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A FUNCTIONAL GENERALIZATION OF DIAMOND–α INTEGRAL MINKOWSKI'S TYPE INEQUALITY ON TIME SCALES

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Abstract. In this paper, we establish a functional generalization of Minkowski's type inequality on time scales based on diamond-α integral, which is introduced as a linear combination of the delta and nabla integrals. Some related results are also obtained.

1. Introduction

The well-known inequality due to Minkowski can be stated as follows (see [1]).

THEOREM 1.1. *Let* $f(x) \geqslant 0$, $g(x) \geqslant 0$ *and* $p > 1$. *Then*

$$\left(\int_a^b (f(x)+g(x))^p dx\right)^{1/p} \leqslant \left(\int_a^b f^p(x)dx\right)^{1/p} + \left(\int_a^b g^p(x)dx\right)^{1/p}, \tag{1.1}$$

with equality if and only if the functions $|f|$ *and* $|g|$ *are proportional.*

The above inequality has many significant applications in different branches of modern mathematics such as classical real and complex analyses, Hilbert space theory, and so forth. A proof of Minkowski's inequality as well as some related results, several extensions, and interesting geometrical interpretations can be found in [2]. Applications of Minkowski's inequality have attracted many authors, for example Agahi et al. [3] applied Minkowski's inequality for Sugeno integrals and Lu et al. [4] applied Minkowski's inequality for fast full search in motion estimation. So it is of considerable interest to develop its counterpart on time scales.

The calculus of time scales was introduced by Stefan Hilger in his PhD thesis in order to unify continuous and discrete analysis [5]. More details related to time scales and dynamic equations on time scales can be found in the literature in [6–12]. Since then, integral inequalities on time scales have been studied by many authors, and lots of integral inequalities on time scales have been obtained (see [13–18] and the references therein). In [13, 14], the authors gave the delta integral Minkowski's inequality on time scales as follows.

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THEOREM 1.2. *Let $f,g,h \in C_{rd}([a,b],\mathbb{R})$ and $p>1$. Then*

$$\begin{aligned} &\left(\int_a^b |h(x)||f(x)+g(x)|^p \Delta x\right)^{\frac{1}{p}} \\ &\leqslant \left(\int_a^b |h(x)||f(x)|^p \Delta x\right)^{\frac{1}{p}} + \left(\int_a^b |h(x)||g(x)|^p \Delta x\right)^{\frac{1}{p}}, \end{aligned} \tag{1.2}$$

with equality if and only if the functions $|f|$ and $|g|$ are proportional.

Özkan et al. [15] established the nabla and diamond-α integral Minkowski's inequality on time scales which can be stated as follows.

THEOREM 1.3. *Let $f,g,h \in C_{ld}([a,b],\mathbb{R})$ and $p>1$. Then*

$$\begin{aligned} &\left(\int_a^b |h(x)||f(x)+g(x)|^p \nabla x\right)^{\frac{1}{p}} \\ &\leqslant \left(\int_a^b |h(x)||f(x)|^p \nabla x\right)^{\frac{1}{p}} + \left(\int_a^b |h(x)||g(x)|^p \nabla x\right)^{\frac{1}{p}}, \end{aligned} \tag{1.3}$$

with equality if and only if the functions $|f|$ and $|g|$ are proportional.

THEOREM 1.4. *Let $f,g,h:[a,b]\to\mathbb{R}$ be $\diamondsuit_\alpha$-integrable functions,and $p>1$. Then*

$$\begin{aligned} &\left(\int_a^b |h(x)||f(x)+g(x)|^p \diamondsuit_\alpha x\right)^{\frac{1}{p}} \\ &\leqslant \left(\int_a^b |h(x)||f(x)|^p \diamondsuit_\alpha x\right)^{\frac{1}{p}} + \left(\int_a^b |h(x)||g(x)|^p \diamondsuit_\alpha x\right)^{\frac{1}{p}}, \end{aligned} \tag{1.4}$$

with equality if and only if the functions $|f|$ and $|g|$ are proportional.

Recently, Chen [16] further generalized inequality (1.4) as follows.

THEOREM 1.5. *Let $f,g,h:[a,b]\to\mathbb{R}$ be $\diamondsuit_\alpha$-integrable functions, $p>0$, $s,t\in\mathbb{R}\setminus\{0\}$, and $s\neq t$. Let $p,s,t\in\mathbb{R}$ be different, such that $s,t>1$ and $(s-t)/(p-t)>1$. Then*

$$\begin{aligned} &\int_a^b |h(x)||f(x)+g(x)|^p \diamondsuit_\alpha x \\ &\leqslant \left[\left(\int_a^b |h(x)||f(x)|^s \diamondsuit_\alpha x\right)^{\frac{1}{s}} + \left(\int_a^b |h(x)||g(x)|^s \diamondsuit_\alpha x\right)^{\frac{1}{s}}\right]^{s(p-t)/(s-t)} \\ &\quad\times \left[\left(\int_a^b |h(x)||f(x)|^t \diamondsuit_\alpha x\right)^{\frac{1}{t}} + \left(\int_a^b |h(x)||g(x)|^t \diamondsuit_\alpha x\right)^{\frac{1}{t}}\right]^{t(p-t)/(s-t)}, \end{aligned} \tag{1.5}$$

with equality if and only if the functions $|f|$ and $|g|$ are proportional.

REMARK 1.1. For Theorem 1.5, for $p>1$, letting $s=p+\varepsilon$, $t=p-\varepsilon$, when p,s,t are different, $s,t>1$, and $(s-t)/(p-t)/2>1$, and letting $\varepsilon\to 0$, we obtain (1.4).

Inspired by Yang [18], in the present paper, we intend to give a functional generalization of diamond-α integral Minkowski's type inequality on time scales. Its reverse form is also presented. The paper is organized as follows. In Section 2, we briefly give basic definitions and some preliminary results which are necessary in the sequel. Namely, we briefly introduce the nabla and the delta calculus [8, 9]. We also introduce the notions of diamond-α derivative and integral [10-12]; In Section 3, we present our main results.

2. Preliminaries

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of real numbers. Let $\mathbb{T}$ be a time scale. $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology. Some important examples of time scales are, $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{N}$.

DEFINITION 2.1. The functions $\sigma,\rho:\mathbb{T}\to\mathbb{T}$ defined by

$$\sigma(t):=\inf\{s\in\mathbb{T}:s>t\} \quad \text{and} \quad \sigma(t):=\sup\{s\in\mathbb{T}:s<t\}$$

are called the jump operators.

In this definition, the convention is $\inf\emptyset=\sup\mathbb{T}$ and $\sup\emptyset=\inf\mathbb{T}$. If $\sup\mathbb{T}$ is finite and left-scattered, then we define $\mathbb{T}^{\kappa}:=\mathbb{T}\setminus\{\sup T\}$, otherwise $\mathbb{T}^{\kappa}:=T$; if $\inf\mathbb{T}$ is finite and right-scattered, then $\mathbb{T}_{\kappa}:=\mathbb{T}\setminus\{\inf\mathbb{T}\}$, otherwise $\mathbb{T}^{\kappa}:=\mathbb{T}$. We set $\mathbb{T}_{\kappa}^{\kappa}:=\mathbb{T}_{\kappa}\cap\mathbb{T}^{\kappa}$.

DEFINITION 2.2. For some $t\in\mathbb{T}^{\kappa}$, and a function $f:\mathbb{T}\to\mathbb{R}$, the delta derivative of f is denoted by $f^{\Delta}(t)$ and satisfies

$$|f(\sigma(t))-f(s)-f^{\Delta}(t)(\sigma(t)-s)|\leqslant\varepsilon|\sigma(t)-s|$$

for all $s\in U$, and U is a neighborhood of t. The function $f(t)$ is called delta differential on $\mathbb{T}^{\kappa}$.

DEFINITION 2.3. For some $t\in\mathbb{T}_{\kappa}$, and a function $f:\mathbb{T}\to\mathbb{R}$, the nabla derivative of f is denoted by $f^{\nabla}(t)$ and satisfies

$$|f(\rho(t))-f(s)-f^{\nabla}(t)(\rho(t)-s)|\leqslant\varepsilon|\rho(t)-s|$$

for all $s\in V$, and V is a neighborhood of t. The function $f(t)$ is called nabla differential on $\mathbb{T}_{\kappa}$.

REMARK 2.1. Assume that $\mathbb{T}=\mathbb{R}$, then $f^{\Delta}(t)=f^{\nabla}(t)=f'$, where f' denotes the usual derivative on $\mathbb{R}$. Assume that $\mathbb{T}=\mathbb{Z}$, then $f^{\Delta}(t)=f(t+1)-f(t)$ and $f^{\nabla}(t)=f(t)-f(t-1)$, i.e., f^{Δ} and f^{∇} are, respectively, the usual forward and backward difference operators.

Let $a,b\in\mathbb{T}$, $a<b$. In what follows we denote $[a,b]_{\mathbb{T}}=\{t\in\mathbb{T}:a\leqslant t\leqslant b\}$.

DEFINITION 2.4. If $F^{\Delta}(t)=f(t)$, $t\in\mathbb{T}^{\kappa}$, then F is called an antiderivative of f, and the delta integral of f from a to b (or on $[a,b]_{\mathbb{T}}$) is defined by

$$\int_a^b f(t)\Delta t=F(b)-F(a).$$

DEFINITION 2.5. If $G^{\Delta}(t) = g(t)$, $t \in \mathbb{T}_{\kappa}$, then G is called an antiderivative of g, and the delta integral of g from a to b (or on $[a,b]_{\mathbb{T}}$) is defined by

$$\int_a^b g(t)\nabla t = G(b) - G(a).$$

For the properties of the delta and nabla integrals we refer the readers to [8, 9].

REMARK 2.2. Assume that $\mathbb{T} = \mathbb{R}$, then $\int_a^b f(t)\Delta t = \int_a^b f(t)\nabla t = \int_a^b f(t)dt$, where the last integral is the usual Riemman integral. Assume that $\mathbb{T} = h\mathbb{Z}$, for some $h > 0$, and $a,b \in \mathbb{T}$, $a < b$, then

$$\int_a^b f(t)\Delta t = \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} hf(kh) \quad \text{and} \quad \int_a^b f(t)\nabla t = \sum_{k=\frac{a}{h}+1}^{\frac{b}{h}} hf(kh).$$

To provide a shorthand notation, for a function $f : \mathbb{T} \to \mathbb{R}$ we let $f^{\sigma}(t) = f(\sigma(t))$ and $f^{\rho}(t) = f(\rho(t))$.

DEFINITION 2.6. Let $t,s \in \mathbb{T}$ and define $\mu_{t,s} = \sigma(t) - s$ and $\eta_{t,s} = \rho(t) - s$. For some $t \in \mathbb{T}_{\kappa}^{\kappa}$, and a function $f : \mathbb{T} \to \mathbb{R}$, the diamond-$\alpha$ derivative of f is denoted by $f^{\diamondsuit_\alpha}(t)$ and satisfies

$$|\alpha[f^{\sigma}(t) - f(s)]\eta_{t,s} + (1-\alpha)[f^{\rho}(t) - f(s)]\mu_{t,s} - f^{\diamondsuit_\alpha}(t)\mu_{t,s}\eta_{t,s}| \leqslant \varepsilon|\mu_{t,s}\eta_{t,s}|,$$

where $s \in U$, and U is a neighborhood of t. The function $f(t)$ is called diamond-α differential on $t \in \mathbb{T}_{\kappa}^{\kappa}$.

THEOREM 2.1. *Let $0 \leqslant \alpha \leqslant 1$ and let f be both nabla and delta differentiable at $t \in \mathbb{T}_{\kappa}^{\kappa}$. Then f is diamond-α differentiable at t and*

$$f^{\diamondsuit_\alpha}(t) = \alpha f^{\Delta}(t) + (1-\alpha)f^{\nabla}(t). \tag{2.1}$$

REMARK 2.3. If $\alpha = 1$, then the diamond-α derivative reduces to the delta derivative; if $\alpha = 0$, then the diamond-α derivative coincides with the nabla derivative.

REMARK 2.4. The equality (2.1) is given as the definition of the diamond-α derivative in [10].

DEFINITION 2.7. Assume that $a,b \in \mathbb{T}$, $a < b$, $h : \mathbb{T} \to \mathbb{R}$ and $\alpha \in [0,1]$. The diamond-α integral (or $\diamondsuit_\alpha$-integral) of h from a to b (or on $[a,b]_{\mathbb{T}}$) is defined by

$$\int_a^b h(t)\diamondsuit_\alpha t = \alpha\int_a^b h(t)\Delta t + (1-\alpha)\int_a^b h(t)\nabla t,$$

provided h is delta and nabla integrable on $[a,b]_{\mathbb{T}}$.

For properties, results, and integral inequalities concerning the diamond-α integral, please refer to [10–12, 15] and references therein.

3. Main results

In this section, Our main results are given in the following theorems.

THEOREM 3.1. (Minkowski's type inequality) *Let $\mathbb{T}$ be a time scale $a,b \in \mathbb{T}$ with $a < b$ and $p > 0, s,t \in \mathbb{R} \setminus \{0\}$, and $s \neq t$. Let $p,s,t \in \mathbb{R}$ be different, such that $s,t > 1$ and $(s-t)/(p-t) > 1$. Let $H_l(x_1,x_2,\dots,x_l) > 0$, $F_m(x_1,x_2,\dots,x_m)$ and $G_k(x_1,x_2,\dots,x_k)$ be three arbitrary functions of l, m and k variables, respectively. Assume that $\{f_i(x)\}_{i=1}^m$, $\{g_i(x)\}_{i=1}^k$ and $\{h_i(x)\}_{i=1}^l$ are continuous real-valued functions on $[a,b]_{\mathbb{T}}$, then*

$$\begin{aligned}
&\int_a^b H_l(h_1,h_2,\dots,h_l)|F_m(f_1,f_2,\dots,f_m)+G_k(g_1,g_2,\dots,g_k)|^p \diamondsuit_\alpha x \\
&\leqslant \left[\left(\int_a^b H_l(h_1,\dots,h_l)|F_m(f_1,\dots,f_m)|^s \diamondsuit_\alpha x\right)^{\frac{1}{s}} + \left(\int_a^b H_l(h_1,\dots,h_l)|G_k(g_1,\dots,g_k)|^s \diamondsuit_\alpha x\right)^{\frac{1}{s}}\right]^{\frac{s(p-t)}{s-t}} \\
&\quad\times \left[\left(\int_a^b H_l(h_1,\dots,h_l)|F_m(f_1,\dots,f_m)|^t \diamondsuit_\alpha x\right)^{\frac{1}{t}} + \left(\int_a^b H_l(h_1,\dots,h_l)|G_k(g_1,\dots,g_k)|^t \diamondsuit_\alpha x\right)^{\frac{1}{t}}\right]^{\frac{t(s-p)}{s-t}},
\end{aligned} \tag{3.1}$$

with equality if and only if $F_m(f_1,f_2,\dots,f_m)$ and $G_k(g_1,g_2,\dots,g_k)$ are constant, or $1/p = (1/s+1/t)/2$ and $F_m(f_1,f_2,\dots,f_m)$ and $G_k(g_1,g_2,\dots,g_k)$ are proportional.

Proof. Clearly,

$$\begin{aligned}
&\int_a^b H_l(h_1,\dots,h_l)|F_m(f_1,\dots,f_m)+G_k(g_1,\dots,g_k)|^p \diamondsuit_\alpha x \\
&= \int_a^b H_l(|F_m+G_k|^s)^{(p-t)/(s-t)}(|F_m+G_k|^t)^{(s-p)/(s-t)} \diamondsuit_\alpha x.
\end{aligned}$$

Since $(s-t)/(p-t) > 1$, by using Hölder's inequality (see [15]) with indices $(s-t)/(p-t)$ and $(s-t)/(s-p)$, we have

$$\begin{aligned}
&\int_a^b H_l(h_1,\dots,h_l)|F_m(f_1,\dots,f_m)+G_k(g_1,\dots,g_k)|^p \diamondsuit_\alpha x \\
&\leqslant \left(\int_a^b H_l|F_m+G_k|^s \diamondsuit_\alpha x\right)^{(p-t)/(s-t)} \left(\int_a^b H_l|F_m+G_k|^t \diamondsuit_\alpha x\right)^{(s-p)/(s-t)}.
\end{aligned} \tag{3.2}$$

On the other hand, by applying Minkowski's inequality in [15] for $s > 1$ and $t > 1$, respectively, we have

$$\left(\int_a^b H_l|F_m+G_k|^s \diamondsuit_\alpha x\right)^{\frac{1}{s}} \leqslant \left(\int_a^b H_l|F_m|^s \diamondsuit_\alpha x\right)^{\frac{1}{s}} + \left(\int_a^b H_l|G_k|^s \diamondsuit_\alpha x\right)^{\frac{1}{s}} \tag{3.3}$$

and

$$\left(\int_a^b H_l|F_m+G_k|^t \diamondsuit_\alpha x\right)^{\frac{1}{t}} \leqslant \left(\int_a^b H_l|F_m|^t \diamondsuit_\alpha x\right)^{\frac{1}{t}} + \left(\int_a^b H_l|G_k|^t \diamondsuit_\alpha x\right)^{\frac{1}{t}}. \tag{3.4}$$

From (3.2), (3.3) and (3.4), we get the desired result. □

COROLLARY 3.1. ($\mathbb{T}=\mathbb{R}$). *Let* $p>0$, $s,t\in\mathbb{R}\setminus\{0\}$, *and* $s\neq t$. *Let* $p,s,t\in\mathbb{R}$ *be different, such that* $s,t>1$ *and* $(s-t)/(p-t)>1$. *Let* $H_l(x_1,x_2,\ldots,x_l)>0$, $F_m(x_1,x_2,\ldots,x_m)$ *and* $G_k(x_1,x_2,\ldots,x_k)$ *be three arbitrary functions of* l, m *and* k *variables, respectively. Assume that* $\{f_i(x)\}_{i=1}^m$, $\{g_i(x)\}_{i=1}^k$ *and* $\{h_i(x)\}_{i=1}^l$ *are continuous real-valued functions on* $[a,b]$, *then*

$$\begin{aligned}
&\int_a^b H_l(h_1,h_2,\ldots,h_l)|F_m(f_1,f_2,\ldots,f_m)+G_k(g_1,g_2,\ldots,g_k)|^p dx\\
&\leqslant\left[\left(\int_a^b H_l(h_1,\ldots,h_l)|F_m(f_1,\ldots,f_m)|^s dx\right)^{\frac{1}{s}}+\left(\int_a^b H_l(h_1,\ldots,h_l)|G_k(g_1,\ldots,g_k)|^s dx\right)^{\frac{1}{s}}\right]^{\frac{s(p-t)}{s-t}}\\
&\quad\times\left[\left(\int_a^b H_l(h_1,\ldots,h_l)|F_m(f_1,\ldots,f_m)|^t dx\right)^{\frac{1}{t}}+\left(\int_a^b H_l(h_1,\ldots,h_l)|G_k(g_1,\ldots,g_k)|^t dx\right)^{\frac{1}{t}}\right]^{\frac{t(s-p)}{s-t}},
\end{aligned}\tag{3.5}$$

with equality if and only if $F_m(f_1,f_2,\ldots,f_m)$ *and* $G_k(g_1,g_2,\ldots,g_k)$ *are constant, or* $1/p=(1/s+1/t)/2$ *and* $F_m(f_1,f_2,\ldots,f_m)$ *and* $G_k(g_1,g_2,\ldots,g_k)$ *are proportional.*

COROLLARY 3.2. ($\mathbb{T}=\mathbb{Z}$). *Let* $p>0,s,t\in\mathbb{R}\setminus\{0\}$, *and* $s\neq t$. *Let* $p,s,t\in\mathbb{R}$ *be different, such that* $s,t>1$ *and* $(s-t)/(p-t)>1$. *Let* $H_l(x_1,x_2,\ldots,x_l)>0$, $F_m(x_1,x_2,\ldots,x_m)$ *and* $G_k(x_1,x_2,\ldots,x_k)$ *be three arbitrary functions of* l, m *and* k *variables, respectively. Assume that* $\{a_{i1},a_{i2},\ldots,a_{im}\}_{i=1}^n$, $\{b_{i1},b_{i2},\ldots,b_{ik}\}_{i=1}^n$ *and* $\{c_{i1},c_{i2},\ldots,c_{il}\}_{i=1}^n$ *are real numbers for any* $m,k,l\in\mathbb{N}$, *then*

$$\begin{aligned}
&\sum_{i=1}^n H_l(c_{i1},c_{i2},\ldots,c_{il})|F_m(a_{i1},a_{i2},\ldots,a_{im})+G_k(b_{i1},b_{i2},\ldots,b_{ik})|^p\\
&\leqslant\left[\left(\sum_{i=1}^n H_l(c_{i1},c_{i2},\ldots,c_{il})|F_m(a_{i1},\ldots,a_{im})|^s\right)^{\frac{1}{s}}+\left(\sum_{i=1}^n H_l(c_{i1},\ldots,c_{il})|G_k(b_{i1},\ldots,b_{ik})|^s\right)^{\frac{1}{s}}\right]^{\frac{s(p-t)}{s-t}}\\
&\quad\times\left[\left(\sum_{i=1}^n H_l(c_{i1},c_{i2},\ldots,c_{il})|F_m(a_{i1},\ldots,a_{im})|^t\right)^{\frac{1}{t}}+\left(\sum_{i=1}^n H_l(c_{i1},\ldots,c_{il})|G_k(b_{i1},\ldots,b_{ik})|^t\right)^{\frac{1}{t}}\right]^{\frac{t(s-p)}{s-t}},
\end{aligned}\tag{3.6}$$

with equality if and only if the functions $F_m(a_{i1},a_{i2},\ldots,a_{im})$ *and* $G_k(b_{i1},b_{i2},\ldots,b_{ik})$ *are are constant, or* $1/p{=}(1/s{+}1/t)/2$ *and* $F_m(a_{i1},a_{i2},\ldots,a_{im})$ *and* $G_k(b_{i1},b_{i2},\ldots,b_{ik})$ *are proportional.*

THEOREM 3.2. (Reverse Minkowski's type inequality) *Let* $\mathbb{T}$ *be a time scale* $a,b\in\mathbb{T}$ *with* $a<b$ *and* $p>0$, $s,t\in\mathbb{R}\setminus\{0\}$, *and* $s\neq t$. *Let* $p,s,t\in\mathbb{R}$ *be different, such that* $s,t<1$, $s,t\neq 0$, *and* $(s-t)/(p-t)<1$. *Let* $H_l(x_1,x_2,\ldots,x_l)>0$, $F_m(x_1,x_2,\ldots,x_m)$ *and* $G_k(x_1,x_2,\ldots,x_k)$ *be three arbitrary functions of* l; m *and* k *variables, respectively. Assume that* $\{f_i(x)\}_{i=1}^m$, $\{g_i(x)\}_{i=1}^k$ *and* $\{h_i(x)\}_{i=1}^l$ *are continuous real-valued functions on* $[a,b]_{\mathbb{T}}$, *then*

$$\begin{aligned}&\int_a^b H_l(h_1,h_2,\dots,h_l)|F_m(f_1,f_2,\dots,f_m)+G_k(g_1,g_2,\dots,g_k)|^p\diamondsuit_\alpha x\\&\geqslant\left[\left(\int_a^b H_l(h_1,\dots,h_l)|F_m(f_1,\dots,f_m)|^s\diamondsuit_\alpha x\right)^{\frac{1}{s}}+\left(\int_a^b H_l(h_1,\dots,h_l)|G_k(g_1,\dots,g_k)|^s\diamondsuit_\alpha x\right)^{\frac{1}{s}}\right]^{\frac{s(p-t)}{s-t}}\\&\quad\times\left[\left(\int_a^b H_l(h_1,\dots,h_l)|F_m(f_1,\dots,f_m)|^t\diamondsuit_\alpha x\right)^{\frac{1}{t}}+\left(\int_a^b H_l(h_1,\dots,h_l)|G_k(g_1,\dots,g_k)|^t\diamondsuit_\alpha x\right)^{\frac{1}{t}}\right]^{\frac{t(s-p)}{s-t}},\end{aligned} \tag{3.7}$$

with equality if and only if $F_m(f_1,f_2,\dots,f_m)$ *and* $G_k(g_1,g_2,\dots,g_k)$ *are constant, or* $1/p=(1/s+1/t)/2$ *and* $F_m(f_1,f_2,\dots,f_m)$ *and* $G_k(g_1,g_2,\dots,g_k)$ *are proportional.*

Proof. We have $(s-t)/(p-t)<1$. Similar to the proof of Theorem 3.1, in view of

$$\begin{aligned}&\int_a^b H_l(h_1,\dots,h_l)|F_m(f_1,\dots,f_m)+G_k(g_1,\dots,g_k)|^p\diamondsuit_\alpha x\\&=\int_a^b H_l(|F_m+G_k|^s)^{(p-t)/(s-t)}(|F_m+G_k|^t)^{(s-p)/(s-t)}\diamondsuit_\alpha x.\end{aligned}$$

By using reverse Hölder's inequality (see [15]) with indices $(s-t)/(p-t)$ and $(s-t)/(s-p)$, we have

$$\begin{aligned}&\int_a^b H_l(h_1,\dots,h_l)|F_m(f_1,\dots,f_m)+G_k(g_1,\dots,g_k)|^p\diamondsuit_\alpha x\\&\geqslant\left(\int_a^b H_l|F_m+G_k|^s\diamondsuit_\alpha x\right)^{(p-t)/(s-t)}\left(\int_a^b H_l|F_m+G_k|^t\diamondsuit_\alpha x\right)^{(s-p)/(s-t)}.\end{aligned} \tag{3.8}$$

On the other hand, by applying reverse Minkowski's inequality in [16] for the cases of $0<s<1$ and $0<t<1$, we have

$$\left(\int_a^b H_l|F_m+G_k|^s\diamondsuit_\alpha x\right)^{\frac{1}{s}}\geqslant\left(\int_a^b H_l|F_m|^s\diamondsuit_\alpha x\right)^{\frac{1}{s}}+\left(\int_a^b H_l|G_k|^s\diamondsuit_\alpha x\right)^{\frac{1}{s}} \tag{3.9}$$

and

$$\left(\int_a^b H_l|F_m+G_k|^t\diamondsuit_\alpha x\right)^{\frac{1}{t}}\geqslant\left(\int_a^b H_l|F_m|^t\diamondsuit_\alpha x\right)^{\frac{1}{t}}+\left(\int_a^b H_l|G_k|^t\diamondsuit_\alpha x\right)^{\frac{1}{t}}. \tag{3.10}$$

From (3.8), (3.9) and (3.10), we get the desired result. □

COROLLARY 3.3. ($\mathbb{T}=\mathbb{R}$). *Let* $p>0$, $s,t\in\mathbb{R}\setminus\{0\}$, *and* $s\neq t$. *Let* $p,s,t\in\mathbb{R}$ *be different, such that* $s,t<1$, $s,t\neq 0$, *and* $(s-t)/(p-t)<1$. *Let* $H_l(x_1,x_2,\dots,x_l)>0$, $F_m(x_1,x_2,\dots,x_m)$ *and* $G_k(x_1,x_2,\dots,x_k)$ *be three arbitrary functions of* l; m *and* k *variables, respectively. Assume that* $\{f_i(x)\}_{i=1}^m$, $\{g_i(x)\}_{i=1}^k$ *and* $\{h_i(x)\}_{i=1}^l$ *are contin-*

uous real-valued functions on $[a,b]$*, then*

$$\int_a^b H_l(h_1,h_2,\dots,h_l)|F_m(f_1,f_2,\dots,f_m)+G_k(g_1,g_2,\dots,g_k)|^p dx$$
$$\geqslant\left[\left(\int_a^b H_l(h_1,\dots,h_l)|F_m(f_1,\dots,f_m)|^s\diamondsuit_\alpha x\right)^{\frac{1}{s}}+\left(\int_a^b H_l(h_1,\dots,h_l)|G_k(g_1,\dots,g_k)|^s dx\right)^{\frac{1}{s}}\right]^{\frac{s(p-t)}{s-t}}$$
$$\times\left[\left(\int_a^b H_l(h_1,\dots,h_l)|F_m(f_1,\dots,f_m)|^t dx\right)^{\frac{1}{t}}+\left(\int_a^b H_l(h_1,\dots,h_l)|G_k(g_1,\dots,g_k)|^t dx\right)^{\frac{1}{t}}\right]^{\frac{t(s-p)}{s-t}}, \tag{3.11}$$

with equality if and only if $F_m(f_1,f_2,\dots,f_m)$ *and* $G_k(g_1,g_2,\dots,g_k)$ *are constant, or* $1/p=(1/s+1/t)/2$ *and* $F_m(f_1,f_2,\dots,f_m)$ *and* $G_k(g_1,g_2,\dots,g_k)$ *are proportional.*

COROLLARY 3.4. ($\mathbb{T}=\mathbb{Z}$). *Let* $p>0$, $s,t\in\mathbb{R}\setminus\{0\}$*, and* $s\neq t$*. Let* $p,s,t\in\mathbb{R}$ *be different, such that* $s,t<1$, $s,t\neq 0$*, and* $(s-t)/(p-t)<1$*. Let* $H_l(x_1,x_2,\dots,x_l)>0$, $F_m(x_1,x_2,\dots,x_m)$ *and* $G_k(x_1,x_2,\dots,x_k)$ *be three arbitrary functions of* l*;* m *and* k *variables, respectively. Assume that* $\{a_{i1},a_{i2},\dots,a_{im}\}_{i=1}^n$, $\{b_{i1},b_{i2},\dots,b_{ik}\}_{i=1}^n$ *and* $\{c_{i1},c_{i2},\dots,c_{il}\}_{i=1}^n$ *are real numbers for any* $m,k,l\in\mathbb{N}$*, then*

$$\sum_{i=1}^n H_l(c_{i1},c_{i2},\dots,c_{il})|F_m(a_{i1},a_{i2},\dots,a_{im})+G_k(b_{i1},b_{i2},\dots,b_{ik})|^p$$
$$\geqslant\left[\left(\sum_{i=1}^n H_l(c_{i1},c_{i2},\dots,c_{il})|F_m(a_{i1},\dots,a_{im})|^s\right)^{\frac{1}{s}}+\left(\sum_{i=1}^n H_l(c_{i1},\dots,c_{il})|G_k(b_{i1},\dots,b_{ik})|^s\right)^{\frac{1}{s}}\right]^{\frac{s(p-t)}{s-t}}$$
$$\times\left[\left(\sum_{i=1}^n H_l(c_{i1},c_{i2},\dots,c_{il})|F_m(a_{i1},\dots,a_{im})|^t\right)^{\frac{1}{t}}+\left(\sum_{i=1}^n H_l(c_{i1},\dots,c_{il})|G_k(b_{i1},\dots,b_{ik})|^t\right)^{\frac{1}{t}}\right]^{\frac{t(s-p)}{s-t}}, \tag{3.12}$$

with equality if and only if the functions $F_m(a_{i1},a_{i2},\dots,a_{im})$ *and* $G_k(b_{i1},b_{i2},\dots,b_{ik})$ *are are constant, or* $1/p=(1/s+1/t)/2$ *and* $F_m(a_{i1},a_{i2},\dots,a_{im})$ *and* $G_k(b_{i1},b_{i2},\dots,b_{ik})$ *are proportional.*

Obviously, Corollaries 3.2 and 3.4 are well known for the integers.

REMARK 3.1. For Theorem 3.1, for $p>1$, letting $s=p+\varepsilon$, $t=p-\varepsilon$, when p,s,t are different, $s,t>1$, and $(s-t)/(p-t)/2>1$, and letting $\varepsilon\to 0$, Theorem 3.1 reduces to Theorem 2.2 obtained by Yang [18].

REMARK 3.2. Assume that $\{f_i(x,y)\}_{i=1}^m$, $\{g_i(x,y)\}_{i=1}^k$ and $\{h_i(x,y)\}_{i=1}^l$ are continuous real-valued functions on $[a,b]_{\mathbb{T}}\times[a,b]_{\mathbb{T}}$, and H_l, F_m and G_k are defined as in Theorem 3.1, then by Theorems 3.1 and 3.2, we obtain functional generalizations of two dimensional diamond-α integral Minkowski's type inequality and reverse Minkowski's inequality on time scales.

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