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ON SOME INEQUALITIES EQUIVALENT TO THE WRIGHT–CONVEXITY

ANDRZEJ OLBRYŚ

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Abstract. In the present paper we establish some conditions and inequalities equivalent to the Wright-convexity.

1. Introduction and terminology

Let X be a real linear space, and let $D \subset X$ be a convex set. A function $f : D \to \mathbb{R}$ is called convex if

$$\bigwedge_{x,y\in D} \bigwedge_{\lambda\in[0,1]} f(\lambda x+(1-\lambda)y) \leqslant \lambda f(x)+(1-\lambda)f(y). \tag{1}$$

If the above inequality holds for all $x,y \in D$ with $\lambda = \frac{1}{2}$ then f is said to be Jensen convex. In 1954 E.M. Wright [23] introduced a new convexity property. A function $f : D \to \mathbb{R}$ is called Wright-convex if

$$\bigwedge_{x,y\in D} \bigwedge_{\lambda\in[0,1]} f(\lambda x+(1-\lambda)y)+f((1-\lambda)x+\lambda y) \leqslant f(x)+f(y). \tag{2}$$

One can easily see that convex functions are Wright-convex, and Wright-convex functions are Jensen-convex. On the other hand, if $f : X \to \mathbb{R}$ is additive, that is,

$$f(x+y) = f(x)+f(y), \quad x,y \in X;$$

then f is also Wright-convex. The main result concerning Wright-convex functions is the much more surprising statement that any Wright-convex function can be decomposed as the sum of such functions. The following theorem has been proved by Ng [14] for functions defined on convex subsets of $\mathbb{R}^n$ and was extended by Kominek [10] for functions defined on convex subsets of more general structures (see also [13]).

THEOREM 1. *Let X be a real linear space, and let $D \subset X$ be an algebraically open and convex set. A function $f : D \to \mathbb{R}$ is Wright-convex if and only if there exist a convex function $F : D \to \mathbb{R}$ and an additive function $a : X \to \mathbb{R}$ such that*

$$f(x) = F(x)+a(x), \quad x \in D. \tag{3}$$

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It follows immediately from the above representation that Wright-convex functions are either very regular or very irregular. It is known that quite week conditions e.g. locally boundedness from above or below at some point, Christensen measurability, continuity at least one point, imply the continuity of such functions. On the other hand, if they are discontinuous, they have a dense graph.

A survey of other results concerning Wright-convex functions may be found in the papers [10], [13], [14], [16], [17], [24].

2. Regularity properties

Recall (see [22]) that a function $f : D \to \mathbb{R}$, where D is an open subset of topological space X, is said to be symmetric at a point $x \in D$, if

$$\lim_{h \to 0}[f(x+h) + f(x-h) - 2f(x)] = 0.$$

Of course, every continuous function is symmetric, but the converse is not true.

Let us start with the following

THEOREM 2. *Let $D \subset \mathbb{R}^n$ be a convex and open set. If $f : D \to \mathbb{R}$ is a Wright-convex function, then it is symmetric at every point $x \in D$.*

Proof. By Theorem 1 f has the form

$$f(x) = a(x) + F(x), \quad x \in D,$$

where $F : D \to \mathbb{R}$ is a convex function and $a : \mathbb{R}^n \to \mathbb{R}$ an additive function. Because convex functions defined on a subset of finite dimensional real linear space are continuous, then by additivity of a we get

$$\lim_{h \to 0}[f(x+h) + f(x-h) - 2f(x)] = \lim_{h \to 0}[F(x+h) + F(x-h) - 2F(x)] = 0. \quad \square$$

As we know from the classical theory of convex functions the convexity of a twice differentiable function can be inferred from the sign of its second derivative. Namely, the following theorem holds true (see [1], [12], [15], [19], [20]).

THEOREM 3. *Let $I \subset \mathbb{R}$ be an open interval, and let $f : I \to \mathbb{R}$ be a convex function. Then f is twice differentiable almost everywhere in I. Whenever the second derivative exists, $f''(x) \geqslant 0$. Moreover, if f is twice differentiable, then it is convex if and only if the function f'' is non-negative in I.*

Let us recall that the upper and the lower second symmetric derivative of f at x, are respectively defined by the formulas

$$\overline{D}_s^2 f(x) := \limsup_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2},$$

and

$$\underline{D}_s^2 f(x) := \liminf_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2}.$$

If the two extreme derivatives are equal and finite at a point x then f is said to have a symmetric derivative at this point and the common value is denoted by $D_s^2 f(x)$.

It is easy to check that if f is twice differentiable at x, then

$$\overline{D}_s^2 f(x) = \underline{D}_s^2 f(x) = f''(x),$$

however, $D_s^2 f(x)$ can exist even at points of discontinuity. For example, for an arbitrary additive function $a : \mathbb{R} \to \mathbb{R}$ we have

$$D_s^2 a(x) = 0, \quad x \in \mathbb{R},$$

and as we know (see [12]) such functions may be discontinuous at every point.

For Wright-convex functions we have the following counterpart of Theorem 3

THEOREM 4. *Let $I \subset \mathbb{R}$ be an open interval, and let $f : I \to \mathbb{R}$ be a Wright-convex function. Then there exists a second symmetric derivative almost everywhere in I. Whenever it exists, $D_s^2 f(x) \geqslant 0$.*

Proof. Let $f : I \to \mathbb{R}$ be a Wright-convex function. By the representation (3) f has the form $f = a + F$, where $a : \mathbb{R} \to \mathbb{R}$ is an additive and $F : I \to \mathbb{R}$ is a convex function. Observe that for all $x \in I$ and $h \in \mathbb{R}$ such that $x-h,\ x+h \in I$ we have

$$f(x+h)+f(x-h)-2f(x) = F(x+h)+F(x-h)-2F(x) \geqslant 0.$$

By the above equality, convexity of F and on account of Theorem 3 we obtain

$$D_s^2 f(x) = D_s^2 F(x) = F''(x) \geqslant 0,$$

almost everywhere in I. The proof of our theorem is complete. $\square$

Since the second symmetric derivative of a Wright-convex function may not exist, we need to use the concept of the lower and upper second symmetric derivative to characterize this kind of convexity. Inspired by methods contained in [15, p. 24] we prove the following

THEOREM 5. *Let $I \subset \mathbb{R}$ be an open interval. A function $f : I \to \mathbb{R}$ is Wright-convex if and only if the map*

$$I \times I \ni (x,y) \longrightarrow f(x)+f(y)-2f\left(\frac{x+y}{2}\right), \tag{4}$$

is continuous, and $\overline{D}_s^2 f(x) \geqslant 0$, $x \in I$.

Proof. If f is Wright-convex then $f = a + F$ for some additive function $a : \mathbb{R} \to \mathbb{R}$ and a convex function $F : I \to \mathbb{R}$. Because

$$f(x+h) + f(x-h) - 2f(x) = F(x+h) + F(x-h) - 2F(x)$$

then by convexity of F we get $\overline{D}_s^2 f(x) \geqslant 0$, $x \in I$. The continuity of a map given by (4) follows from the continuity of F.

Conversely, suppose that the map given by (4) is continuous and $\overline{D}_s^2 f(x) \geqslant 0$, $x \in I$. Let us consider the sequence of functions $f_n : I \to \mathbb{R}$ given by the formula

$$f_n(x) := f(x) + \frac{1}{n} x^2, \quad n \in \mathbb{N}.$$

Clearly, the mapping

$$I \times I \ni (x,y) \longrightarrow f_n(x) + f_n(y) - 2f_n\Big(\frac{x+y}{2}\Big),$$

is continuous, for all $n \in \mathbb{N}$, moreover,

$$f_n(x) \longrightarrow_{n\to\infty} f(x), \quad \text{and,} \quad \overline{D}_s^2 f_n(x) > 0,\ x \in I,\ n \in \mathbb{N}.$$

We shall show that f_n is a Wright-convex function for all $n \in \mathbb{N}$. On account of Theorem 12 from [17] it is enough to show that f_n is a convex function in the sense of Jensen, $n \in \mathbb{N}$.

Assume the contrary, that there exists a subinterval $I_0 = [a_0, b_0] \subset I$ such that

$$J_{f_n}(a_0, b_0) := \frac{f_n(a_0) + f_n(b_0)}{2} - f_n\Big(\frac{a_0+b_0}{2}\Big) < 0.$$

An easy calculation shows that

$$J_{f_n}(a_0,b_0) = J_{f_n}\Big(a_0, \frac{a_0+b_0}{2}\Big) + J_{f_n}\Big(\frac{a_0+b_0}{2}, b_0\Big) + 2J_{f_n}\Big(\frac{a_0+3b_0}{4}, \frac{3a_0+b_0}{4}\Big) < 0,$$

consequently, one of the intervals

$$\Big[a_0, \frac{a_0+b_0}{2}\Big], \qquad \Big[\frac{a_0+b_0}{2}, b_0\Big], \qquad \Big[\frac{a_0+3b_0}{4}, \frac{3a_0+b_0}{4}\Big]$$

can be chosen to replace I_0 by a smaller interval $I_1 = [a_1, b_1]$, with $b_1 - a_1 = \frac{b_0 - a_0}{2}$ and $J_{f_n}(a_1, b_1) < 0$.

Using induction, we can construct a sequence of intervals $I_k = [a_k, b_k]$, $k \in \mathbb{N}$ such that

$$I_{k+1} \subset I_k, \qquad b_k - a_k = \frac{b_0 - a_0}{2^k}, \qquad J_{f_n}(a_k, b_k) < 0, \quad k \in \mathbb{N}.$$

Denote by x_0 the unique element of the intersection

$$\{x_0\} = \cap_{k=1}^{\infty} [a_k, b_k].$$

Obviously, from the choice of the sequence $\{I_k\}_{k\in\mathbb{N}}$ and by continuity of the function given by (4) we obtain

$$\overline{D}_s^2 f_n(x_0) \leqslant 0$$

which is a contradiction. $\square$

3. Characterizations

Let start this section with the following

THEOREM 6. *A function* $f : D \to \mathbb{R}$ *is Wright-convex if and only if for all* $x,y \in D$, $\lambda \in [0,1]$, $k,n \in \mathbb{N}$, $s_1,\ldots,s_k$, $t_1,\ldots,t_n \in [0,1]$ *such that* $\sum_{j=1}^k s_j = \sum_{i=1}^n t_i = \lambda$ *the following inequality holds*

$$\sum_{j=1}^{k} f(s_jx+(1-s_j)y)+\sum_{i=1}^{n} f((1-t_i)x+t_iy) \leqslant kf(y)+nf(x). \tag{5}$$

Proof. If (5) holds true then putting $s_1 = t_1 = \lambda \in [0,1]$, we obtain the Wright-convexity of f. Conversely, assume that f is a Wright-convex function. By Theorem 1 it has the form

$$f(x) = a(x)+F(x), \quad x \in D,$$

where $a : X \to \mathbb{R}$ is an additive and $F : D \to \mathbb{R}$ a convex function. By additivity of a and convexity of F we obtain

$$\begin{aligned}
&\textstyle\sum_{j=1}^k f(s_jx+(1-s_j)y)+\sum_{i=1}^n f((1-t_i)x+t_iy)\\
&\qquad = \textstyle\sum_{j=1}^k [F(s_jx+(1-s_j)y)+a(s_jx+(1-s_j)y)]\\
&\qquad\quad + \textstyle\sum_{i=1}^n [F((1-t_i)x+t_iy)+a((1-t_i)x+t_iy)]\\
&\qquad \leqslant ka(y)+a(\lambda(x-y))+\textstyle\sum_{j=1}^k [s_jF(x)+(1-s_j)F(y)]\\
&\qquad\quad + na(x)-a(\lambda(x-y))+\textstyle\sum_{i=1}^n [(1-t_i)F(x)+t_iF(y)]\\
&\qquad = k[a(y)+F(y)]+n[a(x)+F(x)] = kf(y)+nf(x).
\end{aligned}$$

The proof of our theorem is finished. □

The following two theorems give another characterization of Wright-convexity and are crucial in the future consideration.

THEOREM 7. *Let* X *be a real linear space, and let* $D \subset X$ *be an algebraically open and convex set. A function* $f : D \to \mathbb{R}$ *is Wright-convex if and only if for each* $x,y \in D$ *the function* $f_{x,y}$ *given by the formula*

$$f_{x,y}(s) := f(sx+(1-s)y)+f((1-s)x+sy), \tag{6}$$

is convex on the set $D_{x,y} := \{s \in \mathbb{R} :\ sx+(1-s)y,\ (1-s)x+sy \in D\}$.

Proof. Assume that $f : D \to \mathbb{R}$ is a Wright-convex function. On account of Theorem 1 there exist a convex function $F : D \to \mathbb{R}$ and additive function $a : X \to \mathbb{R}$ such that

$$f(x) = a(x)+F(x), \quad x \in D.$$

Fix arbitrary $x,y \in D$. By the above representation we obtain

$$f_{x,y}(s) = F(sx+(1-s)y) + F((1-s)x+sy) + a(x+y),$$

hence it is convex, as a sum of two convex functions.

Conversely, suppose that for all $x,y \in D$ the function $f_{x,y}$ given by (6) is convex. By convexity, for all $s \in [0,1]$ we get

$$\begin{aligned} f(sx+(1-s)y) + f((1-s)x+sy) &= f_{x,y}(s) = f_{x,y}(s1+(1-s)0) \\ &\leqslant sf_{x,y}(1) + (1-s)f_{x,y}(0) \\ &= s[f(x)+f(y)] + (1-s)[f(x)+f(y)] \\ &= f(x)+f(y), \end{aligned}$$

which ends the proof. □

REMARK 1. The function $f_{x,y}$ given by the formula (6) is continuous, symmetric with respect to $\frac{1}{2}$, decreasing on $[0,\frac{1}{2}]$, increasing on $[\frac{1}{2},1]$ and attains a global minimum at $\frac{1}{2}$.

THEOREM 8. *Let X be a real linear space, and let $D \subset X$ be an algebraically open and convex set. A function $f : D \to \mathbb{R}$ is Wright-convex if and only if for all $y \in D$ the function $f_y : D_y \to \mathbb{R}$ given by the formula*

$$f_y(x) := f(x) + f(2y-x), \tag{7}$$

is convex, where $D_y := D \cap (2y-D)$.

Proof. Suppose that f is a Wright-convex function. We use the representation

$$f(x) = F(x) + a(x), \quad x \in D,$$

where $a : X \to \mathbb{R}$ is an additive and $F : D \to \mathbb{R}$ is a convex function. Therefore

$$\begin{aligned} f_y(x) &= F(x) + a(x) + F(2y-x) + a(2y-x) \\ &= F(x) + F(2y-x) + 2a(y), \quad x \in D_y, \end{aligned}$$

is a convex function, as a sum of two convex functions, where $y \in D$.

Now, assume that for all $y \in D$ the function $f_y : D_y \to \mathbb{R}$ given by formula (7) is convex. Fix arbitrary $x,z \in D$ and $\alpha \in [0,1]$. By convexity of the function $f_{\frac{x+z}{2}}$ and because $x,z \in D_{\frac{x+z}{2}}$ it follows that

$$\begin{aligned} &f(\alpha x + (1-\alpha)z) + f((1-\alpha)x + \alpha z) \\ &\quad = f(\alpha x + (1-\alpha)z) + f(2 \cdot \tfrac{x+z}{2} - (\alpha x + (1-\alpha)z)) \\ &\quad = f_{\frac{x+z}{2}}(\alpha x + (1-\alpha)z) \\ &\quad \leqslant \alpha f_{\frac{x+z}{2}}(x) + (1-\alpha) f_{\frac{x+z}{2}}(z) \\ &\quad = \alpha[f(x)+f(z)] + (1-\alpha)[f(x)+f(z)] = f(x)+f(z). \end{aligned}$$

The proof of our theorem is complete. $\square$

It follows from the proof of Theorem 7 and Theorem 8 that for a function defined on an interval one can prove the joint generalization of these theorems.

THEOREM 9. *Let $I \subset \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is Wright-convex if and only if for all $a,b \in I$, $a < b$ the function $H : [0,1] \times [a,b] \to \mathbb{R}$ given by the formula*

$$H(s,x) := f(sx + (1-s)(a+b-x)) + f((1-s)x + s(a+b-x)), \tag{8}$$

is convex separately in each variable.

4. Some integral inequalities

There are many inequalities valid for convex functions. Probably two of the most well-known ones are the Hermite-Hadamard ([6], [8]) inequalities

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}, \quad a,b \in \mathbb{R}, \quad a < b. \tag{9}$$

In fact, in the class of continuous functions, each of the above inequalities is equivalent to convexity. The Hermite-Hadamard inequalities have been the subject of intensive research, many applications, generalizations and improvements of them can be found in literature (see, for instance, [4], [12], [15], [18], [25]).

Some results concerning the Hermite-Hadamard inequalities for Wright-convex functions are also known in the literature (see [2], [9], [23]). Unfortunately, the authors of all these articles assume that the considered Wright-convex functions are measurable, so in particular continuous and convex. In fact, these results refer to convex functions. Below we give several inequalities of Hermite-Hadamard's type for Wright-convex functions without any regularity assumptions.

THEOREM 10. *Let X be a real linear space, and let $D \subset X$ be an algebraically open and convex set. If $f : D \to \mathbb{R}$ is a Wright-convex function, then for all $x,y \in D$, and $s,t \in [0,1]$, such that $s < t$, the following inequalities hold true*

$$\begin{aligned} f(\tfrac{s+t}{2}x + (1-\tfrac{s+t}{2})y) + f((1-\tfrac{s+t}{2})x + \tfrac{s+t}{2}y) \\ \leqslant \tfrac{1}{t-s}\textstyle\int_s^t [f(ux+(1-u)y) + f((1-u)x+uy)]du \\ \leqslant \tfrac{1}{2}\big[f(sx+(1-s)y) + f((1-s)x+sy) \\ + f(tx+(1-t)y) + f((1-t)x+ty)\big]. \end{aligned}$$

Proof. Fix arbitrary $x,y \in D$. By Theorem 7 the function $f_{x,y} : [0,1] \to \mathbb{R}$ given by formula (6) is continuous and convex. Using the Hermite-Hadamard inequality for this function we obtain, for $s < t$

$$f_{x,y}\left(\frac{s+t}{2}\right) \leqslant \frac{1}{t-s}\int_s^t f_{x,y}(u)du \leqslant \frac{f_{x,y}(s)+f_{x,y}(t)}{2}, \tag{10}$$

which coincides with the above inequalities. □

As an immediate consequence of the above theorem putting $s=0$, $t=1$ we obtain

COROLLARY 1. *Let $f : D \to \mathbb{R}$ be a Wright-convex function. Then for all $x, y \in D$*

$$2f\left(\frac{x+y}{2}\right) \leqslant \int_0^1 [f(sx+(1-s)y) + f((1-s)x+sy)]ds \leqslant f(x)+f(y).$$

For a function defined on an interval we have the following

THEOREM 11. *Let $I \subset \mathbb{R}$ be an interval, and let $f : I \to \mathbb{R}$ be a Wright-convex function. Then for all $a, b \in I$, $a < b$ the following inequalities hold true*

$$2f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b [f(x)+f(a+b-x)]dx \leqslant f(a)+f(b). \tag{11}$$

Proof. Take arbitrary $a, b \in I$, $a < b$. Put in Theorem 8 $y := \frac{a+b}{2}$ and $D := [a,b]$. Since D is symmetric with respect to y then $D_y = [a,b]$. By Theorem 8 we infer that the function

$$[a,b] \ni x \longrightarrow f(x)+f(a+b-x),$$

is continuous and convex. Using the Hermite-Hadamard inequalities for the above function we obtain

$$2f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b [f(x)+f(a+b-x)]dx \leqslant f(a)+f(b),$$

which ends the proof. □

REMARK 2. Similarly as for convex functions:

(i) Each of the inequalities (10) is equivalent to the Wright-convexity in the class of functions $f : D \to \mathbb{R}$ for which for all $x, y \in D$ the mapping given by formula

$$[0,1] \ni s \longrightarrow f(sx+(1-s)y) + f((1-s)x+sy),$$

is continuous.

(ii) Each of the inequalities (11) is equivalent to the Wright-convexity in the class of functions $f : I \to \mathbb{R}$ for which for all $a, b \in I$, $a < b$ the mapping

$$[a,b] \ni x \longrightarrow f(x)+f(a+b-x),$$

is continuous.

As an immediate consequence of Theorem 11 we obtain the following

COROLLARY 2. *Let $I \subset \mathbb{R}$ be an open interval, and let $f : I \to \mathbb{R}$ be a Wright-convex function. Then for every $x \in I$ we have*

$$f(x) = \lim_{h \to 0_+} \frac{1}{4h}\int_{x-h}^{x+h} [f(s)+f(2x-s)]ds.$$

Proof. By Theorem 11 for all $a,b \in I$, $a<b$ f satisfies the following inequalities

$$2f\Big(\frac{a+b}{2}\Big) \leqslant \frac{1}{b-a}\int_a^b [f(x)+f(a+b-x)]dx \leqslant f(a)+f(b).$$

Take an arbitrary $x \in I$. There exists a number $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subset I$. Let $h \in (0,\varepsilon)$ and put in the above inequalities $a := x-h$, $b := x+h$. Then we get

$$2f(x) \leqslant \frac{1}{2h}\int_{x-h}^{x+h} [f(s)+f(2x-s)]ds \leqslant f(x-h)+f(x+h). \tag{12}$$

Letting $h \to 0_+$ in (12) and applying Theorem 2 we obtain

$$f(x) = \lim_{h\to 0_+} \frac{1}{4h}\int_{x-h}^{x+h} [f(s)+f(2x-s)]ds.$$

This concludes the proof. $\square$

In [3] S. S. Dragomir established the following theorem which is a refinement of the first inequality of (9).

THEOREM 12. *If $f : [a,b] \to \mathbb{R}$ is a convex function, and G is defined on $[0,1]$ by*

$$G(t) := \frac{1}{b-a}\int_a^b f\Big(tx+(1-t)\frac{a+b}{2}\Big)dx,$$

then G is convex, increasing on $[0,1]$, and for all $t \in [0,1]$, we have

$$f\Big(\frac{a+b}{2}\Big) = G(0) \leqslant G(t) \leqslant G(1) = \frac{1}{b-a}\int_a^b f(x)dx.$$

Below we present the following two theorems which are refinements of the first inequality of (10) and (11)

THEOREM 13. *If $f : [a,b] \to \mathbb{R}$ is a Wright-convex function, and P is defined on $[0,1]$ by*

$$P(s) := \frac{1}{b-a}\int_a^b H(s,x)dx,$$

where H is given by (8), then P is convex, $P(s) = P(1-s)$, $s \in [0,1]$, moreover,

$$2f\Big(\frac{a+b}{2}\Big) = P\Big(\frac{1}{2}\Big) \leqslant \int_0^1 P(t)dt \leqslant \frac{P(0)+P(1)}{2} = \frac{1}{b-a}\int_a^b [f(x)+f(a+b-x)]dx.$$

Proof. Fix arbitrary $s,t,\alpha \in [0,1]$. By convexity of the function $[0,1] \ni s \longrightarrow H(s,x)$ we obtain

$$H(\alpha s+(1-\alpha)t,x) \leqslant \alpha H(s,x)+(1-\alpha)H(t,x), \quad x \in [a,b].$$

Integrating the above inequality over $[a,b]$ and dividing by $b-a$ we get

$$P(\alpha s+(1-\alpha)t)\leqslant \alpha P(s)+(1-\alpha)P(t),$$

hence from the classical Hermite-Hadamard inequalities, and because $P(0)=P(1)$ we obtain

$$2f\Big(\frac{a+b}{2}\Big)=P\Big(\frac{0+1}{2}\Big)\leqslant\int_0^1 P(t)dt\leqslant\frac{P(0)+P(1)}{2}=\frac{1}{b-a}\int_a^b[f(x)+f(a+b-x)]dx,$$

which ends the proof. □

The proof of the following theorem runs in a similar way

THEOREM 14. *If $f:[a,b]\to\mathbb{R}$ is a Wright-convex function, and Q is defined on $[a,b]$ by*

$$Q(x):=\int_0^1 H(s,x)ds,$$

where H is given by (8), then Q is convex, $Q(x)=Q(a+b-x)$, $x\in[a,b]$, moreover,

$$\begin{aligned}2f\left(\tfrac{a+b}{2}\right)=Q\left(\tfrac{a+b}{2}\right)\leqslant\tfrac{1}{b-a}\textstyle\int_a^b Q(x)dx\leqslant\tfrac{Q(a)+Q(b)}{2}\\ =\textstyle\int_0^1[f(sa+(1-s)b)+f((1-s)a+sb)]ds.\end{aligned}$$

5. Schur-convexity of the upper and the lower limit of some integral

The Schur-convex function was introduced by I. Schur in 1923 [21] and has many important applications in analytic inequalities. The following definitions can be found in many references such as [1], [7], [14], [18], [19].

DEFINITION 1. An $n\times n$ matrix $S=[s_{ij}]$ is doubly stochastic if

$$s_{ij}\geqslant 0,\qquad \text{for}\quad i,j=1,\dots,n,$$

and

$$\sum_{i=1}^n s_{ij}=1,\quad j=1,\dots,n,\qquad \sum_{j=1}^n s_{ij}=1\quad i=1,\dots,n.$$

Particularly interesting examples of doubly stochastic matrices are provided by the permutation matrices. Recall that matrix P is said to be a permutation matrix if each row and column has a single unite entry and all other entries are zero.

DEFINITION 2. A real valued function f defined on a set I^n, where I is an interval is said to be Schur-convex, if for every doubly stochastic matrix S,

$$f(Sx)\leqslant f(x).$$

Similarly f is said to be Schur-concave on I^n, if for every doubly stochastic matrix S,

$$f(Sx)\geqslant f(x).$$

Of course, f is Schur-concave if and only if $-f$ is Schur-convex.

Every Schur-convex function is a symmetric function, because if P is a permutation matrix, so is its inverse P^{-1}. Hence if f is Schur-convex, then

$$f(x) = f(P^{-1}(Px)) \leqslant f(Px) \leqslant f(x).$$

It shows that $f(Px) = f(x)$, for every permutation matrix P.

A survey of results concerning Schur-convex functions may be found in the papers [1], [5], [7], [14], [18], [19], [21].

In [5] N. Elezović and J. Pečarić researched the Schur-convexity of the upper and the lower limit of the integral for the mean of the convex functions and established the following result

THEOREM 15. *Let I be an interval with non-empty interior and let f be a continuous on I. Then*

$$\Phi(a,b) = \begin{cases} \frac{1}{b-a}\int_a^b f(t)dt, & a,b \in I,\ a \neq b \\ f(a), & a = b \end{cases}$$

is Schur-convex (Schur-concave) on I^2 if and only if f is convex (concave) on I.

The aim of this part of our paper is to establish the result which is similar to Theorem 15 for Wright-convex functions.

THEOREM 16. *Let I be an interval with non-empty interior and let $f : I \to \mathbb{R}$ be a function such that for all $a,b \in I$, $a < b$ the map*

$$[a,b] \ni x \longrightarrow f(x) + f(a+b-x), \tag{13}$$

is continuous. Then the mapping $\Phi : I \times I \to \mathbb{R}$ given by formula

$$\Phi(a,b) = \begin{cases} \frac{1}{b-a}\int_a^b [f(x) + f(a+b-x)]dx, & a,b \in I,\ a \neq b \\ 2f(a), & a = b \end{cases}$$

is Schur-convex (Schur-concave) on I^2 if and only if f is Wright-convex (Wright-concave) on I.

Proof. Evidently the map Φ is symmetric. For arbitrary $a,b \in I$ and $s \in (\frac{1}{2}, 1]$ we have

$$\Phi(sa + (1-s)b, (1-s)a + sb) = \frac{1}{(2s-1)(b-a)} \int_{sa+(1-s)b}^{(1-s)a+sb} [f(x) + f(a+b-x)]dx.$$

By the transformation

$$x(u) := (2s-1)u + (1-s)(a+b), \qquad x'(u) = 2s-1 > 0,$$

we obtain

$$u \in [a,b] \Longleftrightarrow x(u) \in [sa+(1-s)b,(1-s)a+sb],$$

and consequently,

$$\begin{aligned}&\Phi(sa+(1-s)b,(1-s)a+sb)\\&\qquad = \tfrac{1}{b-a}\int_a^b [f(su+(1-s)(a+b-u))+f((1-s)u+s(a+b-u))]du.\end{aligned}$$

Assume that f is a Wright-convex function. Fix arbitrary $a,b \in I$, $a<b$ and $s \in (\frac{1}{2},1]$. By the definition, for all $u \in [a,b]$ we obtain

$$f(su+(1-s)(a+b-u))+f((1-s)u+s(a+b-u)) \leqslant f(u)+f(a+b-u),$$

so integrating the above inequality over $[a,b]$ and dividing by $b-a$ we get

$$\begin{aligned}&\Phi(sa+(1-s)b,(1-s)a+sb)\\&\qquad = \tfrac{1}{b-a}\int_a^b [f(su+(1-s)(a+b-u))+f((1-s)u+s(a+b-u))]du\\&\qquad \leqslant \tfrac{1}{b-a}\int_a^b [f(u)+f(a+b-u)]du = \Phi(a,b).\end{aligned}$$

In the case $s=\frac{1}{2}$ on account of Theorem 11 the above inequality is a consequence of the first inequality from (11), which together with symmetry of Φ shows that Φ is a Schur-convex.

Conversely, suppose that Φ is a Schur-convex, and f is not convex in the sense of Wright. Then there exist $a,b \in I$, $a<b$, $s \in (0,1)$ and $u_0 \in (a,b)$ such that

$$f(u_0)+f(a+b-u_0)-f(su_0+(1-s)(a+b-u_0))-f((1-s)u_0+s(a+b-u_0))<0.$$

By the continuity of the map given by (13) we infer that there exists an $\varepsilon>0$ such that $(u_0-\varepsilon,u_0+\varepsilon) \subset (a,b)$ and for all $u \in (u_0-\varepsilon,u_0+\varepsilon)$ we have

$$f(u)+f(a+b-u)-f(su+(1-s)(a+b-u))-f((1-s)u+s(a+b-u))<0,$$

therefore,

$$\int_{u_0-\varepsilon}^{u_0+\varepsilon} [f(u)+f(a+b-u)-f(su+(1-s)(a+b-u))-f((1-s)u+s(a+b-u))]du<0,$$

and consequently, putting $c:=u_0-\varepsilon$, $d:=u_0+\varepsilon$ and dividing by 2ε we obtain

$$\Phi(sc+(1-s)d,(1-s)c+sd)>\Phi(c,d).$$

This contradiction shows that f is convex in the sense of Wright. The proof of our theorem is completed. $\square$

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Andrzej Olbryś
Institute of Mathematics
Silesian University
Bankowa 14
40-007 Katowice, Poland
e-mail: andrzej.olbrys@wp.pl