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A NOTE ON JORDAN–VON NEUMANN CONSTANT FOR $Z_{p,q}$ SPACE

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Abstract. Let $\lambda > 0$, $Z_{p,q}$ denote $\mathbb{R}^2$ endowed with the norm

$$|x|_{p,q} = (\|x\|_p^2 + \lambda \|x\|_q^2)^{\frac{1}{2}}.$$

Recently, von Neumann-Jordan constant $C_{NJ}(Z_{p,q})$ have been investigated under the two cases of a space $2 \leqslant p \leqslant q \leqslant \infty$ and $1 \leqslant p \leqslant q \leqslant 2$. For the case of $1 \leqslant p < 2 < q \leqslant \infty$, we only have shown an inequality on the constant. In this note, the exact value of the Jordan-von Neumann constant about this case is investigated.

1. Introduction and preliminaries

Let X be a non-trivial Banach space, and B_X and S_X denote the unit ball and unit sphere of X, respectively. Many geometric constants for a Banach space X have been investigated.

The von Neumann-Jordan constant (hereafter referred to as the NJ constant) of a Banach space X was introduced by Clarkson [2] as the smallest constant C for which

$$\frac{1}{C} \leqslant \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \leqslant C$$

holds for all $x, y \in X$ with $(x,y) \neq (0,0)$. An equivalent definition of the NJ constant is found in [6, 10] as the following form:

$$C_{NJ}(X) = \sup\left\{ \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x \in S_X, y \in B_X \right\}.$$

The following constant $C'_{NJ}(X)$, called the modified von Neumann-Jordan constant was introduced by Gao and Lau in [4]

$$C'_{NJ}(X) = \sup\left\{ \frac{\|x+y\|^2 + \|x-y\|^2}{4} : x, y \in S_X \right\}.$$

It is well known that $C'_{NJ}(X)$ doesn't necessarily coincide with $C_{NJ}(X)$ [1, 3, 5].

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Recall that a norm $\|\cdot\|$ on $\mathbb{R}^2$ is said to be absolute if $\|(x,y)\| = \|(|x|,|y|)\|$ for arbitrary $(x,y) \in \mathbb{R}^2$, and to be normalized if $\|(1,0)\| = \|(0,1)\| = 1$.

Let $\lambda > 0$, and $Z_{p,q}$ denote $\mathbb{R}^2$ endowed with the norm

$$|x|_{p,q} = (\|x\|_p^2 + \lambda \|x\|_q^2)^{\frac{1}{2}},$$

then by the definition, it is clear that $|\cdot|_{p,q}$ is absolute and $\|\cdot\|_{p,q} =: \frac{|\cdot|_{p,q}}{\sqrt{1+\lambda}}$ is an absolute normalized norm.

Let Ψ_2 denote the family of all continuous convex function ψ on $[0,1]$ such that $\psi(0) = \psi(1) = 1$ and $\max\{1-t,t\} \leqslant \psi(t) \leqslant 1$. It is well known that the set of all absolute normalized norm on $\mathbb{R}^2$ and Ψ_2 are in a one-to-one correspondence under the equation $\psi(t) = \|(1-t,t)\|$ [7, 9].

Recently, the exact values of NJ constant $C_{NJ}(Z_{p,q})$ under the cases of $2 \leqslant p \leqslant q \leqslant \infty$ and $1 \leqslant p \leqslant q \leqslant 2$ have been given as follows [12].

(i) If $2 \leqslant p \leqslant q \leqslant \infty$, then

$$C_{NJ}(Z_{p,q}) = \frac{2(\lambda+1)}{2^{\frac{2}{p}} + \lambda 2^{\frac{2}{q}}}.$$

(ii) If $1 \leqslant p \leqslant q \leqslant 2$, then

$$C_{NJ}(Z_{p,q}) = \frac{2^{\frac{2}{p}} + \lambda 2^{\frac{2}{q}}}{2(\lambda+1)}.$$

For the case of $1 \leqslant p \leqslant 2 \leqslant q \leqslant \infty$, we have the following inequalities on $C_{NJ}(Z_{p,q})$ [11].

THEOREM 1.1. *Let $\lambda > 0, Z_{p,q} = \mathbb{R}^2$ endowed with the norm*

$$|x|_{p,q} = (\|x\|_p^2 + \lambda \|x\|_q^2)^{\frac{1}{2}}.$$

If $1 \leqslant p \leqslant 2 \leqslant q \leqslant \infty$, then

$$\max\left\{\frac{2(\lambda+1)}{2^{\frac{2}{p}} + \lambda 2^{\frac{2}{q}}}, \frac{2^{\frac{2}{p}} + \lambda 2^{\frac{2}{q}}}{2(\lambda+1)}\right\} \leqslant C_{NJ}(Z_{p,q}) \leqslant \frac{2^{\frac{2}{p}} + 2\lambda}{2^{\frac{2}{q}}\lambda + 2}.$$

In this note, we consider the exact values of von Neumann-Jordan constants under the case of $1 \leqslant p < 2 < q \leqslant \infty$.

2. Main results and proofs

Before giving of our main results, we have the following Lemmas first.

LEMMA 2.1. *If $1 \leqslant p < 2 < q < \infty$, then*

$$\frac{(2-p)2^{\frac{2}{p}}}{(q-2)2^{\frac{2}{q}}} < \frac{2^{\frac{2}{p}} - 2}{2 - 2^{\frac{2}{q}}}.$$

Proof. We only need to show that

$$(2-p)+(q-2)2^{\frac{2}{q}-\frac{2}{p}}-(q-p)2^{\frac{2}{q}-1}<0.$$

Now letting $f(p)=(2-p)+(q-2)2^{\frac{2}{q}-\frac{2}{p}}-(q-p)2^{\frac{2}{q}-1}$, we have

$$f'(p)=-1+(q-2)2^{\frac{2}{q}-\frac{2}{p}}\frac{2\ln 2}{p^2}+2^{\frac{2}{q}-1},$$

and

$$f''(p)=(q-2)2^{\frac{2}{q}-\frac{2}{p}}\frac{4\ln 2}{p^4}(\ln 2-p)<0.$$

Hence

$$f'(p)>f'(2)=\frac{1}{4}[-4+2^{1+\frac{2}{q}}+2^{\frac{2}{q}}(q-2)\ln 2].$$

Assume that $h(q)=-4+2^{1+\frac{2}{q}}+2^{\frac{2}{q}}(q-2)\ln 2$, and we have $h'(q)=2^{\frac{2}{q}}\frac{(q-2)\ln 2}{q^2}(q+2-2\ln 2)>0$, so $h(q)>h(2)=0$. Therefore, $f'(p)>0$ and $f(p)<f(2)=0$. □

LEMMA 2.2. *If $1\leqslant p<2<q<\infty$ and $v>1$, then*
(i) $(2-p)(1+v^q)(1-v^{2-q})-(q-2)(1+v^p)(v^{2-p}-1)>0$;
(ii) $J(v)\equiv\frac{(1+v^p)^{\frac{2}{p}-1}(v^p-v^2)}{(1+v^q)^{\frac{2}{q}-1}(v^2-v^q)}$ *is increasing on* $(1,+\infty)$ *and* $J(v)>\frac{(2-p)2^{\frac{2}{p}}}{(q-2)2^{\frac{2}{q}}}$.

Proof. (i) Letting $F_v(t)=v^t-v^{2-t}$ and for any $v>1$ to be fixed, we can easily prove that $F_v(t)$ is a convex function of t. So we have

$$\frac{F_v(q)-F_v(2)}{q-2}>\frac{F_v(2)-F_v(p)}{2-p}$$

for $1\leqslant p<2\leqslant q<\infty$. Therefore (i) is valid.

(ii) Since $J(v)=\frac{(1+v^p)^{\frac{2}{p}-1}(v^{p-2}-1)}{(1+v^q)^{\frac{2}{q}-1}(1-v^{q-2})}$ and (i), we have

$$\begin{aligned}J'(v)=&\frac{(1+v^p)^{\frac{2}{p}-2}}{(1+v^q)^{\frac{2}{q}}}(1-v^{q-2})^{-2}\{(1+v^q)(1-v^{q-2})[(2-p)(v^{2p-3}-v^{p-1})\\&+(1+v^p)(p-2)v^{p-3}]-(1+v^p)(v^{p-2}-1)[(2-q)v^{q-1}(1-v^{q-2})\\&-(q-2)v^{q-3}(1+v^q)]\}\\=&\frac{(1+v^p)^{\frac{2}{p}-2}}{(1+v^q)^{\frac{2}{q}}}(1+v^2)(v^2-v^q)^{-2}v^{p+q-1}[(2-p)(1+v^q)(1-v^{2-q})\\&-(q-2)(1+v^p)(v^{2-p}-1)]>0.\end{aligned}$$

Hence $J(v)$ is increasing and $J(v)>\lim_{v\to+1}J(v)=\frac{(2-p)2^{\frac{2}{p}}}{(q-2)2^{\frac{2}{q}}}$. □

LEMMA 2.3. [8, 9] *Let* $\psi \in \Psi_2$ *and let* $M_1 = \max_{0 \leqslant t \leqslant 1} \frac{\psi(t)}{\psi_2(t)}$ *and* $M_2 = \max_{0 \leqslant t \leqslant 1} \frac{\psi_2(t)}{\psi(t)}$.

(i) Assume that $\psi \geqslant \psi_2$. *Then* $C_{NJ}(\|\cdot\|_\psi) = \max_{0 \leqslant t \leqslant 1} \frac{\psi(t)^2}{\psi_2(t)^2}$.

(ii) Let $\psi \in \Psi_2$ *and let* $\psi(t) = \psi(1-t)$ *for all* $t \in [0,1]$. *If* ψ/ψ_2 *attains a maximum or a minimum at* $t = 1/2$, *then* $C_{NJ}(\|\cdot\|_\psi) = M_1^2 M_2^2$.

(iii) Let $\psi \in \Psi_2$ *and let* $\psi(t) = \psi(1-t)$ *for all* $t \in [0,1]$. *If* ψ/ψ_2 *attains a maximum at* $t = 1/2$, *then* $C_{NJ}(\|\cdot\|_\psi) = C'_{NJ}(\|\cdot\|_\psi) = M_1^2 M_2^2$.

Now, we prove the following equalities on $C_{NJ}(Z_{p,q})$.

THEOREM 2.1. *Let* $\lambda > 0$, $1 \leqslant p < 2 < q < \infty$ *and let* $Z_{p,q} = \mathbb{R}^2$ *endowed with the norm*

$$|x|_{p,q} = (\|x\|_p^2 + \lambda \|x\|_q^2)^{\frac{1}{2}}.$$

(i) If $0 < \lambda \leqslant \frac{(2-p)2^{\frac{2}{p}}}{(q-2)2^{\frac{2}{q}}}$, *then*

$$C_{NJ}(Z_{p,q}) = C'_{NJ}(Z_{p,q}) = \frac{2^{\frac{2}{p}} + \lambda 2^{\frac{2}{q}}}{2(1+\lambda)}. \tag{2.1}$$

(ii) If $\frac{(2-p)2^{\frac{2}{p}}}{(q-2)2^{\frac{2}{q}}} < \lambda \leqslant \frac{2^{\frac{2}{p}}-2}{2-2^{\frac{2}{q}}}$, *then*

$$C_{NJ}(Z_{p,q}) = \frac{(1+v^p)^{\frac{2}{p}} + \lambda(1+v^q)^{\frac{2}{q}}}{(1+\lambda)(1+v^2)}, \tag{2.2}$$

where v *is the unique solution of the following equation*

$$\lambda = \frac{(1+v^p)^{\frac{2}{p}-1}(v^2 - v^p)}{(1+v^q)^{\frac{2}{q}-1}(v^q - v^2)}. \tag{2.3}$$

(iii) If $\frac{2^{\frac{2}{p}}-2}{2-2^{\frac{2}{q}}} \leqslant \lambda < \infty$, *then*

$$C_{NJ}(Z_{p,q}) = \frac{2[(1+v^p)^{\frac{2}{p}} + \lambda(1+v^q)^{\frac{2}{q}}]}{(2^{\frac{2}{p}} + \lambda 2^{\frac{2}{q}})(1+v^2)},$$

where v *is also the unique solution of the equation (2.3).*

Proof. Letting $f(u) = \frac{(1+u^p)^{\frac{2}{p}} + \lambda(1+u^q)^{\frac{2}{q}}}{1+u^2}$ for $u \in (0,1]$ and $w = \frac{1}{u}$, we have

$$f'(u) = \frac{2u}{(1+u^2)^2}\{(1+u^p)^{\frac{2}{p}-1}(u^{p-2}+u^p) + \lambda(1+u^q)^{\frac{2}{q}-1}(u^{q-2}+u^q)$$
$$-(1+u^p)^{\frac{2}{p}} - \lambda(1+u^q)^{\frac{2}{q}}\}$$

$$= \frac{2u}{(1+u^2)^2}\{\lambda(1+u^q)^{\frac{2}{q}-1}(u^{q-2}-1)-(1+u^p)^{\frac{2}{p}-1}(1-u^{p-2})\}$$
$$= \frac{2w}{(1+w^2)^2}(1+w^q)^{\frac{2}{q}-1}(w^2-w^q)\left\{\lambda-\frac{(1+w^p)^{\frac{2}{p}-1}(w^2-w^p)}{(1+w^q)^{\frac{2}{q}-1}(w^q-w^2)}\right\}$$

(i) If $0<\lambda\leqslant\frac{(2-p)2^{\frac{2}{p}}}{(q-2)2^{\frac{2}{q}}}$, then $f'(u)>0$ in $[0,1]$ by Lemma 2.2 (ii), so $f(u)$ attains a maximum at $u=1$ and a minimum at $u=0$. Since

$$\frac{\psi_{p,q}^2(t)}{\psi_2^2(t)} = \frac{(t^p+(1-t)^p)^{\frac{2}{p}}+\lambda(t^q+(1-t)^q)^{\frac{2}{q}}}{(t^2+(1-t)^2)(1+\lambda)}$$
$$= \frac{(1+u^p)^{\frac{2}{p}}+\lambda(1+u^q)^{\frac{2}{q}}}{(1+u^2)(1+\lambda)},$$

holds for $u=\frac{t}{1-t}$, and hence $\psi_{p,q}(t)\geqslant\psi_2(t)$ for any $0\leqslant t\leqslant 1$. So Lemma 2.3 (i) and (iii) imply (2.1).

(ii) If $\frac{(2-p)2^{\frac{2}{p}}}{(q-2)2^{\frac{2}{q}}}<\lambda\leqslant\frac{2^{\frac{2}{p}}-2}{2-2^{\frac{2}{q}}}$, then $f'(u)>0$ for $u\in(0,\frac{1}{v})$ (i.e $w\in(v,+\infty)$) and $f'(u)<0$ for $u\in(\frac{1}{v},1)$ (i.e $w\in(1,v)$) by Lemma 2.2 (ii). Hence $f(u)$ attains a maximum at $u=\frac{1}{v}$ and a minimum 1 at $u=0$, where v is the unique solution of (2.3). So Lemma 2.3 (i) implies

$$C_{NJ}(Z_{p,q}) = \frac{(1+v^p)^{\frac{2}{p}}+\lambda(1+v^q)^{\frac{2}{q}}}{(1+\lambda)(1+v^2)}.$$

Hence (2.2) is valid.

(iii) If $\frac{2^{\frac{2}{p}}-2}{2-2^{\frac{2}{q}}}\leqslant\lambda<\infty$, then $f(u)$ attains a maximum at $u=\frac{1}{v}$ again and a minimum at $u=1$, where v is the unique solution of (2.3). So ψ/ψ_2 attains a minimum at $t=1/2$, and hence we have

$$C_{NJ}(Z_{p,q}) = \frac{2[(1+v^p)^{\frac{2}{p}}+\lambda(1+v^q)^{\frac{2}{q}}]}{(2^{\frac{2}{p}}+\lambda 2^{\frac{2}{q}})(1+v^2)},$$

by Lemma 2.3 (ii). □

For $q=\infty$, we can prove similarly that

THEOREM 2.2. *Let $\lambda>0$, $1\leqslant p<2<q=\infty$ and let $Z_{p,q}=\mathbb{R}^2$ endowed with the norm*

$$|x|_{p,q} = (\|x\|_p^2+\lambda\|x\|_q^2)^{\frac{1}{2}}.$$

(i) If $0<\lambda\leqslant 2^{\frac{2}{p}}-2$, then

$$C_{NJ}(Z_{p,q}) = \frac{(1+v^p)^{\frac{2}{p}}+\lambda v^2}{(1+\lambda)(1+v^2)},$$

where v *is the unique solution of the following equation*

$$\lambda = \frac{(1+v^p)^{\frac{2}{p}-1}(v^2-v^p)}{v^2}. \tag{2.4}$$

(ii) If $2^{\frac{2}{p}} - 2 \leqslant \lambda < \infty$, *then*

$$C_{NJ}(Z_{p,q}) = \frac{2[(1+v^p)^{\frac{2}{p}} + \lambda v^2]}{(2^{\frac{2}{p}}+\lambda)(1+v^2)},$$

where v *is also the unique solution of the equation (2.4).*

In particular, we have

$$C_{NJ}(Z_{1,+\infty}) = \begin{cases} \frac{2+\lambda+\sqrt{\lambda^2+4}}{2(1+\lambda)} & \text{for} \quad 0 < \lambda \leqslant 2; \\ \frac{2+\lambda+\sqrt{\lambda^2+4}}{4+\lambda} & \text{for} \quad 2 \leqslant \lambda < \infty. \end{cases}$$

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