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REVERSED VERSIONS OF ACZÉL–TYPE INEQUALITY AND BELLMAN–TYPE INEQUALITY

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Abstract. In this note, using the method of Farid et al. (2010) [G. Farid, J. Pečarić, Atiq Ur Rehman, On Refinements of Aczél's, Popoviciu, Bellman's Inequalities and Related Results, J. Inequal. Appl. **2010** (2010), Article ID 579567], some reversed versions of Aczél-type inequality and Bellman-type inequality proposed by Farid, Pečarić and Ur Rehman are established. The obtained results are the refinements of reversed Aczél-Popoviciu inequality, Aczél-Bjelica inequality and Bellman's inequality.

1. Introduction

The famous Aczél's inequality, which is of wide application in the theory of functional equations in non-Euclidean geometry, was presented by Aczél [1] as follows.

THEOREM A. *Let n be a fixed positive integer, and let A, B, a_i, b_i $(i=1,2,\cdots,n)$ be real numbers such that $A^2-\sum_{i=1}^n a_i^2>0$ and $B^2-\sum_{i=1}^n b_i^2>0$. Then*

$$\left(A^2-\sum_{i=1}^n a_i^2\right)\left(B^2-\sum_{i=1}^n b_i^2\right)\leqslant\left(a_1b_1-\sum_{i=2}^n a_ib_i\right)^2. \tag{1}$$

Later in 1959 Popoviciu [6] gave a generalization of the above inequality in the following theorem.

THEOREM B. *Let n be a fixed positive integer, let $p>1$, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$, and let A, B, a_i, b_i $(i=1,2,\cdots,n)$ be nonnegative real numbers such that $A^p-\sum_{i=1}^n a_i^p>0$ and $B^q-\sum_{i=1}^n b_i^q>0$. Then*

$$\left(A^p-\sum_{i=1}^n a_i^p\right)^{\frac{1}{p}}\left(B^q-\sum_{i=1}^n b_i^q\right)^{\frac{1}{q}}\leqslant AB-\sum_{i=1}^n a_ib_i, \tag{2}$$

which is called as Aczél-Popoviciu inequality.

In 1982, Vasić and Pečarić [9] presented the following reversed version of Aczél-Popoviciu inequality (2).

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THEOREM C. *Let n be a fixed positive integer, let $p<1$ $(p\neq 0)$, $\frac{1}{p}+\frac{1}{q}=1$, and let A, B, a_i, b_i $(i=1,2,\cdots,n)$ be positive numbers such that $A^p-\sum_{i=1}^n a_i^p>0$ and $B^q-\sum_{i=1}^n b_i^q>0$. Then*

$$\left(A^p-\sum_{i=1}^n a_i^p\right)^{\frac{1}{p}}\left(B^q-\sum_{i=1}^n b_i^q\right)^{\frac{1}{q}}\geqslant AB-\sum_{i=1}^n a_ib_i, \tag{3}$$

which is called as Aczél-Vasić-Pečarić inequality.

In 1990, Bjelica [3] obtained an interesting Aczél-type inequality as follows.

THEOREM D. *Let n be a fixed positive integer, let $0<p\leqslant 2$, and let A, B, a_i, b_i $(i=1,2,\cdots,n)$ be nonnegative real numbers such that $A^p-\sum_{i=1}^n a_i^p>0$ and $B^p-\sum_{i=1}^n b_i^p>0$. Then*

$$\left(A^p-\sum_{i=1}^n a_i^p\right)^{\frac{1}{p}}\left(B^p-\sum_{i=1}^n b_i^p\right)^{\frac{1}{p}}\leqslant AB-\sum_{i=1}^n a_ib_i, \tag{4}$$

which is called as Aczél-Bjelica inequality.

Bellman inequality [2] related with Aczél's inequality is stated as follows.

THEOREM E. *Let n is a fixed positive integer, and let a_i, b_i $(i=1,2,\cdots,n)$ be positive numbers such that $a_1^p-\sum_{i=2}^n a_i^p>0$ and $b_1^q-\sum_{i=2}^n b_i^q>0$. If $p\geqslant 1$ (or $p<0$), then*

$$\left[\left(a_1^p-\sum_{i=2}^n a_i^p\right)^{\frac{1}{p}}+\left(b_1^p-\sum_{i=2}^n b_i^p\right)^{\frac{1}{p}}\right]^p\leqslant (a_1+b_1)^p-\sum_{i=2}^n (a_i+b_i)^p; \tag{5}$$

If $0<p<1$, then the reverse inequality is valid.

REMARK 1.1. The case $p\geqslant 1$ of Theorem E was given by Bellman [2]. The cases $0<p<1$ and $p<0$ were proved in [9] by Vasić and Pečarić.

In 2010, Farid, Pečarić and Ur Rehman [5] established some interesting refinements of (2), (3), (4), and (5) as follows.

THEOREM F. *Let n and j be fixed positive integers with $1\leqslant j<n$, let $1<p\leqslant 2$, and let A, B, a_i, b_i $(i=1,2,\cdots,n)$ be nonnegative real numbers such that $A^p-\sum_{i=1}^n a_i^p>0$ and $B^p-\sum_{i=1}^n b_i^p>0$. Then*

$$\begin{aligned}
&\left(A^p-\sum_{i=1}^n a_i^p\right)^{\frac{1}{p}}\left(B^p-\sum_{i=1}^n b_i^p\right)^{\frac{1}{p}}\\
&\leqslant \alpha\beta-\left(\sum_{i=k+1}^n a_i^p\right)^{\frac{1}{p}}\left(\sum_{i=k+1}^n b_i^p\right)^{\frac{1}{p}}\\
&\leqslant \alpha\beta-\sum_{i=k+1}^n a_ib_i\\
&\leqslant AB-\sum_{i=1}^n a_ib_i,
\end{aligned} \tag{6}$$

where $\alpha = \left(A^p - \sum_{i=1}^k a_i^p\right)^{\frac{1}{p}}$, $\beta = \left(B^p - \sum_{i=1}^k b_i^p\right)^{\frac{1}{p}}$.

THEOREM G. *Let* n *and* j *be fixed positive integers with* $1 \leqslant j < n$, *let* $p > 1$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, *and let* A, B, a_i, b_i $(i = 1, 2, \cdots, n)$ *be nonnegative real numbers such that* $A^p - \sum_{i=1}^n a_i^p > 0$ *and* $B^q - \sum_{i=1}^n b_i^q > 0$. *Then*

$$\begin{aligned}
&\left(A^p - \sum_{i=1}^n a_i^p\right)^{\frac{1}{p}} \left(B^q - \sum_{i=1}^n b_i^q\right)^{\frac{1}{q}} \\
&\leqslant \alpha\widetilde{\beta} - \left(\sum_{i=k+1}^n a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=k+1}^n b_i^q\right)^{\frac{1}{q}} \\
&\leqslant \alpha\widetilde{\beta} - \sum_{i=k+1}^n a_i b_i \\
&\leqslant AB - \sum_{i=1}^n a_i b_i, \qquad (7)
\end{aligned}$$

where $\alpha = \left(A^p - \sum_{i=1}^k a_i^p\right)^{\frac{1}{p}}$, $\widetilde{\beta} = \left(B^q - \sum_{i=1}^k b_i^q\right)^{\frac{1}{q}}$.

THEOREM H. *Let* n *and* j *be fixed positive integers with* $1 \leqslant j < n$, *let* $p \geqslant 1$, *and let* A, B, a_i, b_i $(i = 1, 2, \cdots, n)$ *be nonnegative real numbers such that* $A^p - \sum_{i=1}^n a_i^p > 0$ *and* $B^p - \sum_{i=1}^n b_i^p > 0$. *Then*

$$\begin{aligned}
&\left(A^p - \sum_{i=1}^n a_i^p\right)^{\frac{1}{p}} + \left(B^p - \sum_{i=1}^n b_i^p\right)^{\frac{1}{p}} \\
&\leqslant \left\{(\alpha + \beta)^p - \left[\left(\sum_{i=k+1}^n a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=k+1}^n b_i^p\right)^{\frac{1}{p}}\right]^p\right\}^{\frac{1}{p}} \\
&\leqslant \left[(\alpha + \beta)^p - \sum_{i=k+1}^n (a_i + b_i)^p\right]^{\frac{1}{p}} \\
&\leqslant \left[(A + B)^p - \sum_{i=1}^n (a_i + b_i)^p\right]^{\frac{1}{p}}, \qquad (8)
\end{aligned}$$

where $\alpha = \left(A^p - \sum_{i=1}^k a_i^p\right)^{\frac{1}{p}}$, $\beta = \left(B^p - \sum_{i=1}^k b_i^p\right)^{\frac{1}{p}}$.

Moreover, in 2012, Tian [7] presented the reversed version of Aczél-Bjelica inequality (4) as follows.

THEOREM I. *Let* n *be a fixed positive integer, let* $p < 0$, *and let* A, B, a_i, b_i $(i = 1, 2, \cdots, n)$ *be positive real numbers such that* $A^p - \sum_{i=1}^n a_i^p > 0$ *and* $B^p - \sum_{i=1}^n b_i^p > 0$. *Then*

$$\left(A^p - \sum_{i=1}^n a_i^p\right)^{\frac{1}{p}} \left(B^p - \sum_{i=1}^n b_i^p\right)^{\frac{1}{p}} \geqslant AB - \sum_{i=1}^n a_i b_i. \qquad (9)$$

In this note, using the method of Farid, Pečarić and Ur Rehman [5], we obtain the reversed versions of inequalities (6), (7) and (8). The obtained results are the refinements of reversed Aczél-Popoviciu inequality, Aczél-Bjelica inequality and Bellman's inequality.

2. Reversed versions of Aczél-type inequality and Bellman-type inequality

LEMMA 2.1. [8] *If* $b_i > 0$ $(i = 1, 2, \ldots, n)$, *and* $p < 0 < q$, *then*

$$\left(\sum_{i=1}^{n} b_i^p\right)^{\frac{1}{p}} \leqslant \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}}. \tag{10}$$

LEMMA 2.2. (Generalized Hölder's inequality) [8] *Let* $a_i,\ b_i > 0$ $(i = 1, 2, \cdots, n)$.
(a) If $p, q > 0$, *and* $\frac{1}{p} + \frac{1}{q} \geqslant 1$, *then*

$$\sum_{i=1}^{n} a_i b_i \leqslant \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}}. \tag{11}$$

(b) If $p < 0,\ q > 0$, *and* $\frac{1}{p} + \frac{1}{q} \leqslant 1$, *then*

$$\sum_{i=1}^{n} a_i b_i \geqslant \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}}. \tag{12}$$

(c) If $p, q < 0$, *then*

$$\sum_{i=1}^{n} a_i b_i \geqslant \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}}. \tag{13}$$

LEMMA 2.3. [7] *Let* n *be a fixed positive integer, let* $p < 0,\ q > 0,\ \frac{1}{p} + \frac{1}{q} \leqslant 1$, *and let* $A,\ B,\ a_i,\ b_i$ $(i = 1, 2, \cdots, n)$ *be positive real numbers such that* $A^p - \sum_{i=1}^{n} a_i^p > 0$ *and* $B^q - \sum_{i=1}^{n} b_i^q > 0$. *Then*

$$\left(A^p - \sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(B^q - \sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}} \geqslant AB - \sum_{i=1}^{n} a_i b_i. \tag{14}$$

Nextly, we give the reversed version of inequality (6), i. e., a new refinement of inequality (9).

THEOREM 2.4. *Let* n *and* j *be fixed positive integers with* $1 \leqslant j < n$, *let* $p < 0$, *and let* $A,\ B,\ a_i,\ b_i$ $(i = 1, 2, \cdots, n)$ *be positive real numbers such that* $A^p - \sum_{i=1}^{n} a_i^p > 0$

and $B^p - \sum_{i=1}^n b_i^p > 0$. *Then*

$$\begin{aligned}
&\left(A^p - \sum_{i=1}^n a_i^p\right)^{\frac{1}{p}} \left(B^p - \sum_{i=1}^n b_i^p\right)^{\frac{1}{p}} \\
&\geqslant \alpha\beta - \left(\sum_{i=k+1}^n a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=k+1}^n b_i^p\right)^{\frac{1}{p}} \\
&\geqslant \alpha\beta - \sum_{i=k+1}^n a_i b_i \\
&\geqslant AB - \sum_{i=1}^n a_i b_i, \qquad (15)
\end{aligned}$$

where $\alpha = \left(A^p - \sum_{i=1}^k a_i^p\right)^{\frac{1}{p}}$, $\beta = \left(B^p - \sum_{i=1}^k b_i^p\right)^{\frac{1}{p}}$.

Proof. By using inequality (9) we have

$$\alpha\beta \geqslant AB - \sum_{i=1}^k a_i b_i. \qquad (16)$$

If we denote $C = \left(\sum_{i=k+1}^n a_i^p\right)^{\frac{1}{p}}$, $D = \left(\sum_{i=k+1}^n b_i^p\right)^{\frac{1}{p}}$, then applying inequality (9) with $n = 1$, we have

$$\begin{aligned}
&\left(A^p - \sum_{i=1}^n a_i^p\right)^{\frac{1}{p}} \left(B^p - \sum_{i=1}^n b_i^p\right)^{\frac{1}{p}} \\
&= \left(\alpha^p - C^p\right)^{\frac{1}{p}} \left(\beta^p - D^p\right)^{\frac{1}{p}} \\
&\geqslant \alpha\beta - CD \\
&= \alpha\beta - \left(\sum_{i=k+1}^n a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=k+1}^n b_i^p\right)^{\frac{1}{p}}. \qquad (17)
\end{aligned}$$

On the other hand, from Lemma 2.1 for $p < 0 < q$, $\frac{1}{p} + \frac{1}{q} = 1$, and using generalized Hölder's inequality (12) we get

$$\left(\sum_{i=k+1}^n a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=k+1}^n b_i^p\right)^{\frac{1}{p}} \leqslant \left(\sum_{i=k+1}^n a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=k+1}^n b_i^q\right)^{\frac{1}{q}} \leqslant \sum_{i=k+1}^n a_i b_i. \qquad (18)$$

Therefore, combining (16), (17) and (18) yields immediately the desired inequality (15). □

REMARK 2.5. Farid, Pečarić and Ur Rehman [5] obtained the inequality (6) for $1 < p \leqslant 2$. In fact, the inequality is also valid for $0 < p \leqslant 2$. Similar to the proof of the inequality (6) for $1 < p \leqslant 2$ in [5] but using (11) in place of Hölder's inequality, we immediately obtain the inequality (6) for $0 < p \leqslant 2$.

Now, we present the reversed version of inequality (7), i.e., a new refinement of Aczél-Vasić-Pečarić inequality (3).

THEOREM 2.6. *Let n and j be fixed positive integers with $1 \leqslant j < n$, let $p < 1$, $p \neq 0$, $\frac{1}{p} + \frac{1}{q} \leqslant 1$, and let A, B, a_i, b_i $(i = 1, 2, \cdots, n)$ be positive real numbers such that $A^p - \sum_{i=1}^n a_i^p > 0$ and $B^q - \sum_{i=1}^n b_i^q > 0$. Then*

$$\begin{aligned}
&\left(A^p - \sum_{i=1}^n a_i^p\right)^{\frac{1}{p}} \left(B^q - \sum_{i=1}^n b_i^q\right)^{\frac{1}{q}} \\
&\geqslant \alpha\widetilde{\beta} - \left(\sum_{i=k+1}^n a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=k+1}^n b_i^q\right)^{\frac{1}{q}} \\
&\geqslant \alpha\widetilde{\beta} - \sum_{i=k+1}^n a_i b_i \\
&\geqslant AB - \sum_{i=1}^n a_i b_i, \qquad (19)
\end{aligned}$$

where $\alpha = \left(A^p - \sum_{i=1}^k a_i^p\right)^{\frac{1}{p}}$, $\widetilde{\beta} = \left(B^q - \sum_{i=1}^k b_i^q\right)^{\frac{1}{q}}$.

Proof. From Lemma 2.3 we have

$$\alpha\widetilde{\beta} \geqslant AB - \sum_{i=1}^k a_i b_i. \qquad (20)$$

If we denote $C = \left(\sum_{i=k+1}^n a_i^p\right)^{\frac{1}{p}}$, $\widetilde{D} = \left(\sum_{i=k+1}^n b_i^q\right)^{\frac{1}{q}}$, then by applying Lemma 2.3, for $n = 1$, we obtain

$$\begin{aligned}
&\left(A^p - \sum_{i=1}^n a_i^p\right)^{\frac{1}{p}} \left(B^q - \sum_{i=1}^n b_i^q\right)^{\frac{1}{q}} \\
&= \left(\alpha^p - C^p\right)^{\frac{1}{p}} \left(\widetilde{\beta}^q - \widetilde{D}^q\right)^{\frac{1}{q}} \\
&\geqslant \alpha\widetilde{\beta} - C\widetilde{D} \\
&= \alpha\widetilde{\beta} - \left(\sum_{i=k+1}^n a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=k+1}^n b_i^q\right)^{\frac{1}{q}}. \qquad (21)
\end{aligned}$$

Moreover, using generalized Hölder's inequality (12) we get

$$\left(\sum_{i=k+1}^n a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=k+1}^n b_i^q\right)^{\frac{1}{q}} \leqslant \sum_{i=k+1}^n a_i b_i. \qquad (22)$$

Combining (20), (21) and (22) yields immediately the desired result. □

Finally, we give the reversed version of inequality (8), i.e., a new refinement of reversed Bellman inequality.

THEOREM 2.7. *Let n and j be fixed positive integers with $1 \leqslant j < n$, let $0 < p < 1$, and let A, B, a_i, b_i $(i = 1, 2, \cdots, n)$ be positive real numbers such that $A^p - \sum_{i=1}^n a_i^p > 0$ and $B^p - \sum_{i=1}^n b_i^p > 0$. Then*

$$\begin{aligned}
&\left(A^p - \sum_{i=1}^n a_i^p\right)^{\frac{1}{p}} + \left(B^p - \sum_{i=1}^n b_i^p\right)^{\frac{1}{p}} \\
&\geqslant \left\{(\alpha+\beta)^p - \left[\left(\sum_{i=k+1}^n a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=k+1}^n b_i^p\right)^{\frac{1}{p}}\right]^p\right\}^{\frac{1}{p}} \\
&\geqslant \left[(\alpha+\beta)^p - \sum_{i=k+1}^n (a_i+b_i)^p\right]^{\frac{1}{p}} \\
&\geqslant \left[(A+B)^p - \sum_{i=1}^n (a_i+b_i)^p\right]^{\frac{1}{p}},
\end{aligned} \tag{23}$$

where $\alpha = \left(A^p - \sum_{i=1}^k a_i^p\right)^{\frac{1}{p}}$, $\beta = \left(B^p - \sum_{i=1}^k b_i^p\right)^{\frac{1}{p}}$.

Proof. By using Bellman's inequality (5), we have

$$\alpha + \beta \geqslant \left[(A+B)^p - \sum_{i=1}^k (a_i+b_i)^p\right]^{\frac{1}{p}}. \tag{24}$$

If we denote $C = \left(\sum_{i=k+1}^n a_i^p\right)^{\frac{1}{p}}$, $D = \left(\sum_{i=k+1}^n b_i^p\right)^{\frac{1}{p}}$, then using Bellman's inequality (5) for $n = 1$, we have

$$\begin{aligned}
&\left(A^p - \sum_{i=1}^n a_i^p\right)^{\frac{1}{p}} + \left(B^q - \sum_{i=1}^n b_i^q\right)^{\frac{1}{q}} \\
&= \left(\alpha^p - C^p\right)^{\frac{1}{p}} + \left(\beta^p - D^p\right)^{\frac{1}{p}} \\
&\geqslant \left[(\alpha+\beta)^p - (C+D)^p\right]^{\frac{1}{p}} \\
&= \left\{(\alpha+\beta)^p - \left[\left(\sum_{i=k+1}^n a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=k+1}^n b_i^p\right)^{\frac{1}{p}}\right]^p\right\}^{\frac{1}{p}}.
\end{aligned} \tag{25}$$

Moreover, by using Minkowski inequality with $0 < p < 1$, we get

$$\left[\left(\sum_{i=k+1}^n a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=k+1}^n b_i^p\right)^{\frac{1}{p}}\right]^p \leqslant \sum_{i=k+1}^n (a_i+b_i)^p. \tag{26}$$

Combining (23), (24) and (25) yields immediately the desired result. □

REMARK 2.8. Farid, Pečarić and Ur Rehman [5] obtained the inequality (8) for $p \geqslant 1$. In fact, inequality (8) is also valid for $p < 0$. Similar to the proof of the inequality (8) under the assumption $p \geqslant 1$ in [5], it is easy to obtain the inequality (8) for $p < 0$.

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