Journal of
Mathematical
Inequalities
Volume 9, Number 2 (2015), 493–498
doi:10.7153/jmi-09-43

INEQUALITIES FOR THE FROBENIUS NORM

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(*Communicated by J. Pečarić*)

Abstract. In this note, we present a refinement of Heinz inequality for the Frobenius norm and discuss the relationship between our result and some existing inequalities

1. Introduction

Let M_n be the space of $n\times n$ complex matrices and $\|\cdot\|$ stand for any unitarily invariant norm on M_n. So, $\|UAV\| = \|A\|$ for all $A\in M_n$ and for all unitary matrices U, $V\in M_n$. For $A=(a_{ij})\in M_n$, the Frobenius norm of A is defined by

$$\|A\|_F = \left(\sum_{i,j=1}^{n} \left|a_{ij}\right|^2\right)^{1/2}.$$

The Frobenius norm is also called Hilbert-Schmidt norm. It plays a basic role in matrix analysis and it is known that the Frobenius norm is unitarily invariant.

Let a and b be nonnegative real numbers. The geometric and arithmetic means are defined as follows:

$$G(a,b)=\sqrt{ab}, \quad A(a,b)=\frac{a+b}{2}.$$

The Heinz means are defined as

$$H_v(a,b)=\frac{a^v b^{1-v}+a^{1-v}b^v}{2}, \quad 0\leqslant v\leqslant 1.$$

It is easy to see that as a function of v, $H_v(a,b)$ is convex and attains its minimum at $v=\frac{1}{2}$. So,

$$G(a,b)\leqslant H_v(a,b)\leqslant A(a,b), \quad 0\leqslant v\leqslant 1. \tag{1}$$

A matrix version of inequality (1) was proved in [2, Theorem 2] which says that if A, B, $X\in M_n$ such that A and B are positive semidefinite and if $0\leqslant v\leqslant 1$, then

$$2\left\|A^{1/2}XB^{1/2}\right\| \leqslant \left\|A^v XB^{1-v}+A^{1-v}XB^v\right\| \leqslant \|AX+XB\|. \tag{2}$$

Mathematics subject classification (2010): 15A15, 15A42, 15A60.
Keywords and phrases: Positive semidefinite matrix, matrix means, Frobenius norm.

Paper JMI-09-43

The second part of inequality (2) is known as Heinz inequality for matrices. For more information on Heinz inequality for matrices the reader is referred to [3–5].

Let A, B, $X \in M_n$ such that A and B are positive semidefinite and suppose that

$$f(v) = \left\| A^v X B^{1-v} + A^{1-v} X B^v \right\|_F, \quad 0 \leqslant v \leqslant 1.$$

It is known [1, p. 265] that f is a continuous convex function on $[0,1]$. Kittaneh and Manasrah [4, Theorem 3.4] proved that if $0 \leqslant v \leqslant 1$, then

$$f(v) + 2r_0 \left(\sqrt{\|AX\|_F} - \sqrt{\|XB\|_F} \right)^2 \leqslant \|AX + XB\|_F, \tag{3}$$

where $r_0 = \min\{v,\ 1-v\}$. Inequality (3) is a refinement of Heinz inequality for the Frobenius norm.

In section 2, we first show a refinement of Heinz inequality for the Frobenius norm. After that, we discuss the relationship between our result and inequality (3).

2. Main results

THEOREM 2.1. *Let $A, X, B \in M_n$ such that A and B are positive semidefinite. If $0 \leqslant v \leqslant 1$, then*

$$f(v) + 4r_0 \left(\int_0^1 f(v)\,dv - 2\left\| A^{1/2} X B^{1/2} \right\|_F \right) \leqslant \|AX + XB\|_F, \tag{4}$$

where $r_0 = \min\{v,\ 1-v\}$.

Proof. Let

$$a = \left(\sqrt{\|AX\|_F} - \sqrt{\|XB\|_F} \right)^2.$$

Since f is continuous convex on $[0,1]$, we have:

Step 1. By inequality (3), we obtain

$$\int_0^1 f(v)\,dv + a \int_0^1 2r_0\,dv \leqslant \|AX + XB\|_F.$$

That is,

$$\int_0^1 f(v)\,dv + \frac{a}{2} \leqslant \|AX + XB\|_F,$$

which is equivalent to

$$2\left\| A^{1/2} X B^{1/2} \right\|_F + \left(\int_0^1 f(v)\,dv - 2\left\| A^{1/2} X B^{1/2} \right\|_F + \frac{a}{2} \right) \leqslant \|AX + XB\|_F. \tag{5}$$

Step 2. By inequality (5) and a similar argument as presented in [4, Theorem 3.4], we have

$$f(v)+2r_0\left(\int_0^1 f(v)\,dv-2\left\|A^{1/2}XB^{1/2}\right\|_F+\frac{a}{2}\right)\leqslant\|AX+XB\|_F. \tag{6}$$

Step 3. By inequality (6), we get

$$\int_0^1 f(v)\,dv+\left(\int_0^1 f(v)\,dv-2\left\|A^{1/2}XB^{1/2}\right\|_F+\frac{a}{2}\right)\int_0^1 2r_0dv\leqslant\|AX+XB\|_F$$

That is,

$$\frac{3}{2}\int_0^1 f(v)\,dv-\left\|A^{1/2}XB^{1/2}\right\|_F+\frac{a}{4}\leqslant\|AX+XB\|_F,$$

which is equivalent to

$$2\left\|A^{1/2}XB^{1/2}\right\|_2+\left(\frac{3}{2}\int_0^1 f(v)\,dv-3\left\|A^{1/2}XB^{1/2}\right\|_F+\frac{a}{4}\right)\leqslant\|AX+XB\|_F. \tag{7}$$

Step 4. By inequality (7) and a similar argument as presented in [4, Theorem 3.4], we have

$$f(v)+2r_0\left(\frac{3}{2}\int_0^1 f(v)\,dv-3\left\|A^{1/2}XB^{1/2}\right\|_F+\frac{a}{4}\right)\leqslant\|AX+XB\|_F. \tag{8}$$

Step 5. By inequality (8), we obtain

$$\int_0^1 f(v)\,dv+\left(\frac{3}{2}\int_0^1 f(v)\,dv-3\left\|A^{1/2}XB^{1/2}\right\|_F+\frac{a}{4}\right)\int_0^1 2r_0dv\leqslant\|AX+XB\|_F.$$

That is,

$$\frac{7}{4}\int_0^1 f(v)\,dv-\frac{3}{2}\left\|A^{1/2}XB^{1/2}\right\|_F+\frac{a}{8}\leqslant\|AX+XB\|_F,$$

which is equivalent to

$$2\left\|A^{1/2}XB^{1/2}\right\|_F+\left(\frac{7}{4}\int_0^1 f(v)\,dv-\frac{7}{2}\left\|A^{1/2}XB^{1/2}\right\|_F+\frac{a}{8}\right)\leqslant\|AX+XB\|_F. \tag{9}$$

Step 6. By inequality (9) and a similar argument as presented in [4, Theorem 3.4], we have

$$f(v)+2r_0\left(\frac{7}{4}\int_0^1 f(v)\,dv-\frac{7}{2}\left\|A^{1/2}XB^{1/2}\right\|_F+\frac{a}{8}\right)\leqslant\|AX+XB\|_F. \tag{10}$$

$$\vdots$$

Step $n = 2k+1$. By the same method above, we get

$$2\left\|A^{1/2}XB^{1/2}\right\|_F + \left(\frac{2^n-1}{2^{n-1}}\int_0^1 f(v)\,dv - \frac{2^{n+1}-2}{2^{n-1}}\left\|A^{1/2}XB^{1/2}\right\|_F + \frac{a}{2^n}\right) \leqslant \|AX+XB\|_F. \tag{11}$$

Step $n = 2k+2$. By inequality (11) and a similar argument as presented in [4, Theorem 3.4], we have

$$f(v) + 2r_0\left(\frac{2^n-1}{2^{n-1}}\int_0^1 f(v)\,dv - \frac{2^{n+1}-2}{2^{n-1}}\left\|A^{1/2}XB^{1/2}\right\|_F + \frac{a}{2^n}\right) \leqslant \|AX+XB\|_F. \tag{12}$$

Then, taking the limit $n \to \infty$ side by side in (12), we have

$$f(v) + 4r_0\left(\int_0^1 f(v)\,dv - 2\left\|A^{1/2}XB^{1/2}\right\|_F\right) \leqslant \|AX+XB\|_F.$$

This completes the proof. $\square$

REMARK 2.1. By inequality (2), we know that

$$\int_0^1 f(v)\,dv \geqslant 2\left\|A^{1/2}XB^{1/2}\right\|_F.$$

So, inequality (4) is a refinement of Heinz inequality.

REMARK 2.2. Inequality (4) has been obtained by Zou and He [6, Theorem 2.1]. Here, we give a new proof.

In view of inequalities (3) and (4), we want to know the relationship between them. We may ask whether one of the the following inequalities holds:

$$2\left(\int_0^1 f(v)\,dv - 2\left\|A^{1/2}XB^{1/2}\right\|_F\right) \leqslant \left(\sqrt{\|AX\|_F} - \sqrt{\|XB\|_F}\right)^2, \tag{13}$$

$$2\left(\int_0^1 f(v)\,dv - 2\left\|A^{1/2}XB^{1/2}\right\|_F\right) \geqslant \left(\sqrt{\|AX\|_F} - \sqrt{\|XB\|_F}\right)^2. \tag{14}$$

The answer is no. We have the following result.

THEOREM 2.2. *Inequalities (13) and (14) are not always true.*

Proof. Firstly, we give an example to show that (13) is not always true. In fact, if we choose

$$A = \begin{bmatrix} 8.6897 & 3.3580 \\ 3.3580 & 1.4082 \end{bmatrix}, \quad X = \begin{bmatrix} 3.6696 & 1.3801 \\ 2.682 & 1.8423 \end{bmatrix}, \quad B = \begin{bmatrix} 0.9331 & 1.6242 \\ 1.6242 & 18.2135 \end{bmatrix}.$$

then, we have

$$2\left(\left\|A^{1/4}XB^{3/4} + A^{3/4}XB^{1/4}\right\|_F - 2\left\|A^{1/2}XB^{1/2}\right\|_F\right) = 5.5623$$

and

$$\left(\sqrt{\|AX\|_F} - \sqrt{\|XB\|_F}\right)^2 = 0.0124.$$

Applying the Hermite-Hadamard inequality for convex function f on each of the subintervals $\left[0, \frac{1}{2}\right]$ and $\left[\frac{1}{2}, 1\right]$ and summing up side by side, we have

$$f\left(\frac{1}{4}\right) \leqslant \int_0^1 f(v)\,dv.$$

So,

$$2\left(\int_0^1 f(v)\,dv - 2\left\|A^{1/2}XB^{1/2}\right\|_F\right) \geqslant 5.5623 \geqslant 0.0124.$$

On the other hand, the following example shows that (14) does not hold. Zou [5, Theorem 3.1] proved that

$$\int_0^1 f(v)\,dv \leqslant \frac{4}{3}\left\|A^{1/2}XB^{1/2}\right\|_F + \frac{1}{3}\|AX + XB\|_F,$$

which implies

$$2\left(\int_0^1 f(v)\,dv - 2\left\|A^{1/2}XB^{1/2}\right\|_F\right) \leqslant \frac{2}{3}\|AX + XB\| - \frac{4}{3}\left\|A^{1/2}XB^{1/2}\right\|_F.$$

Let

$$A = \begin{bmatrix} 0.3486 & 0.3686 \\ 0.3686 & 0.4376 \end{bmatrix}, \quad X = \begin{bmatrix} 0.4756 & 0.7881 \\ 0.3625 & 0.7803 \end{bmatrix}, \quad B = \begin{bmatrix} 0.2574 & 0.2898 \\ 0.2898 & 0.3274 \end{bmatrix}.$$

Then, we have

$$\|AX + XB\|_F = 1.6591 \leqslant 1.8265 = 2\left\|A^{1/2}XB^{1/2}\right\|_F + \frac{3}{2}\left(\sqrt{\|AX\|_F} - \sqrt{\|XB\|_F}\right)^2,$$

it follows that inequality (14) is not true for these matrices. This completes the proof. □

Acknowledgements. The author wishes to express her heartfelt thanks to the referees for their detailed and helpful suggestions for revising the manuscript.

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(Received November 5, 2013)

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