INEQUALITIES FOR THE FROBENIUS NORM

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Abstract. In this note, we present a refinement of Heinz inequality for the Frobenius norm and discuss the relationship between our result and some existing inequalities.

1. Introduction

Let $M_n$ be the space of $n \times n$ complex matrices and $\|\cdot\|$ stand for any unitarily invariant norm on $M_n$. So, $\|UAV\| = \|A\|$ for all $A \in M_n$ and for all unitary matrices $U, V \in M_n$. For $A = (a_{ij}) \in M_n$, the Frobenius norm of $A$ is defined by

$$\|A\|_F = \left(\sum_{i,j=1}^{n} |a_{ij}|^2\right)^{1/2}.$$ 

The Frobenius norm is also called Hilbert-Schmidt norm. It plays a basic role in matrix analysis and it is known that the Frobenius norm is unitarily invariant.

Let $a$ and $b$ be nonnegative real numbers. The geometric and arithmetic means are defined as follows:

$$G(a, b) = \sqrt{ab}, \quad A(a, b) = \frac{a + b}{2}.$$ 

The Heinz means are defined as

$$H_v(a, b) = \frac{a^v b^{1-v} + a^{1-v} b^v}{2}, \quad 0 \leq v \leq 1.$$ 

It is easy to see that as a function of $v$, $H_v(a, b)$ is convex and attains its minimum at $v = \frac{1}{2}$. So,

$$G(a, b) \leq H_v(a, b) \leq A(a, b), \quad 0 \leq v \leq 1. \quad (1)$$

A matrix version of inequality (1) was proved in [2, Theorem 2] which says that if $A, B, X \in M_n$ such that $A$ and $B$ are positive semidefinite and if $0 \leq v \leq 1$, then

$$2 \left\|A^{1/2} XB^{1/2}\right\| \leq \left\|A^{1-v} XB^{1-v} + A^{1-v} XB^v\right\| \leq \left\|AX + XB\right\|. \quad (2)$$


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The second part of inequality (2) is known as Heinz inequality for matrices. For more information on Heinz inequality for matrices the reader is referred to [3–5].

Let \( A, B, X \in M_n \) such that \( A \) and \( B \) are positive semidefinite and suppose that

\[
f(v) = \|A^{1-v}XB^{1-v} + A^{1-v}XB^v\|_F, \quad 0 \leq v \leq 1.
\]

It is known [1, p. 265] that \( f \) is a continuous convex function on \([0,1]\). Kittaneh and Manasrah [4, Theorem 3.4] proved that if \( 0 \leq v \leq 1 \), then

\[
f(v) + 2r_0 \left( \sqrt{\|AX\|_F} - \sqrt{\|XB\|_F} \right)^2 \leq \|AX + XB\|_F,
\]

where \( r_0 = \min \{v, 1 - v\} \). Inequality (3) is a refinement of Heinz inequality for the Frobenius norm.

In section 2, we first show a refinement of Heinz inequality for the Frobenius norm. After that, we discuss the relationship between our result and inequality (3).

### 2. Main results

**Theorem 2.1.** Let \( A,X,B \in M_n \) such that \( A \) and \( B \) are positive semidefinite. If \( 0 \leq v \leq 1 \), then

\[
f(v) + 4r_0 \left( \int_0^1 f(v) \, dv - 2 \left\| A^{1/2}XB^{1/2} \right\|_F \right) \leq \|AX + XB\|_F,
\]

where \( r_0 = \min \{v, 1 - v\} \).

**Proof.** Let

\[
a = \left( \sqrt{\|AX\|_F} - \sqrt{\|XB\|_F} \right)^2.
\]

Since \( f \) is continuous convex on \([0,1]\), we have:

**Step 1.** By inequality (3), we obtain

\[
\int_0^1 f(v) \, dv + a \int_0^1 2r_0 \, dv \leq \|AX + XB\|_F.
\]

That is,

\[
\int_0^1 f(v) \, dv + \frac{a}{2} \leq \|AX + XB\|_F,
\]

which is equivalent to

\[
2 \left\| A^{1/2}XB^{1/2} \right\|_F + \left( \int_0^1 f(v) \, dv - 2 \left\| A^{1/2}XB^{1/2} \right\|_F + \frac{a}{2} \right) \leq \|AX + XB\|_F.
\]
**Step 2.** By inequality (5) and a similar argument as presented in [4, Theorem 3.4], we have

\[
f(v) + 2r_0 \left( \int_0^1 f(v) \, dv - 2 \left\| A^{1/2}XB^{1/2} \right\|_F + \frac{a}{2} \right) \leq \|AX + XB\|_F.
\]  
(6)

**Step 3.** By inequality (6), we get

\[
\int_0^1 f(v) \, dv + \left( \int_0^1 f(v) \, dv - 2 \left\| A^{1/2}XB^{1/2} \right\|_F + \frac{a}{2} \right) \int_0^1 2r_0 \, dv \leq \|AX + XB\|_F.
\]

That is,

\[
\frac{3}{2} \int_0^1 f(v) \, dv - \left\| A^{1/2}XB^{1/2} \right\|_F + \frac{a}{4} \leq \|AX + XB\|_F,
\]

which is equivalent to

\[
2 \left\| A^{1/2}XB^{1/2} \right\|_2 + \left( \frac{3}{2} \int_0^1 f(v) \, dv - 3 \left\| A^{1/2}XB^{1/2} \right\|_F + \frac{a}{4} \right) \leq \|AX + XB\|_F.
\]  
(7)

**Step 4.** By inequality (7) and a similar argument as presented in [4, Theorem 3.4], we have

\[
f(v) + 2r_0 \left( \frac{3}{2} \int_0^1 f(v) \, dv - 3 \left\| A^{1/2}XB^{1/2} \right\|_F + \frac{a}{4} \right) \leq \|AX + XB\|_F.
\]  
(8)

**Step 5.** By inequality (8), we obtain

\[
\int_0^1 f(v) \, dv + \left( \frac{3}{2} \int_0^1 f(v) \, dv - 3 \left\| A^{1/2}XB^{1/2} \right\|_F + \frac{a}{4} \right) \int_0^1 2r_0 \, dv \leq \|AX + XB\|_F.
\]

That is,

\[
\frac{7}{4} \int_0^1 f(v) \, dv - \frac{3}{2} \left\| A^{1/2}XB^{1/2} \right\|_F + \frac{a}{8} \leq \|AX + XB\|_F,
\]

which is equivalent to

\[
2 \left\| A^{1/2}XB^{1/2} \right\|_F + \left( \frac{7}{4} \int_0^1 f(v) \, dv - \frac{7}{2} \left\| A^{1/2}XB^{1/2} \right\|_F + \frac{a}{8} \right) \leq \|AX + XB\|_F.
\]  
(9)

**Step 6.** By inequality (9) and a similar argument as presented in [4, Theorem 3.4], we have

\[
f(v) + 2r_0 \left( \frac{7}{4} \int_0^1 f(v) \, dv - \frac{7}{2} \left\| A^{1/2}XB^{1/2} \right\|_F + \frac{a}{8} \right) \leq \|AX + XB\|_F.
\]  
(10)
\[ \text{Step } n = 2k + 1. \text{ By the same method above, we get} \]
\[ 2 \left\| A^{1/2}XB^{1/2} \right\|_F \left( 1 - \frac{2^n - 1}{2^{n+1} - 2} \right) \left( \frac{A^{1/2}XB^{1/2}}{2} \right) \leq \|AX + XB\|_F. \]

\[ (11) \]

\[ \text{Step } n = 2k + 2. \text{ By inequality } (11) \text{ and a similar argument as presented in } [4, \text{ Theorem 3.4}], \text{ we have} \]
\[ f(v) + 2r_0 \left( \frac{2^n - 1}{2^{n+1} - 2} \right) \left( \frac{A^{1/2}XB^{1/2}}{2} \right) \leq \|AX + XB\|_F. \]

Then, taking the limit \( n \to \infty \) side by side in (12), we have
\[ f(v) + 4r_0 \left( \int_0^1 f(v)\,dv - 2 \left\| A^{1/2}XB^{1/2} \right\|_F \right) \leq \|AX + XB\|_F. \]

This completes the proof. \( \Box \)

**Remark 2.1.** By inequality (2), we know that
\[ \int_0^1 f(v)\,dv \geq 2 \left\| A^{1/2}XB^{1/2} \right\|_F. \]

So, inequality (4) is a refinement of Heinz inequality.

**Remark 2.2.** Inequality (4) has been obtained by Zou and He [6, Theorem 2.1]. Here, we give a new proof.

In view of inequalities (3) and (4), we want to know the relationship between them. We may ask whether one of the the following inequalities holds:
\[ 2 \left( \int_0^1 f(v)\,dv - 2 \left\| A^{1/2}XB^{1/2} \right\|_F \right) \leq \left( \sqrt{\|AX\|_F} - \sqrt{\|XB\|_F} \right)^2, \]

\[ (13) \]

\[ 2 \left( \int_0^1 f(v)\,dv - 2 \left\| A^{1/2}XB^{1/2} \right\|_F \right) \geq \left( \sqrt{\|AX\|_F} - \sqrt{\|XB\|_F} \right)^2. \]

The answer is no. We have the following result.
THEOREM 2.2. Inequalities (13) and (14) are not always true.

Proof. Firstly, we give an example to show that (13) is not always true. In fact, if we choose

\[ A = \begin{bmatrix} 8.6897 & 3.3580 \\ 3.3580 & 1.4082 \end{bmatrix}, \quad X = \begin{bmatrix} 3.6696 & 1.3801 \\ 2.682 & 1.8423 \end{bmatrix}, \quad B = \begin{bmatrix} 0.9331 & 1.6242 \\ 1.6242 & 18.2135 \end{bmatrix}. \]

then, we have

\[ 2 \left( \left\| A^{1/4}XB^{3/4} + A^{3/4}XB^{1/4} \right\|_F - 2 \left\| A^{1/2}XB^{1/2} \right\|_F \right) = 5.5623 \]

and

\[ \left( \sqrt{\left\| AX \right\|_F} - \sqrt{\left\| XB \right\|_F} \right)^2 = 0.0124. \]

Applying the Hermite-Hadamard inequality for convex function \( f \) on each of the subintervals \([0, \frac{1}{2}] \) and \([\frac{1}{2}, 1] \) and summing up side by side, we have

\[ f \left( \frac{1}{4} \right) \leq \int_0^1 f(y) \, dy. \]

So,

\[ 2 \left( \int_0^1 f(y) \, dy - 2 \left\| A^{1/2}XB^{1/2} \right\|_F \right) \geq 5.5623 \geq 0.0124. \]

On the other hand, the following example shows that (14) does not hold. Zou [5, Theorem 3.1] proved that

\[ \int_0^1 f(y) \, dy \leq \frac{4}{3} \left\| A^{1/2}XB^{1/2} \right\|_F + \frac{1}{3} \left\| AX + XB \right\|_F, \]

which implies

\[ 2 \left( \int_0^1 f(y) \, dy - 2 \left\| A^{1/2}XB^{1/2} \right\|_F \right) \leq \frac{2}{3} \left\| AX + XB \right\| - \frac{4}{3} \left\| A^{1/2}XB^{1/2} \right\|_F. \]

Let

\[ A = \begin{bmatrix} 0.3486 & 0.3686 \\ 0.3686 & 0.4376 \end{bmatrix}, \quad X = \begin{bmatrix} 0.4756 & 0.7881 \\ 0.3625 & 0.7803 \end{bmatrix}, \quad B = \begin{bmatrix} 0.2574 & 0.2898 \\ 0.2898 & 0.3274 \end{bmatrix}. \]

Then, we have

\[ \left\| AX + XB \right\|_F = 1.6591 \leq 1.8265 = 2 \left\| A^{1/2}XB^{1/2} \right\|_F + \frac{3}{2} \left( \sqrt{\left\| AX \right\|_F} - \sqrt{\left\| XB \right\|_F} \right)^2, \]

it follows that inequality (14) is not true for these matrices. This completes the proof. □

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REFERENCES


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