\left( A^p - \sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} + \left( B^q - \sum_{i=1}^{n} b_i^q \right)^{\frac{1}{q}} \\
\leq \left\{ (\alpha + \beta)^p - \left( \sum_{i=k+1}^{n} a_i^p \right)^{\frac{1}{p}} + \left( \sum_{i=k+1}^{n} b_i^q \right)^{\frac{1}{q}} \right\}^{\frac{1}{p}} \\
\leq \left[ (\alpha + \beta)^p - \sum_{i=k+1}^{n} (a_i + b_i)^p \right]^{\frac{1}{p}} \\
\leq \left[ (A + B)^p - \sum_{i=1}^{n} (a_i + b_i)^p \right]^{\frac{1}{p}} \tag{8},
\]

where $\alpha = \left( A^p - \sum_{i=1}^{k} a_i^p \right)^{\frac{1}{p}}$, $\beta = \left( B^q - \sum_{i=1}^{k} b_i^q \right)^{\frac{1}{q}}$.

Moreover, in 2012, Tian [7] presented the reversed version of Aczél-Bjelica inequality (4) as follows.

**Theorem I.** Let $n$ be a fixed positive integer, let $p < 0$, and let $A$, $B$, $a_i$, $b_i$ ($i = 1, 2, \cdots, n$) be positive real numbers such that $A^p - \sum_{i=1}^{n} a_i^p > 0$ and $B^q - \sum_{i=1}^{n} b_i^q > 0$. Then

\[
\left( A^p - \sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} \left( B^q - \sum_{i=1}^{n} b_i^q \right)^{\frac{1}{q}} \geq AB - \sum_{i=1}^{n} a_i b_i. \tag{9}
\]
In this note, using the method of Farid, Pečarić and Ur Rehman [5], we obtain the reversed versions of inequalities (6), (7) and (8). The obtained results are the refinements of reversed Aczél-Popoviciu inequality, Aczél-Bjelica inequality and Bellman’s inequality.

2. Reversed versions of Aczél-type inequality and Bellman-type inequality

**Lemma 2.1.** [8] If \( b_i > 0 \) \((i = 1, 2, \ldots, n)\), and \( p < 0 < q \), then

\[
\left( \sum_{i=1}^{n} b_i^p \right)^{\frac{1}{p}} \leq \left( \sum_{i=1}^{n} b_i^q \right)^{\frac{1}{q}}.
\]  

(10)

**Lemma 2.2.** (Generalized Hölder’s inequality) [8] Let \( a_i, b_i > 0 \) \((i = 1, 2, \cdots, n)\).

(a) If \( p, q > 0 \), and \( \frac{1}{p} + \frac{1}{q} \geq 1 \), then

\[
\sum_{i=1}^{n} a_i b_i \leq \left( \sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^{n} b_i^q \right)^{\frac{1}{q}}.
\]  

(11)

(b) If \( p < 0, q > 0 \), and \( \frac{1}{p} + \frac{1}{q} \leq 1 \), then

\[
\sum_{i=1}^{n} a_i b_i \geq \left( \sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^{n} b_i^q \right)^{\frac{1}{q}}.
\]  

(12)

(c) If \( p, q < 0 \), then

\[
\sum_{i=1}^{n} a_i b_i \geq \left( \sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^{n} b_i^q \right)^{\frac{1}{q}}.
\]  

(13)

**Lemma 2.3.** [7] Let \( n \) be a fixed positive integer; let \( p < 0, q > 0, \frac{1}{p} + \frac{1}{q} \leq 1 \), and let \( A, B, a_i, b_i \) \((i = 1, 2, \cdots, n)\) be positive real numbers such that \( A^p - \sum_{i=1}^{n} a_i^p > 0 \) and \( B^q - \sum_{i=1}^{n} b_i^q > 0 \). Then

\[
\left( A^p - \sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} \left( B^q - \sum_{i=1}^{n} b_i^q \right)^{\frac{1}{q}} \geq AB - \sum_{i=1}^{n} a_i b_i.
\]  

(14)

Nextly, we give the reversed version of inequality (6), i.e., a new refinement of inequality (9).

**Theorem 2.4.** Let \( n \) and \( j \) be fixed positive integers with \( 1 \leq j < n \), let \( p < 0 \), and let \( A, B, a_i, b_i \) \((i = 1, 2, \cdots, n)\) be positive real numbers such that \( A^p - \sum_{i=1}^{n} a_i^p > 0 \)
and $B^p - \sum_{i=1}^n b_i^p > 0$. Then
\[
\left( A^p - \sum_{i=1}^n a_i^p \right)^{\frac{1}{p}} \left( B^p - \sum_{i=1}^n b_i^p \right)^{\frac{1}{p}} \\
\geq \alpha \beta - \left( \sum_{i=k+1}^n a_i^p \right)^{\frac{1}{p}} \left( \sum_{i=k+1}^n b_i^p \right)^{\frac{1}{p}} \\
\geq \alpha \beta - \sum_{i=k+1}^n a_i b_i \\
\geq AB - \sum_{i=1}^n a_i b_i, \quad (15)
\]

where $\alpha = \left( A^p - \sum_{i=1}^k a_i^p \right)^{\frac{1}{p}}$, $\beta = \left( B^p - \sum_{i=1}^k b_i^p \right)^{\frac{1}{p}}$.

**Proof.** By using inequality (9) we have
\[
\alpha \beta \geq AB - \sum_{i=1}^k a_i b_i. \quad (16)
\]

If we denote $C = \left( \sum_{i=k+1}^n a_i^p \right)^{\frac{1}{p}}$, $D = \left( \sum_{i=k+1}^n b_i^p \right)^{\frac{1}{p}}$, then applying inequality (9) with $n = 1$, we have
\[
\left( A^p - \sum_{i=1}^n a_i^p \right)^{\frac{1}{p}} \left( B^p - \sum_{i=1}^n b_i^p \right)^{\frac{1}{p}} \\
= \left( \alpha^p - C^p \right)^{\frac{1}{p}} \left( \beta^p - D^p \right)^{\frac{1}{p}} \\
\geq \alpha \beta - CD \\
= \alpha \beta - \left( \sum_{i=k+1}^n a_i^p \right)^{\frac{1}{p}} \left( \sum_{i=k+1}^n b_i^p \right)^{\frac{1}{p}}. \quad (17)
\]

On the other hand, from Lemma 2.1 for $p < 0 < q$, $\frac{1}{p} + \frac{1}{q} = 1$, and using generalized Hölder’s inequality (12) we get
\[
\left( \sum_{i=k+1}^n a_i^p \right)^{\frac{1}{p}} \left( \sum_{i=k+1}^n b_i^p \right)^{\frac{1}{p}} \leq \left( \sum_{i=k+1}^n a_i^p \right)^{\frac{1}{q}} \left( \sum_{i=k+1}^n b_i^q \right)^{\frac{1}{q}} \leq \sum_{i=k+1}^n a_i b_i. \quad (18)
\]

Therefore, combining (16), (17) and (18) yields immediately the desired inequality (15). □

**Remark 2.5.** Farid, Pečarić and Ur Rehman [5] obtained the inequality (6) for $1 < p \leq 2$. In fact, the inequality is also valid for $0 < p \leq 2$. Similar to the proof of the inequality (6) for $1 < p \leq 2$ in [5] but using (11) in place of Hölder’s inequality, we immediately obtain the inequality (6) for $0 < p \leq 2$.  

Now, we present the reversed version of inequality (7), i.e., a new refinement of Aczél-Vasić-Pečarić inequality (3).

**Theorem 2.6.** Let \( n \) and \( j \) be fixed positive integers with \( 1 \leq j < n \), let \( p < 1 \), \( p \neq 0 \), \( \frac{1}{p} + \frac{1}{q} \leq 1 \), and let \( A, B, a_i, b_i \) (\( i = 1, 2, \ldots, n \)) be positive real numbers such that \( A^p - n \sum_{i=1}^{n} a_i^p > 0 \) and \( B^q - n \sum_{i=1}^{n} b_i^q > 0 \). Then

\[
\left( A^p - n \sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} \left( B^q - n \sum_{i=1}^{n} b_i^q \right)^{\frac{1}{q}} \geq \alpha \tilde{\beta} - \left( \sum_{i=k+1}^{n} a_i^p \right)^{\frac{1}{p}} \left( \sum_{i=k+1}^{n} b_i^q \right)^{\frac{1}{q}} \geq AB - n \sum_{i=1}^{n} a_i b_i, \tag{19}
\]

where \( \alpha = \left( A^p - \sum_{i=1}^{k} a_i^p \right)^{\frac{1}{p}} \), \( \tilde{\beta} = \left( B^q - \sum_{i=1}^{k} b_i^q \right)^{\frac{1}{q}} \).

**Proof.** From Lemma 2.3 we have

\[
\alpha \tilde{\beta} \geq AB - \sum_{i=1}^{k} a_i b_i. \tag{20}
\]

If we denote \( C = \left( \sum_{i=k+1}^{n} a_i^p \right)^{\frac{1}{p}} \), \( D = \left( \sum_{i=k+1}^{n} b_i^q \right)^{\frac{1}{q}} \), then by applying Lemma 2.3, for \( n = 1 \), we obtain

\[
\left( A^p - n \sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} \left( B^q - n \sum_{i=1}^{n} b_i^q \right)^{\frac{1}{q}} = (\alpha^p - C^p)^{\frac{1}{p}} (\tilde{\beta}^q - D^q)^{\frac{1}{q}} \geq \alpha \tilde{\beta} - CD \]

\[
= \alpha \tilde{\beta} - \left( \sum_{i=k+1}^{n} a_i^p \right)^{\frac{1}{p}} \left( \sum_{i=k+1}^{n} b_i^q \right)^{\frac{1}{q}}. \tag{21}
\]

Moreover, using generalized Hölder’s inequality (12) we get

\[
\left( \sum_{i=k+1}^{n} a_i^p \right)^{\frac{1}{p}} \left( \sum_{i=k+1}^{n} b_i^q \right)^{\frac{1}{q}} \leq \sum_{i=k+1}^{n} a_i b_i. \tag{22}
\]

Combining (20), (21) and (22) yields immediately the desired result. \( \square \)

Finally, we give the reversed version of inequality (8), i.e., a new refinement of reversed Bellman inequality.
Theorem 2.7. Let \( n \) and \( j \) be fixed positive integers with \( 1 \leq j < n \), let \( 0 < p < 1 \), and let \( A, B, a_i, b_i \ (i = 1, 2, \cdots, n) \) be positive real numbers such that \( A^p - \sum_{i=1}^{n} a_i^p > 0 \) and \( B^p - \sum_{i=1}^{n} b_i^p > 0 \). Then

\[
(A^p - \sum_{i=1}^{n} a_i^p)^{\frac{1}{p}} + (B^p - \sum_{i=1}^{n} b_i^p)^{\frac{1}{p}} \\
\geq \left\{ (\alpha + \beta)^p - \left( \left( \sum_{i=k+1}^{n} a_i^p \right)^{\frac{1}{p}} + \left( \sum_{i=k+1}^{n} b_i^p \right)^{\frac{1}{p}} \right)^{p} \right\}^{\frac{1}{p}}
\]

where \( \alpha = (A^p - \sum_{i=1}^{k} a_i^p)^{\frac{1}{p}}, \beta = (B^p - \sum_{i=1}^{k} b_i^p)^{\frac{1}{p}} \).

Proof. By using Bellman’s inequality (5), we have

\[
\alpha + \beta \geq \left( A + B \right)^p - \sum_{i=1}^{k} (a_i + b_i)^p
\]

(24)

If we denote \( C = \left( \sum_{i=k+1}^{n} a_i^p \right)^{\frac{1}{p}}, \ D = \left( \sum_{i=k+1}^{n} b_i^p \right)^{\frac{1}{p}} \), then using Bellman’s inequality (5) for \( n = 1 \), we have

\[
(A^p - \sum_{i=1}^{n} a_i^p)^{\frac{1}{p}} + (B^p - \sum_{i=1}^{n} b_i^p)^{\frac{1}{q}} \\
= (\alpha^p - C^p)^{\frac{1}{p}} + (\beta^p - D^p)^{\frac{1}{p}} \\
\geq \left[ (\alpha + \beta)^p - (C + D)^p \right]^{\frac{1}{p}}
\]

(25)

Moreover, by using Minkowski inequality with \( 0 < p < 1 \), we get

\[
\left[ \left( \sum_{i=k+1}^{n} a_i^p \right)^{\frac{1}{p}} + \left( \sum_{i=k+1}^{n} b_i^p \right)^{\frac{1}{p}} \right]^p \leq \sum_{i=k+1}^{n} (a_i + b_i)^p.
\]

(26)

Combining (23), (24) and (25) yields immediately the desired result. \( \square \)

Remark 2.8. Farid, Pečarić and Ur Rehman [5] obtained the inequality (8) for \( p \geq 1 \). In fact, inequality (8) is also valid for \( p < 0 \). Similar to the proof of the inequality (8) under the assumption \( p \geq 1 \) in [5], it is easy to obtain the inequality (8) for \( p < 0 \).
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