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# PROPERTIES OF GENERALIZED HÖLDER'S INEQUALITIES

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*Abstract.* In this paper, we present some new properties of generalized Hölder's inequalities proposed by Vasić and Pečarić, and then we obtain some new refinements of generalized Hölder's inequalities.

## 1. Introduction

We begin by recalling here the classical Hölder's inequality as Theorem A below.

THEOREM A. *If* $a_k \geqslant 0,\ b_k \geqslant 0\ (k = 1,2,\dots,n),\ p \geqslant q > 1,\ \frac{1}{p}+\frac{1}{q} = 1$, *then*

$$\sum_{k=1}^{n} a_k b_k \leqslant \left(\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} b_k^q\right)^{\frac{1}{q}}. \tag{1}$$

*The sign of inequality is reversed for* $p < 0$. *(For* $p < 0$, *we assume that* $a_k, b_k > 0$.*)*

It is well known that Hölder's inequality is one of the most important inequalities in analysis. Various generalizations, improvements, and applications of Hölder's inequality have appeared in the literature so far. For example, Matkowski in [3, 4] obtained some important converses of the Hölder inequality. Nikolova and Varošanec [5] derived some new improvements of the Hölder inequality by applying a convex function. For detailed expositions, the interested reader may consult [1], [2], [6] and the references therein. Among various generalizations of (1), Vasić and Pečarić in [7] established the following interesting theorem.

THEOREM B. *Let* $a_{ir} > 0\ (i = 1,2,\dots,n,\ r = 1,2,\dots,k)$.
*(a) If* $\lambda_r > 0$, *and if* $\sum_{r=1}^{k} \frac{1}{\lambda_r} \geqslant 1$, *then*

$$\sum_{i=1}^{n} \prod_{r=1}^{k} a_{ir} \leqslant \prod_{r=1}^{k} \left(\sum_{i=1}^{n} a_{ir}^{\lambda_r}\right)^{\frac{1}{\lambda_r}}. \tag{2}$$

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*(b) If* $\lambda_r < 0 (r = 1,2,\ldots,k)$, *then*

$$\sum_{i=1}^{n}\prod_{r=1}^{k} a_{ir} \geqslant \prod_{r=1}^{k}\left(\sum_{i=1}^{n} a_{ir}^{\lambda_r}\right)^{\frac{1}{\lambda_r}}. \tag{3}$$

*(c) If* $\lambda_1 > 0,\ \lambda_r < 0\ \ (r = 2,3,\ldots,k)$, *and if* $\sum_{r=1}^{k}\frac{1}{\lambda_r} \leqslant 1$, *then*

$$\sum_{i=1}^{n}\prod_{r=1}^{k} a_{ir} \geqslant \prod_{r=1}^{k}\left(\sum_{i=1}^{n} a_{ir}^{\lambda_r}\right)^{\frac{1}{\lambda_r}}. \tag{4}$$

*The above inequalities are called as generalized Hölder's inequalities.*

The main purpose of this paper is to give new properties and refinements of (2), (3), (4).

## 2. Properties of generalized Hölder inequalities

THEOREM 2.1. *Let* $a_{nr} > 0\ \ (n = 1,2,\ldots, r = 1,2,\ldots,k)$, *let* $\lambda_1 > \lambda_2 > \ldots > \lambda_k > 0$, $\sum_{r=1}^{k}\frac{1}{\lambda_r} \geqslant 1$, *and let*

$$F_l(n) = \prod_{r=1}^{k}\left(\sum_{i=1}^{n} a_{ir}^{\lambda_r}\right)^{\frac{2l}{\lambda_r}} - \left(\sum_{i=1}^{n}\prod_{r=1}^{k} a_{ir}\right)^{2l} \tag{5}$$

$l = 1,2,\ldots$. *Then*

$$0 \leqslant F_l(n) \leqslant F_l(n+1). \tag{6}$$

*Proof.* Write

$$T_n = \left(\sum_{i=1}^{n}\prod_{r=1}^{k} a_{ir}\right)^2,$$

$$S_n = \prod_{r=1}^{k}\left(\sum_{i=1}^{n} a_{ir}^{\lambda_r}\right)^{\frac{2}{\lambda_r}}.$$

*Case* I. When $l = 1$ and $k$ is even. Performing some simple computations, we have

$$\begin{aligned} S_n = &\left[\left(\sum_{i=1}^{n} a_{i2}^{\lambda_2}\right)\left(\sum_{j=1}^{n} a_{j2}^{\lambda_2}\right)\right]^{\frac{1}{\lambda_2}-\frac{1}{\lambda_1}} \\ &\times\left[\left(\sum_{i=1}^{n} a_{i1}^{\lambda_1}\right)\left(\sum_{j=1}^{n} a_{j2}^{\lambda_2}\right)\right]^{\frac{1}{\lambda_1}}\left[\left(\sum_{i=1}^{n} a_{i2}^{\lambda_2}\right)\left(\sum_{j=1}^{n} a_{j1}^{\lambda_1}\right)\right]^{\frac{1}{\lambda_1}} \\ &\times\left[\left(\sum_{i=1}^{n} a_{i4}^{\lambda_4}\right)\left(\sum_{j=1}^{n} a_{j4}^{\lambda_4}\right)\right]^{\frac{1}{\lambda_4}-\frac{1}{\lambda_3}} \end{aligned}$$

$$\times\left[\left(\sum_{i=1}^{n} a_{i3}^{\lambda_3}\right)\left(\sum_{j=1}^{n} a_{j4}^{\lambda_4}\right)\right]^{\frac{1}{\lambda_3}}\left[\left(\sum_{i=1}^{n} a_{i4}^{\lambda_4}\right)\left(\sum_{j=1}^{n} a_{j3}^{\lambda_3}\right)\right]^{\frac{1}{\lambda_3}}$$
$$\times \cdots\cdots\cdots$$
$$\times\left[\left(\sum_{i=1}^{n} a_{ik}^{\lambda_k}\right)\left(\sum_{j=1}^{n} a_{jk}^{\lambda_k}\right)\right]^{\frac{1}{\lambda_k}-\frac{1}{\lambda_{k-1}}}$$
$$\times\left[\left(\sum_{i=1}^{n} a_{i(k-1)}^{\lambda_{(k-1)}}\right)\left(\sum_{j=1}^{n} a_{jk}^{\lambda_k}\right)\right]^{\frac{1}{\lambda_{k-1}}}\left[\left(\sum_{i=1}^{n} a_{ik}^{\lambda_k}\right)\left(\sum_{j=1}^{n} a_{j(k-1)}^{\lambda_{k-1}}\right)\right]^{\frac{1}{\lambda_{k-1}}}. \tag{7}$$

$$\begin{aligned}T_{n+1}-T_n &= \left(\sum_{i=1}^{n+1}\prod_{r=1}^{k} a_{ir}\right)^2-\left(\sum_{i=1}^{n}\prod_{r=1}^{k} a_{ir}\right)^2\\ &= \left(\sum_{i=1}^{n+1}\prod_{r=1}^{k} a_{ir}-\sum_{i=1}^{n}\prod_{r=1}^{k} a_{ir}\right)\left(\sum_{i=1}^{n+1}\prod_{r=1}^{k} a_{ir}+\sum_{i=1}^{n}\prod_{r=1}^{k} a_{ir}\right)\\ &= \left(\prod_{r=1}^{k} a_{(n+1)r}\right)\left(\sum_{i=1}^{n+1} a_{i1}a_{i2}\ldots a_{ik}\right)+\left(\prod_{r=1}^{k} a_{(n+1)r}\right)\left(\sum_{i=1}^{n} a_{i1}a_{i2}\ldots a_{ik}\right).\end{aligned} \tag{8}$$

Then, from inequality (2) we have

$$T_{n+1}-T_n \leqslant \left(\prod_{r=1}^{k} a_{(n+1)r}\right)\left(\sum_{i=1}^{n+1} a_{i1}^{\lambda_1}\right)^{\frac{1}{\lambda_1}}\left(\sum_{i=1}^{n+1} a_{i2}^{\lambda_2}\right)^{\frac{1}{\lambda_2}}\ldots\left(\sum_{i=1}^{n+1} a_{ik}^{\lambda_k}\right)^{\frac{1}{\lambda_k}}$$
$$+\left(\prod_{r=1}^{k} a_{(n+1)r}\right)\left(\sum_{i=1}^{n} a_{i1}^{\lambda_1}\right)^{\frac{1}{\lambda_1}}\left(\sum_{i=1}^{n} a_{i2}^{\lambda_2}\right)^{\frac{1}{\lambda_2}}\ldots\left(\sum_{i=1}^{n} a_{ik}^{\lambda_k}\right)^{\frac{1}{\lambda_k}}$$
$$=\left[a_{(n+1)2}^{\lambda_2}\left(\sum_{i=1}^{n+1} a_{i2}^{\lambda_2}\right)\right]^{\frac{1}{\lambda_2}-\frac{1}{\lambda_1}}\left[a_{(n+1)1}^{\lambda_1}\left(\sum_{i=1}^{n+1} a_{i2}^{\lambda_2}\right)\right]^{\frac{1}{\lambda_1}}\left[a_{(n+1)2}^{\lambda_2}\left(\sum_{i=1}^{n+1} a_{i1}^{\lambda_1}\right)\right]^{\frac{1}{\lambda_1}}$$
$$\times\left[a_{(n+1)4}^{\lambda_4}\left(\sum_{i=1}^{n+1} a_{i4}^{\lambda_4}\right)\right]^{\frac{1}{\lambda_4}-\frac{1}{\lambda_3}}\left[a_{(n+1)3}^{\lambda_3}\left(\sum_{i=1}^{n+1} a_{i4}^{\lambda_4}\right)\right]^{\frac{1}{\lambda_3}}\left[a_{(n+1)4}^{\lambda_4}\left(\sum_{i=1}^{n+1} a_{i3}^{\lambda_3}\right)\right]^{\frac{1}{\lambda_3}}$$
$$\times \cdots\cdots\cdots$$
$$\times\left[a_{(n+1)k}^{\lambda_k}\left(\sum_{i=1}^{n+1} a_{ik}^{\lambda_k}\right)\right]^{\frac{1}{\lambda_k}-\frac{1}{\lambda_{k-1}}}$$
$$\times\left[a_{(n+1)(k-1)}^{\lambda_{k-1}}\left(\sum_{i=1}^{n+1} a_{ik}^{\lambda_k}\right)\right]^{\frac{1}{\lambda_{k-1}}}\left[a_{(n+1)k}^{\lambda_k}\left(\sum_{i=1}^{n+1} a_{i(k-1)}^{\lambda_{k-1}}\right)\right]^{\frac{1}{\lambda_{k-1}}}$$
$$+\left[a_{(n+1)2}^{\lambda_2}\left(\sum_{i=1}^{n} a_{i2}^{\lambda_2}\right)\right]^{\frac{1}{\lambda_2}-\frac{1}{\lambda_1}}\left[a_{(n+1)2}^{\lambda_2}\left(\sum_{i=1}^{n} a_{i1}^{\lambda_1}\right)\right]^{\frac{1}{\lambda_1}}\left[a_{(n+1)1}^{\lambda_1}\left(\sum_{i=1}^{n} a_{i2}^{\lambda_2}\right)\right]^{\frac{1}{\lambda_1}}$$

$$\times\left[a_{(n+1)4}^{\lambda_4}\left(\sum_{i=1}^{n}a_{i4}^{\lambda_4}\right)\right]^{\frac{1}{\lambda_4}-\frac{1}{\lambda_3}}\left[a_{(n+1)4}^{\lambda_4}\left(\sum_{i=1}^{n}a_{i3}^{\lambda_3}\right)\right]^{\frac{1}{\lambda_3}}\left[a_{(n+1)3}^{\lambda_3}\left(\sum_{i=1}^{n}a_{i4}^{\lambda_4}\right)\right]^{\frac{1}{\lambda_3}}$$
$$\times\cdots\cdots\cdots$$
$$\times\left[a_{(n+1)k}^{\lambda_k}\left(\sum_{i=1}^{n}a_{ik}^{\lambda_k}\right)\right]^{\frac{1}{\lambda_k}-\frac{1}{\lambda_{k-1}}}$$
$$\times\left[a_{(n+1)k}^{\lambda_k}\left(\sum_{i=1}^{n}a_{i(k-1)}^{\lambda_{k-1}}\right)\right]^{\frac{1}{\lambda_{k-1}}}\left[a_{(n+1)(k-1)}^{\lambda_{k-1}}\left(\sum_{i=1}^{n}a_{ik}^{\lambda_k}\right)\right]^{\frac{1}{\lambda_{k-1}}}. \tag{9}$$

Consequently, from (9), (7) and (2) we obtain

$$\begin{aligned}
&T_{n+1}-T_n+S_n\\
&\leqslant\left[a_{(n+1)2}^{\lambda_2}\left(\sum_{i=1}^{n+1}a_{i2}^{\lambda_2}\right)+\sum_{i=1}^{n}\left(a_{(n+1)2}^{\lambda_2}a_{i2}^{\lambda_2}+a_{i2}^{\lambda_2}\sum_{j=1}^{n}a_{j2}^{\lambda_2}\right)\right]^{\frac{1}{\lambda_2}-\frac{1}{\lambda_1}}\\
&\quad\times\left[a_{(n+1)1}^{\lambda_1}\left(\sum_{i=1}^{n+1}a_{i2}^{\lambda_2}\right)+\sum_{i=1}^{n}\left(a_{(n+1)2}^{\lambda_2}a_{i1}^{\lambda_1}+a_{i1}^{\lambda_1}\sum_{j=1}^{n}a_{j2}^{\lambda_2}\right)\right]^{\frac{1}{\lambda_1}}\\
&\quad\times\left[a_{(n+1)2}^{\lambda_2}\left(\sum_{i=1}^{n+1}a_{i1}^{\lambda_1}\right)+\sum_{i=1}^{n}\left(a_{(n+1)1}^{\lambda_1}a_{i2}^{\lambda_2}+a_{i2}^{\lambda_2}\sum_{j=1}^{n}a_{j1}^{\lambda_1}\right)\right]^{\frac{1}{\lambda_1}}\\
&\quad\times\left[a_{(n+1)4}^{\lambda_4}\left(\sum_{i=1}^{n+1}a_{i4}^{\lambda_4}\right)+\sum_{i=1}^{n}\left(a_{(n+1)4}^{\lambda_4}a_{i4}^{\lambda_4}+a_{i4}^{\lambda_4}\sum_{j=1}^{n}a_{j4}^{\lambda_4}\right)\right]^{\frac{1}{\lambda_4}-\frac{1}{\lambda_3}}\\
&\quad\times\left[a_{(n+1)3}^{\lambda_3}\left(\sum_{i=1}^{n+1}a_{i4}^{\lambda_4}\right)+\sum_{i=1}^{n}\left(a_{(n+1)4}^{\lambda_4}a_{i3}^{\lambda_3}+a_{i3}^{\lambda_3}\sum_{j=1}^{n}a_{j4}^{\lambda_4}\right)\right]^{\frac{1}{\lambda_3}}\\
&\quad\times\left[a_{(n+1)4}^{\lambda_4}\left(\sum_{i=1}^{n+1}a_{i3}^{\lambda_3}\right)+\sum_{i=1}^{n}\left(a_{(n+1)3}^{\lambda_3}a_{i4}^{\lambda_4}+a_{i4}^{\lambda_4}\sum_{j=1}^{n}a_{j3}^{\lambda_3}\right)\right]^{\frac{1}{\lambda_3}}\\
&\quad\times\cdots\cdots\cdots\\
&\quad\times\left[a_{(n+1)k}^{\lambda_k}\left(\sum_{i=1}^{n+1}a_{ik}^{\lambda_k}\right)+\sum_{i=1}^{n}\left(a_{(n+1)k}^{\lambda_k}a_{ik}^{\lambda_k}+a_{ik}^{\lambda_k}\sum_{j=1}^{n}a_{jk}^{\lambda_k}\right)\right]^{\frac{1}{\lambda_k}-\frac{1}{\lambda_{k-1}}}\\
&\quad\times\left[a_{(n+1)(k-1)}^{\lambda_{k-1}}\left(\sum_{i=1}^{n+1}a_{ik}^{\lambda_k}\right)+\sum_{i=1}^{n}\left(a_{(n+1)k}^{\lambda_k}a_{i(k-1)}^{\lambda_{k-1}}+a_{i(k-1)}^{\lambda_{k-1}}\sum_{j=1}^{n}a_{jk}^{\lambda_k}\right)\right]^{\frac{1}{\lambda_{k-1}}}\\
&\quad\times\left[a_{(n+1)k}^{\lambda_k}\left(\sum_{i=1}^{n+1}a_{i(k-1)}^{\lambda_{k-1}}\right)+\sum_{i=1}^{n}\left(a_{(n+1)(k-1)}^{\lambda_{k-1}}a_{ik}^{\lambda_k}+a_{ik}^{\lambda_k}\sum_{j=1}^{n}a_{j(k-1)}^{\lambda_{k-1}}\right)\right]^{\frac{1}{\lambda_{k-1}}}
\end{aligned}$$

$$= \left[ a_{(n+1)2}^{\lambda_2} \left( \sum_{i=1}^{n+1} a_{i2}^{\lambda_2} + \sum_{i=1}^{n} a_{i2}^{\lambda_2} \right) + \left( \sum_{i=1}^{n} a_{i2}^{\lambda_2} \right)^2 \right]^{\frac{1}{\lambda_2} - \frac{1}{\lambda_1}}$$

$$\times \left[ a_{(n+1)1}^{\lambda_1} \left( \sum_{i=1}^{n+1} a_{i2}^{\lambda_2} \right) + \sum_{i=1}^{n} a_{i1}^{\lambda_1} \left( \sum_{j=1}^{n+1} a_{j2}^{\lambda_2} \right) \right]^{\frac{1}{\lambda_1}}$$

$$\times \left[ a_{(n+1)2}^{\lambda_2} \left( \sum_{i=1}^{n+1} a_{i1}^{\lambda_1} \right) + \sum_{i=1}^{n} a_{i2}^{\lambda_2} \left( \sum_{j=1}^{n+1} a_{j1}^{\lambda_1} \right) \right]^{\frac{1}{\lambda_1}}$$

$$\times \left[ a_{(n+1)4}^{\lambda_4} \left( \sum_{i=1}^{n+1} a_{i4}^{\lambda_4} + \sum_{i=1}^{n} a_{i4}^{\lambda_4} \right) + \left( \sum_{i=1}^{n} a_{i4}^{\lambda_4} \right)^2 \right]^{\frac{1}{\lambda_4} - \frac{1}{\lambda_3}}$$

$$\times \left[ a_{(n+1)3}^{\lambda_3} \left( \sum_{i=1}^{n+1} a_{i4}^{\lambda_4} \right) + \sum_{i=1}^{n} a_{i3}^{\lambda_3} \left( \sum_{j=1}^{n+1} a_{j4}^{\lambda_4} \right) \right]^{\frac{1}{\lambda_3}}$$

$$\times \left[ a_{(n+1)4}^{\lambda_4} \left( \sum_{i=1}^{n+1} a_{i3}^{\lambda_3} \right) + \sum_{i=1}^{n} a_{i4}^{\lambda_4} \left( \sum_{j=1}^{n+1} a_{j3}^{\lambda_3} \right) \right]^{\frac{1}{\lambda_3}}$$

$$\cdots\cdots\cdots$$

$$\times \left[ a_{(n+1)k}^{\lambda_k} \left( \sum_{i=1}^{n+1} a_{ik}^{\lambda_k} + \sum_{i=1}^{n} a_{ik}^{\lambda_k} \right) + \left( \sum_{i=1}^{n} a_{ik}^{\lambda_k} \right)^2 \right]^{\frac{1}{\lambda_k} - \frac{1}{\lambda_{k-1}}}$$

$$\times \left[ a_{(n+1)(k-1)}^{\lambda_{k-1}} \left( \sum_{i=1}^{n+1} a_{ik}^{\lambda_k} \right) + \sum_{i=1}^{n} a_{i(k-1)}^{\lambda_{k-1}} \left( \sum_{j=1}^{n+1} a_{jk}^{\lambda_k} \right) \right]^{\frac{1}{\lambda_{k-1}}}$$

$$\times \left[ a_{(n+1)k}^{\lambda_k} \left( \sum_{i=1}^{n+1} a_{i(k-1)}^{\lambda_{k-1}} \right) + \sum_{i=1}^{n} a_{ik}^{\lambda_k} \left( \sum_{j=1}^{n+1} a_{j(k-1)}^{\lambda_{k-1}} \right) \right]^{\frac{1}{\lambda_{k-1}}}$$

$$= \left[ \left( a_{(n+1)2}^{\lambda_2} \right)^2 + 2a_{(n+1)2}^{\lambda_2} \left( \sum_{i=1}^{n} a_{i2}^{\lambda_2} \right) + \left( \sum_{i=1}^{n} a_{i2}^{\lambda_2} \right)^2 \right]^{\frac{1}{\lambda_2} - \frac{1}{\lambda_1}}$$

$$\times \left( \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} a_{i1}^{\lambda_1} a_{j2}^{\lambda_2} \right)^{\frac{1}{\lambda_1}} \left( \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} a_{i2}^{\lambda_2} a_{j1}^{\lambda_1} \right)^{\frac{1}{\lambda_1}}$$

$$\times \left[ \left( a_{(n+1)4}^{\lambda_4} \right)^2 + 2a_{(n+1)4}^{\lambda_4} \left( \sum_{i=1}^{n} a_{i4}^{\lambda_4} \right) + \left( \sum_{i=1}^{n} a_{i4}^{\lambda_4} \right)^2 \right]^{\frac{1}{\lambda_4} - \frac{1}{\lambda_3}}$$

$$\times \left( \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} a_{i3}^{\lambda_3} a_{j4}^{\lambda_4} \right)^{\frac{1}{\lambda_3}} \left( \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} a_{i4}^{\lambda_4} a_{j3}^{\lambda_3} \right)^{\frac{1}{\lambda_3}}$$

$$\times \cdots\cdots\cdots$$

$$\times\left[\left(a_{(n+1)k}^{\lambda_k}\right)^2+2a_{(n+1)k}^{\lambda_k}\left(\sum_{i=1}^{n}a_{ik}^{\lambda_k}\right)+\left(\sum_{i=1}^{n}a_{ik}^{\lambda_k}\right)^2\right]^{\frac{1}{\lambda_k}-\frac{1}{\lambda_{k-1}}}$$
$$\times\left(\sum_{i=1}^{n+1}\sum_{j=1}^{n+1}a_{i(k-1)}^{\lambda_{k-1}}a_{jk}^{\lambda_k}\right)^{\frac{1}{\lambda_{k-1}}}\left(\sum_{i=1}^{n+1}\sum_{j=1}^{n+1}a_{ik}^{\lambda_k}a_{j(k-1)}^{\lambda_{k-1}}\right)^{\frac{1}{\lambda_{k-1}}}$$
$$=\prod_{r=1}^{k}\left(\sum_{i=1}^{n+1}a_{ir}^{\lambda_r}\right)^{\frac{2}{\lambda_r}}=S_{n+1}. \tag{10}$$

So

$$F_1(n)\leqslant F_1(n+1). \tag{11}$$

*Case* II. When $l=1$ and $k$ is odd. By the same method as in Case I, we can obtain the inequality (11).

Moreover, from $T_{n+1}-T_n\leqslant S_{n+1}-S_n$, it is easy to find that the following inequality

$$T_{n+1}^l-T_n^l\leqslant S_{n+1}^l-S_n^l$$

holds for $l=2,3,\ldots$, that is, $F_l(n)\leqslant F_l(n+1),\quad l=1,2,\ldots.$

If we set $n=1$ in (5), then $F_l(1)=0$.

Thus

$$0\leqslant F_l(n)\leqslant F_l(n+1),\quad l=1,2,\ldots.$$

The proof of Theorem 2.1 is completed. □

From Theorem 2.1, we obtain the following refinement of generalized Hölder's inequality (2).

COROLLARY 2.2. *Let* $a_{nr}>0$ $(n=1,2,\ldots,r=1,2,\ldots,k)$, *and let* $\lambda_1>\lambda_2>\ldots>\lambda_k>0$, $\sum_{r=1}^{k}\frac{1}{\lambda_r}\geqslant 1$. *Then*

$$\sum_{i=1}^{n}\prod_{r=1}^{k}a_{ir}\leqslant\left(\sum_{i=1}^{n}\prod_{r=1}^{k}a_{ir}\right)\left(1+\frac{F(2)}{(\sum_{i=1}^{n}\prod_{r=1}^{k}a_{ir})^2}\right)^{\frac{1}{2}}\leqslant\prod_{r=1}^{k}\left(\sum_{i=1}^{n}a_{ir}^{\lambda_r}\right)^{\frac{1}{\lambda_r}}, \tag{12}$$

*where* $F(2)=\prod_{r=1}^{k}(a_{1r}^{\lambda_r}+a_{2r}^{\lambda_r})^{\frac{2}{\lambda_r}}-\left(\prod_{r=1}^{k}a_{1r}+\prod_{r=1}^{k}a_{2r}\right)^2\geqslant 0$.

Similar to the proof of Theorem 2.1 but using inequality (3) in place of inequality (2), we immediately obtain the following result.

THEOREM 2.3. *Let* $a_{nr}>0$ $(n=1,2,\ldots,r=1,2,\ldots,k)$, *let* $\lambda_1<\lambda_2<\ldots<\lambda_k<0$, *and let*

$$F_l(n)=\prod_{r=1}^{k}\left(\sum_{i=1}^{n}a_{ir}^{\lambda_r}\right)^{\frac{2l}{\lambda_r}}-\left(\sum_{i=1}^{n}\prod_{r=1}^{k}a_{ir}\right)^{2l} \tag{13}$$

$l=1,2,\ldots.$ *Then*

$$0\geqslant F_l(n)\geqslant F_l(n+1). \tag{14}$$

From Theorem 2.3, we get the refinement of generalized Hölder's inequality (3) as follows.

COROLLARY 2.4. *Let* $a_{nr} > 0$ $(n = 1, 2, \ldots, r = 1, 2, \ldots, k)$*, let* $\lambda_1 < \lambda_2 < \ldots < \lambda_k < 0$*. Then*

$$\sum_{i=1}^{n} \prod_{r=1}^{k} a_{ir} \geqslant \left( \sum_{i=1}^{n} \prod_{r=1}^{k} a_{ir} \right) \left( 1 + \frac{F(2)}{(\sum_{i=1}^{n} \prod_{r=1}^{k} a_{ir})^2} \right)^{\frac{1}{2}} \geqslant \prod_{r=1}^{k} \left( \sum_{i=1}^{n} a_{ir}^{\lambda_r} \right)^{\frac{1}{\lambda_r}}, \tag{15}$$

*where* $F(2) = \prod_{r=1}^{k} (a_{1r}^{\lambda_r} + a_{2r}^{\lambda_r})^{\frac{2}{\lambda_r}} - \left( \prod_{r=1}^{k} a_{1r} + \prod_{r=1}^{k} a_{2r} \right)^2 \leqslant 0$.

Similar to the proof of Theorem 2.1 but using inequality (4) in place of inequality (2), we immediately obtain the following property of generalized Hölder inequality (4).

THEOREM 2.5. *Let* $a_{nr} > 0 (n = 1, 2, \ldots, r = 1, 2, \ldots, k)$*, let* $\lambda_1 > 0$*,* $\lambda_2 < \lambda_3 < \ldots < \lambda_k < 0$*,* $\sum_{r=1}^{k} \frac{1}{\lambda_r} \leqslant 1$*, and let*

$$F_l(n) = \prod_{r=1}^{k} \left( \sum_{i=1}^{n} a_{ir}^{\lambda_r} \right)^{\frac{2l}{\lambda_r}} - \left( \sum_{i=1}^{n} \prod_{r=1}^{k} a_{ir} \right)^{2l} \tag{16}$$

$l = 1, 2, \ldots$*. Then*

$$0 \geqslant F_l(n) \geqslant F_l(n+1). \tag{17}$$

By using Theorem 2.5, we can obtain the following refinement of generalized Hölder's inequality (4).

COROLLARY 2.6. *Let* $a_{nr} > 0$ $(n = 1, 2, \ldots, r = 1, 2, \ldots, k)$*, and let* $\lambda_1 > 0$*,* $\lambda_2 < \lambda_3 < \ldots < \lambda_k < 0$*,* $\sum_{r=1}^{k} \frac{1}{\lambda_r} \leqslant 1$*. Then*

$$\sum_{i=1}^{n} \prod_{r=1}^{k} a_{ir} \geqslant \left( \sum_{i=1}^{n} \prod_{r=1}^{k} a_{ir} \right) \left( 1 + \frac{F(2)}{(\sum_{i=1}^{n} \prod_{r=1}^{k} a_{ir})^2} \right)^{\frac{1}{2}} \geqslant \prod_{r=1}^{k} \left( \sum_{i=1}^{n} a_{ir}^{\lambda_r} \right)^{\frac{1}{\lambda_r}}, \tag{18}$$

*where* $F(2) = \prod_{r=1}^{k} (a_{1r}^{\lambda_r} + a_{2r}^{\lambda_r})^{\frac{2}{\lambda_r}} - \left( \prod_{r=1}^{k} a_{1r} + \prod_{r=1}^{k} a_{2r} \right)^2 \leqslant 0$.

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## REFERENCES

[1] S. ABRAMOVICH, J. PEČARIĆ AND S. VAROŠANEC, *Sharpening Hölder's and Popoviciu's inequalities via functionals*, Rocky Mt. J. Math. **34** (2004), 793–810.

[2] S. ABRAMOVICH, J. PEČARIĆ AND S. VAROŠANEC, *Continuous sharpening of Hölder's and Minkowski's inequalities*, Math. Inequal. Appl. **8**, 2 (2005), 179–190.

[3] J. MATKOWSKI, *The converse of the Hölder inequality and its generalizations*, Studia Math. **109**, 2 (1994), 171–182.

[4] J. MATKOWSKI, *A converse of the Hölder inequality theorem*, Math. Inequal. Appl. **12**, 1 (2009), 21–32.

[5] L. NIKOLOVA AND S. VAROŠANEC, *Refinements of Hölder's inequality derived from functions $\psi_{p,q,\lambda}$ and $\phi_{p,q,\lambda}$*, Ann. Funct. Anal. **2**, 1 (2011), 72–83.

[6] J.-F. TIAN, *Property of a Hölder-type inequality and its application*, Math. Inequal. Appl. **16**, 3 (2013), 831–841.

[7] P. M. VASIĆ, J. E. PEČARIĆ, *On the Jensen inequality for monotone functions*, An. Univ. Timişoara Ser. Şt. Matematice **17**, 1 (1979), 95–104.

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