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GENERALIZED STEFFENSEN'S INEQUALITY

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Abstract. We give a generalization of Steffensen's inequality by extending the results of Pečarić [4] and Rabier [5]. We make use of the n-order Taylor expansion of a composition of functions and Faà di Bruno's formula for higher order derivatives of the composition.

1. Introduction

Steffensen [6] proved the following inequality: if $f,h:[\alpha,\beta]\to\mathbb{R}$, $0\leqslant h\leqslant 1$ and f is decreasing, then

$$\int_\alpha^\beta f(t)h(t)\,dt \leqslant \int_\alpha^{\alpha+\gamma} f(t)\,dt, \qquad \text{where } \gamma=\int_\alpha^\beta h(t)\,dt. \tag{1}$$

A few hundred papers are devoted to studying generalizations of Steffensen's inequality (1). One recent is given by Rabier [5].

THEOREM 1.1. *Let $\phi:[0,\infty)\to\mathbb{R}$ be convex and continuous with $\phi(0)=0$. If $b>0$ and $h\in L^\infty(0,b), h\geqslant 0$ and $\|h\|_\infty\leqslant 1$, then $h\phi^{(1)}\in L^1(0,b)$ and*

$$\phi\Big(\int_0^b h(t)\,dt\Big)\leqslant\int_0^b h(t)\phi^{(1)}(t)\,dt \tag{2}$$

In fact, Rabier's result is closely related to another generalization of Steffensen's inequality given by Pečarić [4].

THEOREM 1.2. *Let $g:[a,b]\to\mathbb{R}$ be a nondecreasing and differentiable function and $f:I\to\mathbb{R}$ be a nondecreasing function (I is an interval in $\mathbb{R}$ such that $a,b,g(a),g(b)\in I$).*

(a) *If $g(x)\leqslant x$, then*

$$\int_a^b f(t)g^{(1)}(t)\,dt\geqslant\int_{g(a)}^{g(b)} f(t)\,dt. \tag{3}$$

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(b) If $g(x) \geqslant x$, then the reverse of the above inequality holds.

REMARK 1.3. The assumptions of Theorem 1.2 can be weakened and differentiability of g can be replaced with absolute continuity. Indeed, for a nondecreasing function f, the function $F(x) = \int_a^x f(t)\,dt$ is well defined and satisfies $F^{(1)} = f$ at all except at most countably many points. For absolutely continuous nondecrasing function g the substitution $z = g(t)$ in the integral is justified (see [2, Corollary 20.5]), so

$$F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(z)\,dz = \int_a^b f(g(t))g^{(1)}(t)\,dt \leqslant \int_a^b f(t)g^{(1)}(t)\,dt, \quad (4)$$

where the last inequality holds when $g(t) \leqslant t$.

Steffensen's inequality (1) follows from Theorem 1.2 by making substitutions $g(x) \mapsto \int_a^x h(t+\alpha-a)\,dt + a$ and $f(x) \mapsto -f(x+\alpha-a)$ and taking $b = \beta - \alpha + a$.

REMARK 1.4. Theorem 1.1 follows from Theorem 1.2 with weaker assumptions. Indeed, the convex function ϕ from Theorem 1.1 has a nondecreasing right-sided derivative $\phi_+^{(1)}$ such that $\phi(x) = \int_0^x \phi_+^{(1)}(t)\,dt$. Furthermore, for a function $h : [0,b] \to [0,1]$, the function $g(x) = \int_0^x h(t)\,dt$ is absolutely continuous and satisfies $g(x) \leqslant x$ and $g^{(1)} = h$ almost everywhere. Therefore, by taking $a = 0$, $f = \phi_+^{(1)}$ and $g(x) = \int_0^x h(t)\,dt$ in Theorem 1.2 (under the weaker assumptions) we get Theorem 1.1.

On the other hand, an absolutely continuous nondecreasing function $g : [0,b] \to \mathbb{R}$ can satisfy $g(x) \leqslant x$ without satisfying $0 \leqslant g^{(1)}(x) \leqslant 1$.

The goal of this paper is to generalize the inequality from Theorem 1.2 by replacing the equality

$$F(g(x)) = F(g(a)) + \int_{g(a)}^{g(x)} f(t)\,dt$$

with the n-th order Taylor expansion of the composition $F \circ g$ with the remainder given in the integral form. In the process we will use Faà di Bruno's formula for higher order derivatives of the composition $F \circ g$. The formula states that

$$\frac{d^m}{dx^m} F(g(x)) = \sum_{k=1}^{m} F^{(k)}(g(x)) B_{m,k}(g^{(1)}(x), \ldots, g^{(m-k+1)}(x)) \quad (5)$$

where $B_{m,k}(x_1, x_2, \ldots, x_{m-k+1})$ are the Bell polynomials

$$B_{m,k}(x_1, x_2, \ldots, x_{m-k+1}) = \sum \frac{m!}{j_1! j_2! \cdots j_{m-k+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \left(\frac{x_2}{2!}\right)^{j_2} \cdots \left(\frac{x_{m-k+1}}{(m-k+1)!}\right)^{j_{m-k+1}}$$

where the sum is taken over all sequences $j_1, j_2, \ldots, j_{m-k+1}$ of non-negative integers such that

$$j_1 + j_2 + \ldots = k \quad \text{and} \quad j_1 + 2j_2 + 3j_3 + \ldots = m.$$

For a historical overview of Faà di Bruno's formula and its various forms see [3].

In this paper we will give a generalization of Steffensen's inequality by making use of the n-order Taylor expansion of the composition $F \circ g$. Moreover, we will give the inequalities for the special case when g is of the form $g(x) = \int_0^x h(t)\,dt$ and use it to obtain a Hardy-type inequality.

2. Main results

The following theorem states our main result.

THEOREM 2.1. *Let $n \in \mathbb{N}$. Let $g : [a,b] \to \mathbb{R}$ and $F : I \to \mathbb{R}$ (where I is an interval in $\mathbb{R}$ such that $a, b, g(a), g(b) \in I$) be two n times differentiable functions such that $g, g^{(1)}, \ldots g^{(n-1)}$, $F^{(1)}, F^{(2)}, \ldots, F^{(n)}$ are nondecreasing functions.*

(a) *If $g(x) \leqslant x$, then*

$$F(g(b)) \leqslant F(g(a)) + \sum_{k=1}^{n-1} F^{(k)}(g(a)) \sum_{i=k}^{n-1} B_{i,k}(g^{(1)}(a), \ldots, g^{(i-k+1)}(a)) \frac{(b-a)^i}{i!} + \int_a^b \frac{(b-t)^{n-1}}{(n-1)!} \sum_{k=1}^{n} F^{(k)}(t) B_{n,k}(g^{(1)}(t), \ldots, g^{(n-k+1)}(t))\,dt.$$

(b) *If $g(x) \geqslant x$, then the reverse of the above inequality holds.*

Proof. The $(n-1)$-th Taylor expansion of a function H with the remainder in the integral form is given by

$$H(x) = \sum_{k=0}^{n-1} H^{(k)}(a) \frac{(x-a)^k}{k!} + \int_a^x H^{(n)}(t) \frac{(x-t)^{n-1}}{(n-1)!}\,dt$$

Applying this formula for the composition $H = F \circ g$ and using Faà di Bruno's formula (5) gives

$$\begin{aligned}
(F\circ g)(b) &= (F\circ g)(a) + \sum_{i=1}^{n-1} \sum_{k=1}^{i} F^{(k)}(g(a)) B_{i,k}(g^{(1)}(a), \ldots, g^{(i-k+1)}(a)) \frac{(b-a)^i}{i!} \\
&\quad + \int_a^b \frac{(b-t)^{n-1}}{(n-1)!} \sum_{k=1}^{n} F^{(k)}(g(t)) B_{n,k}(g^{(1)}(t), \ldots, g^{(n-k+1)}(t))\,dt \\
&= (F\circ g)(a) + \sum_{k=1}^{n-1} F^{(k)}(g(a)) \sum_{i=k}^{n-1} B_{i,k}(g^{(1)}(a), \ldots, g^{(i-k+1)}(a)) \frac{(b-a)^i}{i!} \\
&\quad + \int_a^b \frac{(b-t)^{n-1}}{(n-1)!} \sum_{k=1}^{n} F^{(k)}(g(t)) B_{n,k}(g^{(1)}(t), \ldots, g^{(n-k+1)}(t))\,dt.
\end{aligned}$$

By the assumptions of the theorem, $g^{(i)} \geqslant 0$ for $i = 1, \dots, n$, so the Bell polynomials evaluated at the derivatives of g in the above expression are nonnegative. Therefore, for $g(x) \leqslant x$ the inequality

$$\int_a^b \frac{(b-t)^{n-1}}{(n-1)!} \sum_{k=1}^{n} F^{(k)}(g(t)) B_{n,k}(g^{(1)}(t), \dots, g^{(n-k+1)}(t))\,dt \leqslant \int_a^b \frac{(b-t)^{n-1}}{(n-1)!} \sum_{k=1}^{n} F^{(k)}(t) B_{n,k}(g^{(1)}(t), \dots, g^{(n-k+1)}(t))\,dt$$

holds, while for $g(x) \geqslant x$ the reversed inequality holds. □

A special case of the previous theorem is given by the following corollary.

COROLLARY 2.2. *Let $n \in \mathbb{N}$, let $F : [0,b] \to \mathbb{R}$ be n times differentiable function such that $F^{(1)}, F^{(2)}, \dots, F^{(n)}$ are nondecreasing functions and let $h : [0,b] \to [0,+\infty)$ be $n-1$ times differentiable function such that $h, h^{(1)}, \dots, h^{(n-1)}$ are nonnegative.*

(a) *If $\int_0^x h(t)\,dt \leqslant x$ for every $x \in [0,b]$, then*

$$F\left(\int_0^b h(t)\,dt\right) \leqslant F(0) + \sum_{k=1}^{n-1} F^{(k)}(0) \sum_{i=k}^{n-1} B_{i,k}(h(0), h^{(1)}(0), \dots, h^{(i-k)}(0)) \frac{b^i}{i!} + \int_0^b \frac{(b-t)^{n-1}}{(n-1)!} \sum_{k=1}^{n} F^{(k)}(t) B_{n,k}(h(t), h^{(1)}(t), \dots, h^{(n-k)}(t))\,dt.$$

(b) *If $x \leqslant \int_0^x h(t)\,dt$ for every $x \in [0,b]$, then the reverse of the above inequality holds.*

Proof. Follows from Theorem 2.1 by taking $a = 0$ and $g(x) = \int_0^x h(t)\,dt$. □

By inserting the Bell polynomials we get the explicit forms of the inequalities from Theorem 2.1 and Corollary 2.2. Since $B_{1,1}(x_1) = x_1$, the inequalities (from part (a)) for $n = 1$ are given by (4) and

$$F\left(\int_0^b h(t)\,dt\right) \leqslant F(0) + \int_0^b F^{(1)}(t) h(t)\,dt.$$

The subsequent Bell polynomials are equal to:

$n = 2: B_{2,1}(x_1, x_2) = x_2, \quad B_{2,2}(x_1) = x_1^2,$

$n = 3: B_{3,1}(x_1, x_2, x_3) = x_3, \quad B_{3,2}(x_1, x_2) = 3x_1x_2, \quad B_{3,3}(x_1) = x_1^3,$

$n = 4: B_{4,1}(x_1, x_2, x_3, x_4) = x_4, \quad B_{4,2}(x_1, x_2, x_3) = 3x_1^3x_3 + 3x_2^2$
$B_{4,3}(x_1, x_2) = 6x_1^2x_2, \quad B_{4,4}(x_1) = x_1^4,$

$n = 5: B_{5,1}(x_1, x_2, x_3, x_4, x_5) = x_5, \quad B_{5,2}(x_1, x_2, x_3, x_4) = 5x_1x_4 + 10x_2x_3,$
$B_{5,3}(x_1, x_2, x_3) = 10x_1^2x_3 + 15x_1x_2^2,$
$B_{5,4}(x_1, x_2) = 10x_1^3x_2, \quad B_{5,5}(x_1) = x_1^5.$

Therefore, for $n=2$ the inequality from Theorem 2.1 (a) is of the form

$$F(g(b)) \leqslant F(g(a)) + (b-a)F^{(1)}(g(a))g^{(1)}(a) + \int_a^b (b-t)\left[F^{(1)}(t)g^{(2)}(t) + F^{(2)}(t)g^{(1)}(t)^2\right] dt$$

and the inequality from Corollary 2.2 (a) is of the form

$$F\left(\int_0^b h(t)\,dt\right) \leqslant F(0) + bF^{(1)}(0)h(0) + \int_0^b (b-t)\left[F^{(1)}(t)h^{(1)}(t) + F^{(2)}(t)h(t)^2\right] dt.$$

Next, we will use the inequality from Corollary 2.2 to obtain the following result.

THEOREM 2.3. *Let $n \in \mathbb{N}$, h and F be as in Corollary 2.2, $k:[0,b] \to [0,+\infty)$ and denote $K_i(t) = \int_t^b \frac{(x-t)^{i-1}}{(i-1)!} k(x)\,dx$ for $i \in \mathbb{N}$.*

(a) If $\int_0^x h(t)\,dt \leqslant x$ for every $x \in [0,b]$, then

$$\begin{aligned}
&\int_0^b k(x) F\left(\int_0^x h(t)\,dt\right) dx \\
&\leqslant F(0)K_1(0) + \sum_{k=1}^{n-1} F^{(k)}(0) \sum_{i=k}^{n-1} B_{i,k}(h(0), h^{(1)}(0), \ldots, h^{(i-k)}(0)) K_{i+1}(0) \\
&\quad + \int_0^b K_n(t) \sum_{k=1}^{n} F^{(k)}(t) B_{n,k}(h(t), h^{(1)}(t), \ldots, h^{(n-k)}(t))\,dt.
\end{aligned}$$

(b) If $x \leqslant \int_0^x h(t)\,dt$ for every $x \in [0,b]$, then the reverse of the above inequality holds.

Proof. Applying Corollary 2.2 with $b=x$ and multiplying by $k(x) \geqslant 0$ we get

$$\begin{aligned}
&k(x) F\left(\int_0^x h(t)\,dt\right) \\
&\leqslant F(0)k(x) + \sum_{k=1}^{n-1} F^{(k)}(0) \sum_{i=k}^{n-1} B_{i,k}(h(0), h^{(1)}(0), \ldots, h^{(i-k)}(0)) \frac{x^i}{i!} k(x) \\
&\quad + k(x) \int_0^x \frac{(x-t)^{n-1}}{(n-1)!} \sum_{k=1}^{n} F^{(k)}(t) B_{n,k}(h(t), h^{(1)}(t), \ldots, h^{(n-k)}(t))\,dt.
\end{aligned}$$

Integrating the above inequality with respect to x from 0 to b and applying Fubini's theorem on the right hand side we obtain the stated inequality. □

For $p>1$, $l>1$ and a nonnegative function h such that $x^{1-p/l}h \in L^p(0,b)$, inequality

$$\int_0^b x^{-l}\left(\int_0^x h(t)\,dt\right)^p dx \leqslant \left(\frac{p}{l-1}\right)^p \int_0^b x^{p-l} h(x)^p\,dx \tag{6}$$

holds. The classical Hardy's inequality is inequality (6) with $b=\infty$ and in that case the constant in (6) is sharp, while for finite $b>0$ the constant is not sharp. In fact, for finite b the following inequality due to Čižmešija and Pečarić [1] holds

$$\int_0^b x^{-l}\left(\int_0^x h(t)\,dt\right)^p dx \leqslant \left(\frac{p}{l-1}\right)^p \int_0^b \left[1-\left(\frac{x}{b}\right)^{\frac{l-1}{p}}\right] x^{p-l} h(x)^p\,dx \tag{7}$$

and the constant in (7) is sharp. For $l=p$ inequality (6) can be written as

$$\|Ah\|_p \leqslant \frac{p}{p-1}\|h\|_p,$$

where A is the operator $Ah(x)=\frac{1}{x}\int_0^x h(t)\,dt$. In the following example we will use results obtained in this paper to derive some inequalities similar to (6) and (7).

EXAMPLE 2.4. Applying Theorem 2.3 (a) with $F(t)=t^p$, $p>n$, we obtain the inequality

$$\int_0^b k(x)\left(\int_0^x h(t)\,dt\right)^p dx \leqslant \int_0^b K_n(t)\sum_{k=1}^n (p)_k t^{p-k} B_{n,k}(h(t),h^{(1)}(t),\dots,h^{(n-k)}(t))\,dt, \tag{8}$$

where $(p)_k = p(p-1)\cdots(p-k+1)$ is the Pochhammer symbol.

Furthermore, for $p>1$

$$\begin{aligned}\int_0^b k(x)\left(\int_0^x h(t)\,dt\right)^p dx &\leqslant p\int_0^b t^{p-1}K(t)h(t)\,dt\\ &\leqslant p\left[\int_0^b h(t)^p\,dt\right]^{\frac{1}{p}}\left[\int_0^b t^p K(t)^{\frac{p}{p-1}}\,dt\right]^{\frac{p-1}{p}},\end{aligned}$$

where the first inequality follows from (8) with $n=1$ and the second inequality follows by applying Hölder's inequality. In particular, for $k(x)=x^{-p}$ we have

$$\begin{aligned}\int_0^b\left(\frac{1}{x}\int_0^x h(t)\,dt\right)^p dx &\leqslant \frac{p}{p-1}\int_0^b\left(1-\left(\frac{t}{b}\right)^{p-1}\right)h(t)\,dt\\ &\leqslant \frac{p}{p-1}\left[\int_0^b h(t)^p\,dt\right]^{\frac{1}{p}}\left[\int_0^b\left(1-\left(\frac{t}{b}\right)^{p-1}\right)^{\frac{p}{p-1}}dt\right]^{\frac{p-1}{p}}.\end{aligned} \tag{9}$$

The last inequality can be written as

$$\|Ah\|_p^p \leqslant C\|h\|_p,$$

where the constant C does not depend on h. By applying the substitution $y=(t/b)^{p-1}$ in the last integral on the right hand side of (9), one can calculate

$$C=\frac{pb}{(p-1)^2}B\left(\frac{1}{p-1},\frac{2p-1}{p-1}\right),$$

where B is the beta function. In particular, for $p=2$ we have

$$\int_0^b \left(\frac{1}{x}\int_0^x h(t)\,dt\right)^2 dx \leqslant \frac{2}{3}b\|h\|_2.$$

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