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TWO KINDS OF COMPOSITIONS OF HILBERT–HARDY–TYPE INTEGRAL OPERATORS AND THE RELATED INEQUALITIES

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Abstract. By the use of the way of Real and Functional Analysis and estimating the weight functions, we build some lemmas and deduce some Hilbert-Hardy-type integral inequalities. The equivalent forms and the reverses are all considered. Two kinds of Hilbert-Hardy-type integral operators are defined and the composition formulas of the operators are given.

1. Introduction

If $f(x), g(y) \geqslant 0$, $f, g \in L^2(\mathbf{R}_+) = \{f; ||f||_2 = (\int_0^\infty |f(x)|^2 dx)^{\frac{1}{2}} < \infty\}$, $||f||_2, ||g||_2 > 0$, then we have the following well known Hilbert's integral inequality and the equivalent form (cf. [1])

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \pi ||f||_2 ||g||_2, \tag{1.1}$$

$$\left[\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y} dx\right)^2 dy\right]^{\frac{1}{2}} < \pi ||f||_2, \tag{1.2}$$

where the constant factor π is the best possible.

In 1925, by introducing one pair of conjugate exponents (p,q) $(\frac{1}{p} + \frac{1}{q} = 1)$, Hardy *et al.* [2] gave extensions of (1.1) and (1.2) as follows: For $p > 1$, $f(x), g(y) \geqslant 0$, $f \in L^p(\mathbf{R}_+)$, $g \in L^q(\mathbf{R}_+)$, $||f||_p, ||g||_q > 0$, we have the following Hardy-Hilbert's integral inequality and the equivalent form:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\pi/p)} ||f||_p ||g||_q, \tag{1.3}$$

$$\left[\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y} dx\right)^p dy\right]^{\frac{1}{p}} < \frac{\pi}{\sin(\pi/p)} ||f||_p, \tag{1.4}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible.

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DEFINITION 1.1. If $\lambda \in \mathbf{R} = (-\infty, \infty)$, $\mathbf{R}_+ = (0, \infty)$, $k_\lambda(x,y)$ is a measurable function in $\mathbf{R}_+^2 = \mathbf{R}_+ \times \mathbf{R}_+$, satisfying for any $t, x, y \in \mathbf{R}_+$, $k_\lambda(tx, ty) = t^{-\lambda} k_\lambda(x,y)$, then we call $k_\lambda(x,y)$ as homogeneous function of degree $-\lambda$.

In 1934, by using a general non-negative homogeneous function of degree -1 $k_1(x,y)$, Hardy *et al*. [3] gave extensions of (1.3) and (1.4) as follows: For $p > 1$, $k_p = \int_0^\infty k_1(u,1) u^{\frac{-1}{p}} du \in \mathbf{R}_+$, $f(x), g(y) \geqslant 0$, $f \in L^p(\mathbf{R}_+)$, $g \in L^q(\mathbf{R}_+)$, $||f||_p, ||g||_q > 0$, we have the following Hardy-Hilbert-type integral inequality and the equivalent form:

$$\int_0^\infty \int_0^\infty k_1(x,y) f(x) g(y) dx dy < k_p ||f||_p ||g||_q, \tag{1.5}$$

$$\left[\int_0^\infty \left(\int_0^\infty k_1(x,y) f(x) dx\right)^p dy\right]^{\frac{1}{p}} < k_p ||f||_p, \tag{1.6}$$

where, the constant factor k_p is the best possible. Some applications of Hardy-Hilbert-type inequalities are provided in [4].

In 1998, by introducing an independent parameter $\lambda \in (0,1]$, Yang [5] gave an extension of (1.3) with the homogeneous kernel of degree $-\lambda$ as $\frac{1}{(x+y)^\lambda}$. In 2009, by using a general non-negative homogeneous function $k_\lambda(x,y)$ of degree $-\lambda$ and adding another pair of conjugate exponents (r,s) $(\frac{1}{r} + \frac{1}{s} = 1)$, Yang [6] gave extensions of (1.5) and (1.6) as follows: For $p, r > 1$, $\varphi(x) = x^{p(1-\frac{\lambda}{r})-1}$, $\psi(y) = y^{q(1-\frac{\lambda}{s})-1}$ $(x, y \in \mathbf{R}_+)$, $k_\lambda(r) = \int_0^\infty k_\lambda(u,1) u^{\frac{\lambda}{r}-1} du \in \mathbf{R}_+$, $f(x), g(y) \geqslant 0$,

$$f \in L_{p,\varphi}(\mathbf{R}_+) = \left\{f; ||f||_{p,\varphi} = \left(\int_0^\infty \varphi(x)|f(x)|^p dx\right)^{\frac{1}{p}} < \infty\right\},$$

$g \in L_{q,\psi}(\mathbf{R}_+)$, $||f||_{p,\varphi}, ||g||_{q,\psi} > 0$, we have the following Hilbert-type integral inequality and the equivalent form:

$$\int_0^\infty \int_0^\infty k_\lambda(x,y) f(x) g(y) dx dy < k_\lambda(r) ||f||_{p,\varphi} ||g||_{q,\psi}, \tag{1.7}$$

$$\left[\int_0^\infty y^{\frac{p\lambda}{s}-1} \left(\int_0^\infty k_\lambda(x,y) f(x) dx\right)^p dy\right]^{\frac{1}{p}} < k_\lambda(r) ||f||_{p,\varphi}, \tag{1.8}$$

where the constant factor $k_\lambda(r)$ is the best possible.

REMARK 1.2. When $\lambda = 1$, $r = q$, $s = p$, (1.7) and (1.8) reduce to (1.5) and (1.6). Hence, these Hilbert-type integral inequalities are best extensions of Hardy-Hilbert-type integral inequalities.

Using (1.2), we may define Hilbert's integral operator $T: L^2(\mathbf{R}_+) \to L^2(\mathbf{R}_+)$ as follows (cf. [7]): For any $f \in L^2(\mathbf{R}_+)$, there exists $Tf \in L^2(\mathbf{R}_+)$, satisfying

$$Tf(y) = \int_0^\infty \frac{f(x)}{x+y} dx \quad (y \in \mathbf{R}_+).$$

Then by (1.2), we have $||Tf||_2 \leqslant \pi||f||_2$, and T is a bounded linear operator satisfying $||T|| \leqslant \pi$. Since the constant factor in (1.2) is the best possible, we have $||T|| = \pi$.

About the discrete forms of (1.1) and (1.2), in 1950, Wilhelm [8] gave the operator expression. In 2002, by using the operator theory, Zhang [9] gave some improvements of (1.2) and the discrete form. In 2006–2009, Yang [10] considered a new Hilbert-type operator and its applications, and [11], [12] gave some multiple Hilbert-type operator expressions.

By using (1.8), we can define Hilbert-type integral operator $T : L_{p,\varphi}(\mathbf{R}_+) \to L_{p,\varphi}(\mathbf{R}_+)$ as follows (cf. [6]): For any $f \in L_{p,\varphi}(\mathbf{R}_+)$, there exists $Tf \in L_{p,\varphi}(\mathbf{R}_+)$, satisfying

$$Tf(y) = y^{\lambda-1}\int_0^\infty \frac{f(x)}{x+y}dx \quad (y \in \mathbf{R}_+).$$

Then by (1.8), we have $||Tf||_{p,\varphi} \leqslant k_\lambda(r)||f||_{p,\varphi}$, and T is a bounded linear operator satisfying $||T|| \leqslant k_\lambda(r)$. Since the constant factor in (1.8) is the best possible, we have $||T|| = k_\lambda(r)$.

About the topic of composition of two Hilbert-type operators, the main objective is to build the formula as $||T_1 \cdot T_2|| = ||T_1|| \cdot ||T_2||$. Recently, [13] published a composition of two discrete Hilbert-Hardy-type operators with the particular kernels. [14] published a composition of two half-discrete Hilbert-Hardy-type operators with the particular kernels, and [15], [16] published some composition of two Hilbert-Hardy-type integral operators with the particular kernels. These works are hard and interested.

In this paper, by the use of the way of Real and Functional Analysis and estimating the weight functions, we build some lemmas and deduce some Hilbert-Hardy-type integral inequalities. The equivalent forms and the reverses are all considered. Two kinds of Hilbert-Hardy-type integral operators are defined and the composition formulas are given, which are some extensions of the results of [15] and [16].

2. Some lemmas

In the following of this paper, we agree on that the parameters $p > 0 \ (p \neq 1)$, $\frac{1}{p} + \frac{1}{q} = 1$, $\mu, \sigma, \lambda > 0$, $\mu + \sigma = \lambda$.

LEMMA 2.1. (cf. [17], Lemma 2.2.5) *Suppose that* $\lambda \in A = (0, c) \ (0 < c \leqslant \infty)$, $k_\lambda^{(i)}(x,y) \ (i = 1,2,3)$ *are non-negative homogeneous functions of degree* $-\lambda$ *in* $\mathbf{R}_+^2$,

$$k_\lambda^{(i)}(\mu) := \int_0^\infty k_\lambda^{(i)}(u,1)u^{\mu-1}du, \tag{2.1}$$

there exists a constant $\delta_0 \in (0, \min\{\mu, \sigma\})$, *such that* $k_\lambda^{(i)}(\mu \pm \delta_0) \in \mathbf{R}_+ \ (i = 1,2,3)$. *Then for any* $\delta \in [0, \delta_0)$, *we have* $k_\lambda^{(i)}(\mu \pm \delta) \in \mathbf{R}_+$, *and*

$$\lim_{\delta \to 0^+} k_\lambda^{(i)}(\mu \pm \delta) = k_\lambda^{(i)}(\mu) \quad (i = 1,2,3). \tag{2.2}$$

As the assumptions of Lemma 2.1, we set the following Conditions:

CONDITION (I). For $\lambda \in A$, there exist constants $\delta_1 \in (0, \delta_0)$ and $L_1 > 0$, such that

$$k_\lambda^{(2)}(u,1)u^{\mu-\delta_1} \leqslant L_1, \quad (0 < u < 1) \qquad k_\lambda^{(3)}(u,1)u^{\mu+\delta_1} \leqslant L_1, \quad (u \in (1,\infty)). \quad (2.3)$$

CONDITION (II). For $\lambda \in (0,1) \cap A$, there exists a constant $L_2 > 0$, such that

$$k_\lambda^{(2)}(u,1)(1-u)^\lambda \leqslant L_2(u \in (0,1)), \quad k_\lambda^{(3)}(u,1)(u-1)^\lambda \leqslant L_2(u \in (1,\infty)). \quad (2.4)$$

CONDITION (III). For $\lambda \in (0,1) \cap A$, there exist constants $a \in (0,\lambda)$ and $L_3 > 0$, such that

$$k_\lambda^{(1)}(u,1)u^a \leqslant L_3(u \in (0,\infty)). \quad (2.5)$$

CONDITION (IV). For $\lambda \in (0, \frac{2}{3}) \cap A$, there exists a constant $L_4 > 0$, such that

$$k_\lambda^{(1)}(u,1)|1-u|^\lambda \leqslant L_4(u \in (0,\infty)). \quad (2.6)$$

EXAMPLE 2.2. For $\lambda \in A = \mathbf{R}_+,\ i = 2,3,$ the functions

$$k_\lambda^{(i)}(u,1) = \frac{1}{(u+1)^\lambda}, \ \frac{1}{u^\lambda+1}, \ \frac{\ln u}{u^\lambda - 1}, \ \frac{|\ln u|^{\beta-1}}{(\max\{u,1\})^\lambda} \quad (\beta \geqslant 1)$$

satisfy for using Condition (i) and (iii). In fact, for $b = \mu - \delta_1,\ \mu + \delta_1,\ a \in (0,\lambda)$, we have

$$\lim_{u \to 0^+} k_\lambda^{(i)}(u,1)u^b = \lim_{u \to \infty} k_\lambda^{(i)}(u,1)u^b = 0.$$

In view of the continuity, $k_\lambda^{(i)}(u,1)u^b$ is bounded in $(0,\infty)$.

For $A = (0,1)$, the functions

$$k_\lambda^{(i)}(u,1) = \frac{1}{|u-1|^\lambda} \quad (i = 1,2,3)$$

satisfy for using Condition (ii) and Condition (iv).

Note. In the following lemmas, theorems, corollaries and definitions, we agree on that the parameter λ is the common degree of $k_\lambda^{(i)}(u,1)$ $(i = 1,2,3)$, which satisfies all the possible using Conditions.

DEFINITION 2.3. As the assumptions of Lemma 2.1, define the following sequences of real functions:

$$\widetilde{F}_k(y) := \begin{cases} y^{\lambda-1} \int_1^\infty k_\lambda^{(2)}(x,y)x^{\mu-\frac{1}{pk}-1}dx, & y \in (1,\infty), \\ 0, & y \in (0,1], \end{cases}$$

$$\widetilde{G}_k(x) := \begin{cases} x^{\lambda-1} \int_1^\infty k_\lambda^{(3)}(x,y)y^{\sigma-\frac{1}{qk}-1}dy, & x \in (1,\infty), \\ 0, & x \in (0,1], \end{cases}$$

where, $k > \max\{\frac{1}{|q|\delta_1}, \frac{1}{p\delta_1}\}$ $(k \in \mathbf{N} = \{1,2,\cdots\})$.

Setting $u = x/y(y > 1)$, we find

$$\begin{aligned}\widetilde{F}_k(y) &= y^{\mu-\frac{1}{pk}-1}\int_{\frac{1}{y}}^{\infty} k_\lambda^{(2)}(u,1)u^{\mu-\frac{1}{pk}-1}du \\ &= y^{\mu-\frac{1}{pk}-1}\left(\int_0^{\infty} k_\lambda^{(2)}(u,1)u^{\mu-\frac{1}{pk}-1}du - \int_0^{\frac{1}{y}} k_\lambda^{(2)}(u,1)u^{\mu-\frac{1}{pk}-1}du\right) \\ &= y^{\mu-\frac{1}{pk}-1}k_\lambda^{(2)}\left(\mu-\frac{1}{pk}\right) - F(y), \qquad (2.7)\\ F(y) &:= y^{\mu-\frac{1}{pk}-1}\int_0^{\frac{1}{y}} k_\lambda^{(2)}(u,1)u^{\mu-\frac{1}{pk}-1}du \quad (y \in (1,\infty)).\end{aligned}$$

(a) If $k_\lambda^{(2)}(u,1)$ satisfies Condition (i) ($\lambda \in A$), then by (2.3), we have

$$0 \leqslant F(y) \leqslant L_1 y^{\mu-\frac{1}{pk}-1}\int_0^{\frac{1}{y}} u^{-\mu+\delta_1}u^{\mu-\frac{1}{pk}-1}du = \frac{L_1 y^{\mu-\delta_1-1}}{\delta_1-\frac{1}{pk}} \quad (y \in (1,\infty));$$

(b) if $k_\lambda^{(2)}(u,1)$ satisfies Condition (ii) ($\lambda \in (0,1)\cap A$), then by (2.4), we have

$$\begin{aligned}0 &\leqslant F(y) \leqslant L_2 y^{\mu-\frac{1}{pk}-1}\int_0^{\frac{1}{y}} \frac{u^{\mu-\frac{1}{pk}-1}}{(1-u)^\lambda}du \\ &\overset{v=yu}{=} L_2 y^{\lambda-1}\int_0^1 \frac{1}{(y-v)^\lambda}v^{\mu-\frac{1}{pk}-1}dv \\ &\leqslant \frac{L_2 y^{\lambda-1}}{(y-1)^\lambda}\int_0^1 v^{\mu-\frac{1}{pk}-1}dv = \frac{L_2}{\mu-\frac{1}{pk}}\frac{y^{\lambda-1}}{(y-1)^\lambda} \quad (y \in (1,\infty)).\end{aligned}$$

Still setting $u = x/y(x > 1)$, we obtain

$$\begin{aligned}\widetilde{G}_k(x) &= x^{\sigma-\frac{1}{qk}-1}\int_0^{x} k_\lambda^{(3)}(u,1)u^{\mu+\frac{1}{qk}-1}du \\ &= x^{\sigma-\frac{1}{qk}-1}\left(\int_0^{\infty} k_\lambda^{(3)}(u,1)u^{\mu+\frac{1}{qk}-1}du - \int_x^{\infty} k_\lambda^{(3)}(u,1)u^{\mu+\frac{1}{qk}-1}du\right) \\ &= x^{\sigma-\frac{1}{qk}-1}k_\lambda^{(3)}\left(\mu+\frac{1}{qk}\right) - G(x), \qquad (2.8)\\ G(x) &:= x^{\sigma-\frac{1}{qk}-1}\int_x^{\infty} k_\lambda^{(3)}(u,1)u^{\mu+\frac{1}{qk}-1}du \quad (x \in (1,\infty)).\end{aligned}$$

(c) If $k_\lambda^{(3)}(u,1)$ satisfies Condition (i) ($\lambda \in A$), then by (2.3), we have

$$0 \leqslant G(x) \leqslant L_1 x^{\sigma-\frac{1}{qk}-1}\int_x^{\infty} u^{-\mu-\delta_1}u^{\mu+\frac{1}{qk}-1}du = \frac{L_1 x^{\sigma-\delta_1-1}}{\delta_1-\frac{1}{qk}} \quad (x \in (1,\infty));$$

(d) if $k_\lambda^{(3)}(u,1)$ satisfies Condition (ii) ($\lambda \in (0,1)\cap A$), then by (2.4), we have

$$0 \leqslant G(x) \leqslant L_2 x^{\sigma-\frac{1}{qk}-1}\int_x^\infty \frac{u^{\mu+\frac{1}{qk}-1}}{(u-1)^\lambda}du \overset{v=x/u}{=} L_2 x^{\lambda-1}\int_0^1 \frac{v^{\sigma-\frac{1}{qk}-1}}{(x-v)^\lambda}dv$$
$$\leqslant \frac{L_2 x^{\lambda-1}}{(x-1)^\lambda}\int_0^1 v^{\sigma-\frac{1}{qk}-1}dv = \frac{L_2 x^{\lambda-1}}{(\sigma-\frac{1}{qk})(x-1)^\lambda} \quad (x\in(1,\infty)).$$

REMARK 2.4. In view of the results of the cases (a)–(d), there exists a large constant $L>0$, such that

(a) $F(y) \leqslant L y^{\mu-\delta_1-1}$ $(y\in(1,\infty);\ \lambda\in A)$;
(b) $F(y) \leqslant L\frac{y^{\lambda-1}}{(y-1)^\lambda}$ $(y\in(1,\infty);\ \lambda\in(0,1)\cap A)$;
(c) $G(x) \leqslant L x^{\sigma-\delta_1-1}$ $(x\in(1,\infty);\ \lambda\in A)$;
(d) $G(x) \leqslant L\frac{x^{\lambda-1}}{(x-1)^\lambda}$ $(x\in(1,\infty);\ \lambda\in(0,1)\cap A)$.

LEMMA 2.5. *As the assumptions of Lemma 2.1, if $k_\lambda^{(1)}(x,y)$ is a symmetric function such that for any $x,y\in\mathbf{R}_+$, $k_\lambda^{(1)}(y,x)=k_\lambda^{(1)}(x,y)$, $k_\lambda^{(2)}(u,1)$ $(k_\lambda^{(3)}(u,1))$ satisfies Condition (i) for $\lambda\in A$ or Condition (ii) for $\lambda\in(0,1)\cap A$, then we have*

$$\widetilde{L}_k := \frac{1}{k}\int_0^\infty\int_0^\infty k_\lambda^{(1)}(x,y)\widetilde{F}_k(y)\widetilde{G}_k(x)dydx \geqslant \prod_{i=1}^3 k_\lambda^{(i)}(\mu)+o(1)(k\to\infty). \tag{2.9}$$

Proof. In view of (2.7) and (2.8), we have

$$\widetilde{L}_k = \frac{1}{k}\int_1^\infty\int_1^\infty k_\lambda^{(1)}(x,y)\left[y^{\mu-\frac{1}{pk}-1}k_\lambda^{(2)}\left(\mu-\frac{1}{pk}\right)-F(y)\right]$$
$$\times\left[x^{\sigma-\frac{1}{qk}-1}k_\lambda^{(3)}\left(\mu+\frac{1}{qk}\right)-G(x)\right]dydx = I_1-I_2-I_3+I_4, \tag{2.10}$$

where, we define

$$I_1 := \frac{1}{k}k_\lambda^{(2)}\left(\mu-\frac{1}{pk}\right)k_\lambda^{(3)}\left(\mu+\frac{1}{qk}\right)\int_1^\infty\left(\int_1^\infty k_\lambda^{(1)}(x,y)y^{\mu-\frac{1}{pk}-1}dy\right)x^{\sigma-\frac{1}{qk}-1}dx,$$
$$I_2 := \frac{1}{k}k_\lambda^{(3)}\left(\mu+\frac{1}{qk}\right)\int_1^\infty\left(\int_1^\infty k_\lambda^{(1)}(x,y)F(y)dy\right)x^{\sigma-\frac{1}{qk}-1}dx,$$
$$I_3 := \frac{1}{k}k_\lambda^{(2)}\left(\mu-\frac{1}{pk}\right)\int_1^\infty\left(\int_1^\infty k_\lambda^{(1)}(x,y)y^{\mu-\frac{1}{pk}-1}dy\right)G(x)dx,$$
$$I_4 := \frac{1}{k}\int_1^\infty\left(\int_1^\infty k_\lambda^{(1)}(x,y)F(y)dy\right)G(x)dx.$$

It is evident that

$$I_1-I_2-I_3 \leqslant \widetilde{L}_k \leqslant I_1+I_4. \tag{2.11}$$

Since $k_\lambda^{(1)}(y,x) = k_\lambda^{(1)}(x,y)$, we obtain by Fubini theorem that (cf. [18])

$$\begin{aligned}
&\int_1^\infty \left(\int_1^\infty k_\lambda^{(1)}(x,y)y^{\mu-\frac{1}{pk}-1}dy\right)x^{\sigma-\frac{1}{qk}-1}dx \\
= &\int_1^\infty \left(\int_1^\infty k_\lambda^{(1)}(y,x)y^{\mu-\frac{1}{pk}-1}dy\right)x^{\sigma-\frac{1}{qk}-1}dx \\
\overset{y=xu}{=} &\int_1^\infty \left(\int_{\frac{1}{x}}^\infty k_\lambda^{(1)}(u,1)u^{\mu-\frac{1}{pk}-1}du\right)x^{-\frac{1}{k}-1}dx \\
= &\int_1^\infty \left(\int_{\frac{1}{x}}^1 k_\lambda^{(1)}(u,1)u^{\mu-\frac{1}{pk}-1}du\right)x^{-\frac{1}{k}-1}dx + \int_1^\infty \left(\int_1^\infty k_\lambda^{(1)}(u,1)u^{\mu-\frac{1}{pk}-1}du\right)x^{-\frac{1}{k}-1}dx \\
= &\int_0^1 \left(\int_{\frac{1}{u}}^\infty x^{-\frac{1}{k}-1}dx\right)k_\lambda^{(1)}(u,1)u^{\mu-\frac{1}{pk}-1}du + k\int_1^\infty k_\lambda^{(1)}(u,1)u^{\mu-\frac{1}{pk}-1}du \\
= &\ k\left(\int_0^1 k_\lambda^{(1)}(u,1)u^{\mu+\frac{1}{qk}-1}du + \int_1^\infty k_\lambda^{(1)}(u,1)u^{\mu-\frac{1}{pk}-1}du\right).
\end{aligned}$$

Since $\{k_\lambda^{(1)}(u,1)u^{\mu-\frac{1}{pk}-1}\}_{k=1}^\infty$ $(u \in (1,\infty))$ is increasing, by Levi theorem (cf. [18]), it follows that

$$\int_1^\infty k_\lambda^{(1)}(u,1)u^{\mu-\frac{1}{pk}-1}du \to \int_1^\infty k_\lambda^{(1)}(u,1)u^{\mu-1}du (k \to \infty).$$

Since $k_\lambda^{(1)}(u,1)u^{\mu+\frac{1}{qk}-1} \leqslant k_\lambda^{(1)}(u,1)u^{\mu-\delta_0-1}(u \in (0,1))$, and

$$0 \leqslant \int_0^1 k_\lambda^{(1)}(u,1)u^{\mu-\delta_0-1}du \leqslant \int_0^\infty k_\lambda^{(1)}(u,1)u^{\mu-\delta_0-1}du = k_\lambda^{(1)}(\mu-\delta_0) < \infty,$$

then by Lebesgue convergence control theorem (cf. [18]), we have

$$\int_0^1 k_\lambda^{(1)}(u,1)u^{\mu+\frac{1}{qk}-1}du \to \int_0^1 k_\lambda^{(1)}(u,1)u^{\mu-1}du (k \to \infty).$$

Hence, by Lemma 2.1, we find

$$\begin{aligned}
I_1 = &\ k_\lambda^{(2)}\left(\mu - \frac{1}{pk}\right)k_\lambda^{(3)}\left(\mu + \frac{1}{qk}\right)\left(\int_0^1 k_\lambda^{(1)}(u,1)u^{\mu+\frac{1}{qk}-1}du + \int_1^\infty k_\lambda^{(1)}(u,1)u^{\mu-\frac{1}{pk}-1}du\right) \\
&\to \prod_{i=1}^3 k_\lambda^{(i)}(\mu)(k \to \infty).
\end{aligned} \tag{2.12}$$

(1) We estimate I_2.

(a) If $k_\lambda^{(2)}(u,1)$ satisfies Condition (i) for $\lambda \in A$, then by Remark 2.4(a), we have

$$\begin{aligned} 0 &\leqslant J_2 := \int_1^\infty \left(\int_1^\infty k_\lambda^{(1)}(x,y)F(y)dy \right) x^{\sigma - \frac{1}{qk} - 1} dx \\ &\leqslant L \int_1^\infty \left(\int_1^\infty k_\lambda^{(1)}(y,x) y^{\mu - \delta_1 - 1} dy \right) x^{\sigma - \frac{1}{qk} - 1} dx \\ &\overset{y=xu}{=} L \int_1^\infty \left(\int_{\frac{1}{x}}^\infty k_\lambda^{(1)}(u,1) u^{\mu - \delta_1 - 1} du \right) x^{-\delta_1 - \frac{1}{qk} - 1} dx \\ &\leqslant L \int_1^\infty \left(\int_0^\infty k_\lambda^{(1)}(u,1) u^{\mu - \delta_1 - 1} du \right) x^{-\delta_1 - \frac{1}{qk} - 1} dx = \frac{L k_\lambda^{(1)}(\mu - \delta_1)}{\delta_1 + \frac{1}{qk}} < \infty; \end{aligned}$$

(b) if $k_\lambda^{(2)}(u,1)$ satisfies Condition (ii) for $\lambda \in (0,1) \cap A$, then by Remark 2.4(b), we have

$$\begin{aligned} 0 &\leqslant J_2 \leqslant L \int_1^\infty \left(\int_1^\infty k_\lambda^{(1)}(y,x) x^{\sigma - \frac{1}{qk} - 1} dx \right) \frac{y^{\lambda - 1}}{(y-1)^\lambda} dy \\ &\overset{y=xu}{=} L \int_1^\infty \left(\int_0^y k_\lambda^{(1)}(u,1) u^{\mu + \frac{1}{qk} - 1} du \right) \frac{y^{\sigma - \frac{1}{qk} - 1}}{(y-1)^\lambda} dy \\ &\leqslant L \int_0^1 \left(\int_0^\infty k_\lambda^{(1)}(u,1) u^{\mu + \frac{1}{qk} - 1} du \right) \frac{v^{\mu + \frac{1}{qk} - 1}}{(1-v)^\lambda} dv \\ &= L k_\lambda^{(1)} \left(\mu + \frac{1}{qk} \right) B\left(1 - \lambda, \mu + \frac{1}{qk} \right) < \infty. \end{aligned}$$

Therefore, in view of (a) and (b), we have $I_2 \to 0 \ (k \to \infty)$.

(2) We estimate I_3.

(c) If $k_\lambda^{(3)}(u,1)$ satisfies Condition (i) for $\lambda \in A$, then by Remark 2.4(c), we have

$$\begin{aligned} 0 &\leqslant J_3 := \int_1^\infty \left(\int_1^\infty k_\lambda^{(1)}(x,y) y^{\mu - \frac{1}{pk} - 1} dy \right) G(x) dx \\ &\leqslant L \int_1^\infty \left(\int_1^\infty k_\lambda^{(1)}(y,x) y^{\mu - \frac{1}{pk} - 1} dy \right) x^{\sigma - \delta_1 - 1} dx \\ &\overset{y=xu}{=} L \int_1^\infty \left(\int_{\frac{1}{x}}^\infty k_\lambda^{(1)}(u,1) u^{\mu - \frac{1}{pk} - 1} du \right) x^{-\delta_1 - \frac{1}{pk} - 1} dx \\ &\leqslant L \int_1^\infty \left(\int_0^\infty k_\lambda^{(1)}(u,1) u^{\mu - \frac{1}{pk} - 1} du \right) x^{-\delta_1 - \frac{1}{pk} - 1} dx = \frac{L k_\lambda^{(1)} \left(\mu - \frac{1}{pk} \right)}{\delta_1 + \frac{1}{pk}} < \infty; \end{aligned}$$

(d) if $k_\lambda^{(3)}(u,1)$ satisfies Condition (ii) for $\lambda \in (0,1)\cap A$, then by Remark 2.4(d), we have

$$\begin{aligned} 0 &\leqslant J_3 \leqslant L\int_1^\infty \left(\int_1^\infty k_\lambda^{(1)}(y,x)y^{\mu-\frac{1}{pk}-1}dy\right)\frac{x^{\lambda-1}}{(x-1)^\lambda}dx \\ &\overset{y=xu}{=} L\int_1^\infty \left(\int_{\frac{1}{x}}^\infty k_\lambda^{(1)}(u,1)u^{\mu-\frac{1}{pk}-1}du\right)\frac{x^{\mu-\frac{1}{pk}-1}}{(x-1)^\lambda}dx \\ &\leqslant L\int_0^1 \left(\int_0^\infty k_\lambda^{(1)}(u,1)u^{\mu-\frac{1}{pk}-1}du\right)\frac{v^{\sigma+\frac{1}{pk}-1}}{(1-v)^\lambda}dv \\ &= Lk_\lambda^{(1)}\left(\mu-\frac{1}{pk}\right)B\left(1-\lambda,\sigma+\frac{1}{pk}\right)<\infty. \end{aligned}$$

Therefore, in view of (c) and (d), we have $I_3 \to 0 \ (k\to\infty)$. By (2.11) and the above results, we have (2.9). □

LEMMA 2.6. *As the assumptions of Lemma 2.1, if $k_\lambda^{(1)}(x,y)$ is a symmetric function, $k_\lambda^{(2)}(u,1)$ $(k_\lambda^{(3)}(u,1))$ satisfies Condition (i) for $\lambda\in A$ or Condition (ii) for $\lambda\in(0,1)\cap A$, and if both $k_\lambda^{(2)}(u,1)$ and $k_\lambda^{(3)}(u,1)$ satisfy Condition (ii), then $k_\lambda^{(1)}(u,1)$ satisfies Condition (iii) for $\lambda\in(0,1)\cap A$ or (iv) for $\lambda\in(0,\frac{2}{3})\cap A$, then we have the reverse of (2.9), namely it follows*

$$\widetilde{L}_k=\frac{1}{k}\int_0^\infty\int_0^\infty k_\lambda^{(1)}(x,y)\widetilde{F}_k(y)\widetilde{G}_k(x)dydx=\prod_{i=1}^3 k_\lambda^{(i)}(\mu)+o(1)(k\to\infty). \tag{2.13}$$

Proof. We divide the following five cases to show $I_4\to 0 \ (k\to\infty)$.

Case (i). $\lambda\in A$, $F(y)\leqslant Ly^{\mu-\delta_1-1}$, $G(x)\leqslant Lx^{\sigma-\delta_1-1}$ $(y,x\in(1,\infty))$. We have

$$\begin{aligned} J_4 &:= \int_1^\infty\left(\int_1^\infty k_\lambda^{(1)}(x,y)F(y)dy\right)G(x)dx \\ &\leqslant L^2\int_1^\infty\left(\int_1^\infty k_\lambda^{(1)}(y,x)y^{\mu-\delta_1-1}dy\right)x^{\sigma-\delta_1-1}dx \\ &\overset{y=xu}{=} L^2\int_1^\infty\left(\int_{\frac{1}{x}}^\infty k_\lambda^{(1)}(u,1)u^{\mu-\delta_1-1}du\right)x^{-2\delta_1-1}dx \\ &\leqslant L^2\int_1^\infty\left(\int_0^\infty k_\lambda^{(1)}(u,1)u^{\mu-\delta_1-1}du\right)x^{-2\delta_1-1}dx=\frac{L^2}{2\delta_1}k_\lambda^{(1)}(\mu-\delta_1)<\infty. \end{aligned}$$

Case (ii). $\lambda\in(0,1)\cap A$, $F(y)\leqslant Ly^{\mu-\delta_1-1}$, $G(x)\leqslant L\frac{x^{\lambda-1}}{(x-1)^\lambda}$ $(y,x\in(1,\infty))$. We have

$$\begin{aligned} J_4 &\leqslant L^2\int_1^\infty\left(\int_1^\infty k_\lambda^{(1)}(y,x)y^{\mu-\delta_1-1}dy\right)\frac{x^{\lambda-1}}{(x-1)^\lambda}dx \\ &\overset{y=xu}{=} L^2\int_1^\infty\left(\int_{\frac{1}{x}}^\infty k_\lambda^{(1)}(u,1)u^{\mu-\delta_1-1}du\right)\frac{x^{\mu-\delta_1-1}}{(x-1)^\lambda}dx \end{aligned}$$

$$
\begin{aligned}
&\leqslant L^2 \int_1^\infty \left(\int_0^\infty k_\lambda^{(1)}(u,1) u^{\mu-\delta_1-1} du \right) \frac{x^{\mu-\delta_1-1}}{(x-1)^\lambda} dx \\
&= L^2 k_\lambda^{(1)}(\mu-\delta_1) B(1-\lambda, \sigma+\delta_1) < \infty.
\end{aligned}
$$

Case (iii). $\lambda \in (0,1) \cap A$, $F(y) \leqslant L \frac{y^{\lambda-1}}{(y-1)^\lambda}$, $G(x) \leqslant L x^{\sigma-\delta_1-1}$ $(y, x \in (1,\infty))$. We have

$$
\begin{aligned}
J_4 &\leqslant L^2 \int_1^\infty \left(\int_1^\infty k_\lambda^{(1)}(y,x) x^{\sigma-\delta_1-1} dx \right) \frac{y^{\lambda-1}}{(y-1)^\lambda} dy \\
&\overset{y=xu}{=} L^2 \int_1^\infty \left(\int_0^y k_\lambda^{(1)}(u,1) u^{\mu+\delta_1-1} du \right) \frac{y^{\sigma-\delta_1-1}}{(y-1)^\lambda} dy \\
&\leqslant L^2 \int_1^\infty \left(\int_0^\infty k_\lambda^{(1)}(u,1) u^{\mu+\delta_1-1} du \right) \frac{y^{\sigma-\delta_1-1}}{(y-1)^\lambda} dy \\
&= L^2 k_\lambda^{(1)}(\mu+\delta_1) B(1-\lambda, \mu+\delta_1) < \infty.
\end{aligned}
$$

Case (iv). $\lambda \in (0,1) \cap A$, $F(y) \leqslant L \frac{y^{\lambda-1}}{(y-1)^\lambda}$, $G(x) \leqslant L \frac{x^{\lambda-1}}{(x-1)^\lambda}$ $(y, x \in (1,\infty))$, and $k_\lambda^{(1)}(u,1)$ satisfies Condition (iii). We have

$$
\begin{aligned}
J_4 &\leqslant L^2 \int_1^\infty \left(\int_1^\infty x^{-\lambda} k_\lambda^{(1)}\left(\frac{y}{x},1\right) \frac{y^{\lambda-1}}{(y-1)^\lambda} dy \right) \frac{x^{\lambda-1}}{(x-1)^\lambda} dx \\
&\leqslant L^2 L_3 \int_1^\infty \left(\int_1^\infty x^{-\lambda} \left(\frac{y}{x}\right)^{-a} \frac{y^{\lambda-1}}{(y-1)^\lambda} dy \right) \frac{x^{\lambda-1}}{(x-1)^\lambda} dx \\
&= L^2 L_3 \int_1^\infty \left(\int_1^\infty \frac{y^{\lambda-a-1}}{(y-1)^\lambda} dy \right) \frac{x^{a-1}}{(x-1)^\lambda} dx \\
&= L^2 L_3 B(1-\lambda, a) B(1-\lambda, \lambda-a) < \infty.
\end{aligned}
$$

Case (v). $\lambda \in (0, \frac{2}{3}) \cap A$, $F(y) \leqslant L \frac{y^{\lambda-1}}{(y-1)^\lambda}$, $G(x) \leqslant L \frac{x^{\lambda-1}}{(x-1)^\lambda}$ $(y, x \in (1,\infty))$, and $k_\lambda^{(1)}(u,1)$ satisfies Condition (iv). We have

$$
\begin{aligned}
J_4 &\leqslant L^2 L_4 \int_1^\infty \left(\int_1^\infty \frac{1}{|x-y|^\lambda} \frac{y^{\lambda-1}}{(y-1)^\lambda} dy \right) \frac{x^{\lambda-1}}{(x-1)^\lambda} dx \\
&= L^2 L_4 \int_1^\infty \left(\int_1^x \frac{1}{(x-y)^\lambda} \frac{y^{\lambda-1}}{(y-1)^\lambda} dy \right) \frac{x^{\lambda-1}}{(x-1)^\lambda} dx \\
&\quad + L^2 L_4 \int_1^\infty \left(\int_x^\infty \frac{1}{(y-x)^\lambda} \frac{y^{\lambda-1}}{(y-1)^\lambda} dy \right) \frac{x^{\lambda-1}}{(x-1)^\lambda} dx
\end{aligned}
$$

$$
\begin{aligned}
&= L^2L_4\int_1^\infty\left(\int_y^\infty\frac{1}{(x-y)^\lambda}\frac{x^{\lambda-1}}{(x-1)^\lambda}dx\right)\frac{y^{\lambda-1}}{(y-1)^\lambda}dy\\
&\quad +L^2L_4\int_1^\infty\left(\int_x^\infty\frac{1}{(y-x)^\lambda}\frac{y^{\lambda-1}}{(y-1)^\lambda}dy\right)\frac{x^{\lambda-1}}{(x-1)^\lambda}dx\\
&\overset{x=yu}{=} 2L^2L_4\int_1^\infty\left(\int_1^\infty\frac{1}{(u-1)^\lambda}\frac{u^{\lambda-1}}{(yu-1)^\lambda}du\right)\frac{y^{\lambda-1}}{(y-1)^\lambda}dy\\
&= 2L^2L_4\int_1^\infty\left(\int_1^\infty\frac{1}{(u-1)^\lambda}\frac{u^{\lambda-1}}{(yu-1)^{\lambda/2}(yu-1)^{\lambda/2}}du\right)\frac{y^{\lambda-1}}{(y-1)^\lambda}dy\\
&\leqslant 2L^2L_4\int_1^\infty\left(\int_1^\infty\frac{1}{(u-1)^\lambda}\frac{u^{\lambda-1}}{(u-1)^{\lambda/2}(y-1)^{\lambda/2}}du\right)\frac{y^{\lambda-1}}{(y-1)^\lambda}dy\\
&= 2L^2L_4\left[\int_1^\infty\frac{y^{\lambda-1}}{(y-1)^{(3\lambda)/2}}dy\right]^2\\
&= 2L^2L_4\left[B\left(1-\frac{3\lambda}{2},\frac{\lambda}{2}\right)\right]^2<\infty.
\end{aligned}
$$

Hence, in the above any case, $I_4=\frac{1}{k}J_4\to 0\ (k\to\infty)$. Therefore, by (2.11) and (2.12), we have the reverse of (2.9), and then (2.13) follows. □

We set $z=\frac{1}{y}$ in (2.9), and define the following function:

$$
\widehat{F}_k(z):=\begin{cases} z^{\lambda-1}\int_1^\infty k_\lambda^{(2)}(xz,1)x^{\mu-\frac{1}{pk}-1}dx, & z\in(0,1),\\ 0, & z\in[1,\infty).\end{cases}
$$

In view of Lemma 2.5 and Lemma 2.6, by calculation, we find

LEMMA 2.7. *As the assumptions of Lemma 2.1, if $k_\lambda^{(1)}(x,y)$ is a symmetric function, $k_\lambda^{(2)}(u,1)$ $(k_\lambda^{(3)}(u,1))$ satisfies Condition (i) for $\lambda\in A$ or Condition (ii) for $\lambda\in(0,1)\cap A$, then we have*

$$
\widetilde{L}_k=\frac{1}{k}\int_0^\infty\int_0^\infty k_\lambda^{(1)}(xz,1)\widehat{F}_k(z)\widetilde{G}_k(x)dzdx\geqslant\prod_{i=1}^3 k_\lambda^{(i)}(\mu)+o(1)(k\to\infty). \tag{2.14}
$$

Adding the condition that if both $k_\lambda^{(2)}(u,1)$ and $k_\lambda^{(3)}(u,1)$ satisfy Condition (ii), then $k_\lambda^{(1)}(u,1)$ satisfies Condition (iii) for $\lambda\in(0,1)\cap A$ or (iv) for $\lambda\in(0,\frac{2}{3})\cap A$, then we have the reverse of (2.14), namely it follows

$$
\widetilde{L}_k=\frac{1}{k}\int_0^\infty\int_0^\infty k_\lambda^{(1)}(xz,1)\widehat{F}_k(z)\widetilde{G}_k(x)dzdx=\prod_{i=1}^3 k_\lambda^{(i)}(\mu)+o(1)(k\to\infty). \tag{2.15}
$$

3. First kind of Hilbert-type integral inequalities

We set functions $\varphi(x):=x^{p(1-\mu)-1}$, $\psi(y):=y^{q(1-\sigma)-1}$ $(x,y\in\mathbf{R}_+)$ in the following.

THEOREM 3.1. *As the assumptions of Lemma 2.5, if* $p>1$, $f(x)$, $G(y)\geqslant 0$, $f\in L_{p,\varphi}(\mathbf{R}_+)$, $G\in L_{q,\psi}(\mathbf{R}_+)$, $||f||_{p,\varphi}$, $||G||_{q,\psi}>0$, *and*

$$F_\lambda(y):=\begin{cases} y^{\lambda-1}\int_0^\infty k_\lambda^{(2)}(x,y)f(x)dx, & y\in\{y\in\mathbf{R}_+;f(y)>0\},\\ 0, & y\in\{y\in\mathbf{R}_+;f(y)=0\},\end{cases}$$

then we have the following equivalent inequalities:

$$I:=\int_0^\infty\int_0^\infty k_\lambda^{(1)}(x,y)F_\lambda(y)G(x)dydx<\prod_{i=1}^2 k_\lambda^{(i)}(\mu)||f||_{p,\varphi}||G||_{q,\psi},\tag{3.1}$$

$$J:=\left[\int_0^\infty x^{p\sigma-1}\left(\int_0^\infty k_\lambda^{(1)}(x,y)F_\lambda(y)dy\right)^p dx\right]^{\frac{1}{p}}<\prod_{i=1}^2 k_\lambda^{(i)}(\mu)||f||_{p,\varphi},\tag{3.2}$$

where, the constant factor $\prod_{i=1}^2 k_\lambda^{(i)}(\mu)$ *is the best possible.*

In particular, for $g(y)\geqslant 0$, $g\in L_{q,\psi}(\mathbf{R}_+)$, $||g||_{q,\psi}>0$, and

$$G(x)=G_\lambda(x):=\begin{cases} x^{\lambda-1}\int_0^\infty k_\lambda^{(3)}(x,y)g(y)dy, & x\in\{x\in\mathbf{R}_+;g(x)>0\},\\ 0, & x\in\{x\in\mathbf{R}_+;g(x)=0\},\end{cases}$$

we have the following inequality

$$\int_0^\infty\int_0^\infty k_\lambda^{(1)}(x,y)F_\lambda(y)G_\lambda(x)dydx<\prod_{i=1}^3 k_\lambda^{(i)}(\mu)||f||_{p,\varphi}||g||_{q,\psi},\tag{3.3}$$

where, the constant factor $\prod_{i=1}^3 k_\lambda^{(i)}(\mu)$ is still the best possible. If we only use Condition (i), then $\lambda\in A$; otherwise, $\lambda\in(0,1)\cap A$.

Proof. Since $k_\lambda^{(1)}(y,x)=k_\lambda^{(1)}(x,y)$, by (1.8) (for $\mu=\frac{\lambda}{r}$, $\sigma=\frac{\lambda}{s}$), we have

$$J=\left[\int_0^\infty y^{p\sigma-1}\left(\int_0^\infty k_\lambda^{(1)}(x,y)F_\lambda(x)dx\right)^p dy\right]^{\frac{1}{p}}\leqslant k_\lambda^{(1)}(\mu)||F_\lambda||_{p,\varphi},\tag{3.4}$$

$$\begin{aligned}||F_\lambda||_{p,\varphi}&=\left[\int_0^\infty y^{p(1-\mu)-1}\left(y^{\lambda-1}\int_0^\infty k_\lambda^{(2)}(x,y)f(x)dx\right)^p dy\right]^{\frac{1}{p}}\\ &=\left[\int_0^\infty y^{p\sigma-1}\left(\int_0^\infty k_\lambda^{(2)}(x,y)f(x)dx\right)^p dy\right]^{\frac{1}{p}}<k_\lambda^{(2)}(\mu)||f||_{p,\varphi}.\end{aligned}\tag{3.5}$$

Then we have (3.2). By Hölder's inequality (cf. [19]), we have

$$I = \int_0^\infty \left(x^{\sigma-\frac{1}{p}} \int_0^\infty k_\lambda^{(1)}(x,y)F_\lambda(y)dy \right) (x^{-\sigma+\frac{1}{p}} G(x))dx \leqslant J||G||_{q,\psi}. \tag{3.6}$$

Then by (3.2), we have (3.1).

On the other hand, suppose that (3.1) is valid. Setting

$$G(x) := y^{p\sigma-1} \left(\int_0^\infty k_\lambda^{(1)}(x,y)F_\lambda(x)dx \right)^{p-1} \quad (x \in \mathbf{R}_+),$$

we find $||G||_{q,\psi}^q = J^p$. If $J = 0$, then (3.2) is trivially valid; if $J = \infty$, then by (3.4), we have $||F_\lambda||_{p,\varphi} = \infty$, which contradicts the fact of (3.5). Assuming that $0 < J < \infty$, then by (3.1), we have

$$||G||_{q,\psi}^q = J^p = I < \prod_{i=1}^2 k_\lambda^{(i)}(\mu)||f||_{p,\varphi}||G||_{q,\psi},$$

$$||G||_{q,\psi}^{q-1} = J < \prod_{i=1}^2 k_\lambda^{(i)}(\mu)||f||_{p,\varphi},$$

then we have (3.2), which is equivalent to (3.1).

For any $k > \max\{\frac{1}{|q|\delta_1}, \frac{1}{p\delta_1}\}$ $(k \in \mathbf{N})$, we set $\widetilde{f}(x) = \widetilde{g}(y) = 0$ $(x,y \in (0,1])$; $\widetilde{f}(x) = x^{\mu-\frac{1}{pk}-1}$, $\widetilde{g}(y) = y^{\sigma-\frac{1}{qk}-1}$ $(x,y \in (1,\infty))$. Then we have $\widetilde{F}_k(y) = \widetilde{G}_k(x) = 0$ $(x,y \in (0,1])$;

$$\widetilde{F}_k(y) = y^{\lambda-1} \int_1^\infty k_\lambda^{(2)}(x,y)x^{\mu-\frac{1}{pk}-1}dx = y^{\lambda-1} \int_0^\infty k_\lambda^{(2)}(x,y)\widetilde{f}(x)dx,$$

$$\widetilde{G}_k(x) = x^{\lambda-1} \int_1^\infty k_\lambda^{(3)}(x,y)y^{\sigma-\frac{1}{qk}-1}dy = x^{\lambda-1} \int_0^\infty k_\lambda^{(3)}(x,y)\widetilde{g}(y)dy$$

$$(x,y \in (1,\infty)).$$

If there exists a positive constant $K \leqslant \prod_{i=1}^3 k_\lambda^{(i)}(\mu)$, such that (3.3) is valid when replacing $\prod_{i=1}^3 k_\lambda^{(i)}(\mu)$ by K, then in particular, we have

$$\widetilde{L}_k = \frac{1}{k} \int_0^\infty \int_0^\infty k_\lambda^{(1)}(x,y)\widetilde{F}_k(y)\widetilde{G}_k(x)dydx < \frac{1}{k}K||\widetilde{f}||_{p,\varphi}||\widetilde{g}||_{q,\psi} = K.$$

By (2.9), we find $\prod_{i=1}^3 k_\lambda^{(i)}(\mu) + o(1) \leqslant \widetilde{L}_k < K$, and then $\prod_{i=1}^3 k_\lambda^{(i)}(\mu) \leqslant K$ $(k \to \infty)$. Hence $K = \prod_{i=1}^3 k_\lambda^{(i)}(\mu)$ is the best possible constant factor of (3.3).

The constant factor in (3.1) is the best possible. Otherwise, setting $G(x) = \widetilde{G}_\lambda(x)$, we would reach a contradiction that the constant factor in (3.3) is not the best possible. By the equivalency, if the constant factor in (3.2) is not the best possible, then we would reach a contradiction that the constant factor in (3.1) is not the best possible. □

In this paper, we call (3.3) with the reverse and the related inequalities as Fist kind of Hilbert-type inequalities, which contain three homogeneous kernels $k_\lambda^{(i)}(x,y)$ $(i = 1,2,3)$.

THEOREM 3.2. *As the assumptions of Lemma 2.6, if* $0 < p < 1,\ f(x), G(y) \geqslant 0,$ $f \in L_{p,\varphi}(\mathbf{R}_+),\ G \in L_{q,\psi}(\mathbf{R}_+),\ ||f||_{p,\varphi}, ||G||_{q,\psi} > 0,$ *and*

$$F_\lambda(y) := \begin{cases} y^{\lambda-1}\int_0^\infty k_\lambda^{(2)}(x,y)f(x)dx, & y \in \{y \in \mathbf{R}_+; f(y) > 0\}, \\ 0, & y \in \{y \in \mathbf{R}_+; f(y) = 0\}, \end{cases}$$

then we have the equivalent reverses of (3.1) and (3.2), where, the constant factor $\prod_{i=1}^2 k_\lambda^{(i)}(\mu)$ *is the best possible. In particular, for* $g(y) \geqslant 0,\ g \in L_{q,\psi}(\mathbf{R}_+),\ ||g||_{q,\psi} > 0,$ *and*

$$G(x) = G_\lambda(x) := \begin{cases} x^{\lambda-1}\int_0^\infty k_\lambda^{(3)}(x,y)g(y)dy, & x \in \{x \in \mathbf{R}_+; g(x) > 0\}, \\ 0, & x \in \{x \in \mathbf{R}_+; g(x) = 0\}, \end{cases}$$

we have the reverse of (3.3) with the best possible constant factor $\prod_{i=1}^3 k_\lambda^{(i)}(\mu)$.

Proof. Since $k_\lambda^{(1)}(y,x) = k_\lambda^{(1)}(x,y)$, by the reverse Hölder's inequality, we obtain the reverses of (3.4) and (3.5). Then we deduce to the reverse of (3.2). By the reverse Hölder's inequality (cf. [19]), we have

$$I = \int_0^\infty \left(x^{\sigma-\frac{1}{p}} \int_0^\infty k_\lambda^{(1)}(x,y)F_\lambda(y)dy \right)(x^{-\sigma+\frac{1}{p}}G(x))dx \geqslant J||G||_{q,\psi}. \tag{3.7}$$

Then by the reverse of (3.2), we have the reverse of (3.1).

On the other hand, suppose that the reverse of (3.1) is valid. Setting $G(x)$ as Theorem 3.1, we find $||G||_{q,\psi}^q = J^p$. If $J = \infty$, then the reverse of (3.2) is trivially valid; if $J = 0$, then by reverse of (3.4), we have $||F_\lambda||_{p,\varphi} = 0$, which contradicts the fact of the reverse of (3.5). Assuming that $0 < J < \infty$, then by the reverse of (3.1), we have

$$||G||_{q,\psi}^q = J^p = I > \prod_{i=1}^2 k_\lambda^{(i)}(\mu)||f||_{p,\varphi}||G||_{q,\psi},$$

$$||G||_{q,\psi}^{q-1} = J > \prod_{i=1}^2 k_\lambda^{(i)}(\mu)||f||_{p,\varphi},$$

and the reverse of (3.2) follows, which is equivalent to reverse of (3.1).

For any $k > \max\{\frac{1}{|q|\delta_1}, \frac{1}{p\delta_1}\}$ $(k \in \mathbf{N})$, we set $\widetilde{f}(x)$, $\widetilde{g}(y)$, $\widetilde{F}_k(y)$, $\widetilde{G}_k(x)$ as Theorem 3.1. If there exists a positive constant $K \geqslant \prod_{i=1}^3 k_\lambda^{(i)}(\mu)$, such that the reverse of (3.3) is valid when replacing $\prod_{i=1}^3 k_\lambda^{(i)}(\mu)$ by K, then in particular, we have

$$\widetilde{L}_k = \frac{1}{k}\int_0^\infty \int_0^\infty k_\lambda^{(1)}(x,y)\widetilde{F}_k(y)\widetilde{G}_k(x)dydx > \frac{1}{k}K||\widetilde{f}||_{p,\varphi}||\widetilde{g}||_{q,\psi} = K.$$

By (2.13), we find $\prod_{i=1}^3 k_\lambda^{(i)}(\mu) + o(1) = \widetilde{L}_k > K$, and then $\prod_{i=1}^3 k_\lambda^{(i)}(\mu) \geqslant K$ $(k \to \infty)$. Hence $K = \prod_{i=1}^3 k_\lambda^{(i)}(\mu)$ is the best possible constant factor of the reverse of (3.3).

The constant factor in the reverse of (3.1) is the best possible. Otherwise, setting $G(x) = \widetilde{G}_\lambda(x)$, we would reach a contradiction that the constant factor in the reverse of (3.3) is not the best possible. By the equivalency, if the constant factor in the reverse of (3.2) is not the best possible, then by (3.7), we would reach a contradiction that the constant factor in the reverse of (3.1) is not the best possible. $\square$

4. Second kind of Hilbert-type integral inequalities

In the same way, applying Lemma 2.7, for $\mu = \sigma = \frac{\lambda}{2}$, $\Phi(x) := x^{p(1-\frac{\lambda}{2})-1}$, $\Psi(y) := y^{q(1-\frac{\lambda}{2})-1}$, we have

THEOREM 4.1. *As the assumptions of Lemma 2.5 (for $\mu = \sigma = \frac{\lambda}{2}$, we don't assume that $k_\lambda^{(1)}(x,y)$ is a symmetric function), if $p > 1$, $f(x), G(y) \geqslant 0$, $f \in L_{p,\Phi}\ (\mathbf{R}_+)$, $G \in L_{q,\Psi}\ (\mathbf{R}_+)$, $||f||_{p,\Phi}$, $||G||_{q,\Psi} > 0$, and*

$$\overline{F}_\lambda(y) := \begin{cases} y^{\lambda-1}\int_0^\infty k_\lambda^{(2)}(xy,1)f(x)dx, & y \in \{y \in \mathbf{R}_+; f(\frac{1}{y}) > 0\}, \\ 0, & y \in \{y \in \mathbf{R}_+; f(\frac{1}{y}) = 0\}, \end{cases}$$

then we have the following equivalent inequalities:

$$\int_0^\infty \int_0^\infty k_\lambda^{(1)}(xy,1)\overline{F}_\lambda(y)G(x)dydx < \prod_{i=1}^2 k_\lambda^{(i)}\left(\frac{\lambda}{2}\right)||f||_{p,\Phi}||G||_{q,\Psi} \tag{4.1}$$

$$\left[\int_0^\infty x^{\frac{p\lambda}{2}-1}\left(\int_0^\infty k_\lambda^{(1)}(xy,1)\overline{F}_\lambda(y)dy\right)^p dx\right]^{\frac{1}{p}} < \prod_{i=1}^2 k_\lambda^{(i)}\left(\frac{\lambda}{2}\right)||f||_{p,\Phi}, \tag{4.2}$$

where, the constant factor $\prod_{i=1}^2 k_\lambda^{(i)}(\frac{\lambda}{2})$ is the best possible.

In particular, for $g(y) \geqslant 0$, $g \in L_{q,\Psi}(\mathbf{R}_+)$, $||g||_{q,\Psi} > 0$, and

$$G(x) = G_\lambda(x) := \begin{cases} x^{\lambda-1}\int_0^\infty k_\lambda^{(3)}(x,y)g(y)dy, & x \in \{x \in \mathbf{R}_+; g(x) > 0\}, \\ 0, & x \in \{x \in \mathbf{R}_+; g(x) = 0\}, \end{cases}$$

we have the following inequality

$$\int_0^\infty \int_0^\infty k_\lambda^{(1)}(xy,1)\overline{F}_\lambda(y)G_\lambda(x)dydx < \prod_{i=1}^3 k_\lambda^{(i)}\left(\frac{\lambda}{2}\right)||f||_{p,\Phi}||g||_{q,\Psi}, \tag{4.3}$$

where, the constant factor $\prod_{i=1}^3 k_\lambda^{(i)}(\frac{\lambda}{2})$ is still the best possible.

Proof. We only prove that the constant factor in (4.3) is the best possible. The others are omitted.

For any $k > \max\{\frac{1}{|q|\delta_1}, \frac{1}{p\delta_1}\}$ $(k \in \mathbf{N})$, we set $\widetilde{f}(x) = 0$ $(x \in (0,1])$; $\widetilde{f}(x) = x^{\frac{\lambda}{2}-\frac{1}{pk}-1}$ $(x \in (1,\infty))$, $\widetilde{g}(y) = y^{\frac{\lambda}{2}-\frac{1}{qk}-1}$ $(y \in (0,1))$; $\widetilde{g}(y) = 0$ $(y \in [1,\infty))$. Then it follows

$$\begin{aligned}
\widetilde{G}_k(x) &= 0 \quad (x \in (0,1]);\\
\widetilde{G}_k(x) &= x^{\lambda-1}\int_1^\infty k_\lambda^{(3)}(x,y)y^{\frac{\lambda}{2}-\frac{1}{qk}-1}dy = x^{\lambda-1}\int_0^\infty k_\lambda^{(3)}(x,y)\widetilde{g}(y)dy \quad (x \in (1,\infty)),\\
\widehat{F}_k(y) &:= y^{\lambda-1}\int_1^\infty k_\lambda^{(2)}(xy,1)x^{\frac{\lambda}{2}-\frac{1}{pk}-1}dx = y^{\lambda-1}\int_0^\infty k_\lambda^{(2)}(xy,1)\widetilde{f}(x)dx \quad (y \in (0,1),\\
\widehat{F}_k(y) &:= 0 \quad (y \in [1,\infty)).
\end{aligned}$$

If there exists a positive constant $K \leqslant \prod_{i=1}^3 k_\lambda^{(i)}(\frac{\lambda}{2})$, such that (4.3) is valid when replacing $\prod_{i=1}^3 k_\lambda^{(i)}$ $(\frac{\lambda}{2})$ to K, then in particular, we have

$$\widetilde{L}_k = \frac{1}{k}\int_0^\infty\int_0^\infty k_\lambda^{(1)}(xy,1)\widehat{F}_k(y)\widetilde{G}_k(x)dydx < \frac{1}{k}K||\widetilde{f}||_{p,\Phi}||\widetilde{g}||_{q,\Psi} = K.$$

By (2.14), we find

$$\prod_{i=1}^3 k_\lambda^{(i)}\left(\frac{\lambda}{2}\right) + o(1) \leqslant \widetilde{L}_k < K (k \to \infty),$$

and then $\prod_{i=1}^3 k_\lambda^{(i)}(\frac{\lambda}{2}) \leqslant K$ $(k \to \infty)$. Hence $K = \prod_{i=1}^3 k_\lambda^{(i)}(\frac{\lambda}{2})$ is the best possible constant factor of (4.3). □

In this paper, we call (4.3) with the reverse and the related inequalities as Second kind of Hilbert-type inequalities, which contain two non-homogeneous kernels $k_\lambda^{(i)}(xy,1)$ $(i = 1,2)$ and a homogeneous kernel $k_\lambda^{(3)}(x,y)$.

In the same way, by using (2.15), we still have

THEOREM 4.2. *As the assumptions of Lemma 2.6 (for $\mu = \sigma = \frac{\lambda}{2}$, we don't assume that $k_\lambda^{(1)}(x,y)$ is a symmetric function), if $0 < p < 1$, $f(x), G(y) \geqslant 0$, $f \in L_{p,\Phi}(\mathbf{R}_+)$, $G \in L_{q,\Psi}(\mathbf{R}_+)$, $||f||_{p,\Phi}, ||G||_{q,\Psi} > 0$, and*

$$F_\lambda(y) := \begin{cases} y^{\lambda-1}\int_0^\infty k_\lambda^{(2)}(xy,1)f(x)dx, & y \in \{y \in \mathbf{R}_+; f(\frac{1}{y}) > 0\},\\ 0, & y \in \{y \in \mathbf{R}_+; f(\frac{1}{y}) = 0\}, \end{cases}$$

then we have the equivalent reverses of (4.1) and (4.2), where, the constant factor $\prod_{i=1}^2 k_\lambda^{(i)}(\frac{\lambda}{2})$ is the best possible.

In particular, for $g(y) \geqslant 0$, $g \in L_{q,\Psi}(\mathbf{R}_+)$, $||g||_{q,\Psi} > 0$, and

$$G(x) = G_\lambda(x) := \begin{cases} x^{\lambda-1}\int_0^\infty k_\lambda^{(3)}(x,y)g(y)dy, & x \in \{x \in \mathbf{R}_+; g(x) > 0\},\\ 0, & x \in \{x \in \mathbf{R}_+; g(x) = 0\}, \end{cases}$$

we have the reverse of (4.3) with the best possible constant factor $\prod_{i=1}^3 k_\lambda^{(i)}(\frac{\lambda}{2})$.

5. Some corollaries on Hilbert-Hardy-type inequalities

In this section, if in a Hilbert-type inequality, the best possible constant factor is related to $k_{\lambda,j}^{(1)}(\mu)\ \ (j=1,2)$, then we call this inequality as Hilbert-Hardy-type inequality.

Assuming that $k_{\lambda}^{(1)}(x,y)=0\ \ (0<y\leqslant x)$, then $k_{\lambda}^{(1)}(u,1)=0\ \ (u\geqslant 1)$, and

$$k_{\lambda}^{(1)}(\mu)=k_{\lambda,1}^{(1)}(\mu):=\int_0^1 k_{\lambda}^{(1)}(u,1)u^{\mu-1}du. \tag{5.1}$$

By Theorem 3.1 and Theorem 3.2, we have

COROLLARY 5.1. *As the assumptions of Lemma 2.5, if* $p>1,\ k_{\lambda,1}^{(1)}(\mu)\in\mathbf{R}_+$, $f(x),G(y)\geqslant 0,\ f\in L_{p,\varphi}(\mathbf{R}_+),\ G\in L_{q,\psi}(\mathbf{R}_+),\ ||f||_{p,\varphi},||G||_{q,\psi}>0$, *and*

$$F_{\lambda}(y):=\begin{cases} y^{\lambda-1}\int_0^\infty k_{\lambda}^{(2)}(x,y)f(x)dx, & y\in\{y\in\mathbf{R}_+;f(y)>0\},\\ 0, & y\in\{y\in\mathbf{R}_+;f(y)=0\},\end{cases}$$

then we have the following equivalent inequalities:

$$\int_0^\infty G(x)\int_x^\infty k_{\lambda}^{(1)}(x,y)F_{\lambda}(y)dydx<k_{\lambda,1}^{(1)}(\mu)k_{\lambda}^{(2)}(\mu)||f||_{p,\varphi}||G||_{q,\psi}, \tag{5.2}$$

$$\left[\int_0^\infty x^{p\sigma-1}\left(\int_x^\infty k_{\lambda}^{(1)}(x,y)F_{\lambda}(y)dy\right)^p dx\right]^{\frac{1}{p}}<k_{\lambda,1}^{(1)}(\mu)k_{\lambda}^{(2)}(\mu)||f||_{p,\varphi}, \tag{5.3}$$

where, the constant factor $k_{\lambda,1}^{(1)}(\mu)k_{\lambda}^{(2)}(\mu)$ *is the best possible. In particular, for* $g(y)\geqslant 0,\ g\in L_{q,\psi}(\mathbf{R}_+),\ ||g||_{q,\psi}>0$, *and*

$$G(x)=G_{\lambda}(x):=\begin{cases} x^{\lambda-1}\int_0^\infty k_{\lambda}^{(3)}(x,y)g(y)dy, & x\in\{x\in\mathbf{R}_+;g(x)>0\},\\ 0, & x\in\{x\in\mathbf{R}_+;g(x)=0\},\end{cases}$$

we have the following inequality

$$\int_0^\infty\int_x^\infty k_{\lambda}^{(1)}(x,y)F_{\lambda}(y)G_{\lambda}(x)dydx<k_{\lambda,1}^{(1)}(\mu)\prod_{i=2}^{3}k_{\lambda}^{(i)}(\mu)||f||_{p,\varphi}||g||_{q,\psi}, \tag{5.4}$$

where, the constant factor $k_{\lambda,1}^{(1)}(\mu)\prod_{i=2}^{3}k_{\lambda}^{(i)}(\mu)$ *is still the best possible.*

As the assumptions of Lemma 2.6, if $0<p<1$, $k_{\lambda,1}^{(1)}(\mu)\in\mathbf{R}_+$, $f(x),G(y)\geqslant 0$, $f\in L_{p,\varphi}(\mathbf{R}_+),\ G\in L_{q,\psi}(\mathbf{R}_+),\ ||f||_{p,\varphi},||G||_{q,\psi}>0$, and

$$F_{\lambda}(y):=\begin{cases} y^{\lambda-1}\int_0^\infty k_{\lambda}^{(2)}(x,y)f(x)dx, & y\in\{y\in\mathbf{R}_+;f(y)>0\},\\ 0, & y\in\{y\in\mathbf{R}_+;f(y)=0\},\end{cases}$$

then we have the equivalent reverses of (5.2) and (5.3), where, the constant $k_{\lambda,1}^{(1)}(\mu)k_{\lambda}^{(2)}(\mu)$ is the best possible. In particular, for $g(y) \geqslant 0,\ g \in L_{q,\psi}(\mathbf{R}_+),\ ||g||_{q,\psi} > 0$, and

$$G(x) = G_\lambda(x) := \begin{cases} x^{\lambda-1}\int_0^\infty k_\lambda^{(3)}(x,y)g(y)dy, & x \in \{x \in \mathbf{R}_+; g(x) > 0\}, \\ 0, & x \in \{x \in \mathbf{R}_+; g(x) = 0\}, \end{cases}$$

we have the reverse of (5.4) with the best possible constant factor $k_{\lambda,1}^{(1)}(\mu)\prod_{i=2}^{3}k_\lambda^{(i)}(\mu)$.

Assuming that $k_\lambda^{(1)}(x,y) = 0\ \ (0 < x \leqslant y)$, then $k_\lambda^{(1)}(u,1) = 0\ \ (0 < u \leqslant 1)$, and

$$k_\lambda^{(1)}(\mu) = k_{\lambda,2}^{(1)}(\mu) := \int_1^\infty k_\lambda^{(1)}(u,1)u^{\mu-1}du. \tag{5.5}$$

By Theorem 3.1 and Theorem 3.2, we have

COROLLARY 5.2. *As the assumptions of Lemma 2.5, if* $p > 1,\ k_{\lambda,2}^{(1)}(\mu) \in \mathbf{R}_+$, $f(x), G(y) \geqslant 0,\ f \in L_{p,\varphi}(\mathbf{R}_+),\ G \in L_{q,\psi}(\mathbf{R}_+),\ ||f||_{p,\varphi}, ||G||_{q,\psi} > 0$, *and*

$$F_\lambda(y) := \begin{cases} y^{\lambda-1}\int_0^\infty k_\lambda^{(2)}(x,y)f(x)dx, & y \in \{y \in \mathbf{R}_+; f(y) > 0\}, \\ 0, & y \in \{y \in \mathbf{R}_+; f(y) = 0\}, \end{cases}$$

then we have the following equivalent inequalities:

$$\int_0^\infty G(x)\int_0^x k_\lambda^{(1)}(x,y)F_\lambda(y)dydx < k_{\lambda,2}^{(1)}(\mu)k_\lambda^{(2)}(\mu)||f||_{p,\varphi}||G||_{q,\psi}, \tag{5.6}$$

$$\left[\int_0^\infty x^{p\sigma-1}\left(\int_0^x k_\lambda^{(1)}(x,y)F_\lambda(y)dy\right)^p dx\right]^{\frac{1}{p}} < k_{\lambda,2}^{(1)}(\mu)k_\lambda^{(2)}(\mu)||f||_{p,\varphi}, \tag{5.7}$$

where, the constant factor $k_{\lambda,2}^{(1)}(\mu)k_\lambda^{(2)}(\mu)$ *is the best possible. In particular, for* $g(y) \geqslant 0,\ g \in L_{q,\psi}(\mathbf{R}_+),\ ||g||_{q,\psi} > 0$, *and*

$$G(x) = G_\lambda(x) := \begin{cases} x^{\lambda-1}\int_0^\infty k_\lambda^{(3)}(x,y)g(y)dy, & x \in \{x \in \mathbf{R}_+; g(x) > 0\}, \\ 0, & x \in \{x \in \mathbf{R}_+; g(x) = 0\}, \end{cases}$$

we have the following inequality

$$\int_0^\infty G_\lambda(x)\int_0^x k_\lambda^{(1)}(x,y)F_\lambda(y)dydx < k_{\lambda,2}^{(1)}(\mu)\prod_{i=2}^{3}k_\lambda^{(i)}(\mu)||f||_{p,\varphi}||g||_{q,\psi}, \tag{5.8}$$

where, the constant factor $k_{\lambda,2}^{(1)}(\mu)\prod_{i=2}^{3}k_\lambda^{(i)}(\mu)$ *is still the best possible.*

As the assumptions of Lemma 2.6, if $0 < p < 1,\ k_{\lambda,2}^{(1)}(\mu) \in \mathbf{R}_+,\ f(x), G(y) \geqslant 0$, $f \in L_{p,\varphi}(\mathbf{R}_+),\ G \in L_{q,\psi}(\mathbf{R}_+),\ ||f||_{p,\varphi}, ||G||_{q,\psi} > 0$, and

$$F_\lambda(y) := \begin{cases} y^{\lambda-1}\int_0^\infty k_\lambda^{(2)}(x,y)f(x)dx, & y \in \{y \in \mathbf{R}_+; f(y) > 0\}, \\ 0, & y \in \{y \in \mathbf{R}_+; f(y) = 0\}, \end{cases}$$

then we have the equivalent reverses of (5.6) and (5.7), where, the constant $k_{\lambda,2}^{(1)}(\mu)k_{\lambda}^{(2)}(\mu)$ is the best possible. In particular, for $g(y) \geqslant 0$, $g \in L_{q,\psi}(\mathbf{R}_+)$, $||g||_{q,\psi} > 0$, and

$$G(x) = G_\lambda(x) := \begin{cases} x^{\lambda-1}\int_0^\infty k_\lambda^{(3)}(x,y)g(y)dy, & x \in \{x \in \mathbf{R}_+; g(x) > 0\}, \\ 0, & x \in \{x \in \mathbf{R}_+; g(x) = 0\}, \end{cases}$$

we have the reverse of (5.8) with the best possible constant factor $k_{\lambda,2}^{(1)}(\mu)\prod_{i=2}^{3} k_\lambda^{(i)}(\mu)$.

Assuming that $k_\lambda^{(1)}(xy,1) = 0 \ \ (0 < \frac{1}{x} \leqslant y)$, then $k_\lambda^{(1)}(u,1) = 0 \ \ (u \geqslant 1)$, and $k_\lambda^{(1)}(\frac{\lambda}{2}) = k_{\lambda,1}^{(1)}(\frac{\lambda}{2})$. By Theorem 4.1 and Theorem 4.2, we have

COROLLARY 5.3. *As the assumptions of Theorem 4.1, if* $p > 1, k_{\lambda,1}^{(1)}(\frac{\lambda}{2}) \in \mathbf{R}_+$, $f(x), G(y) \geqslant 0$, $f \in L_{p,\Phi}(\mathbf{R}_+)$, $G \in L_{q,\Psi}(\mathbf{R}_+)$, $||f||_{p,\Phi}, ||G||_{q,\Psi} > 0$, *and*

$$\overline{F}_\lambda(y) := \begin{cases} y^{\lambda-1}\int_0^\infty k_\lambda^{(2)}(xy,1)f(x)dx, & y \in \{y \in \mathbf{R}_+; f(\frac{1}{y}) > 0\}, \\ 0, & y \in \{y \in \mathbf{R}_+; f(\frac{1}{y}) = 0\}, \end{cases}$$

then we have the following equivalent inequalities:

$$\int_0^\infty G(x)\int_0^{\frac{1}{x}} k_\lambda^{(1)}(xy,1)\overline{F}_\lambda(y)dydx < k_{\lambda,1}^{(1)}\left(\frac{\lambda}{2}\right)k_\lambda^{(2)}\left(\frac{\lambda}{2}\right)||f||_{p,\Phi}||G||_{q,\Psi}, \tag{5.9}$$

$$\left[\int_0^\infty x^{\frac{p\lambda}{2}-1}\left(\int_0^{\frac{1}{x}} k_\lambda^{(1)}(xy,1)\overline{F}_\lambda(y)dy\right)^p dx\right]^{\frac{1}{2}} < k_{\lambda,1}^{(1)}\left(\frac{\lambda}{2}\right)k_\lambda^{(2)}\left(\frac{\lambda}{2}\right)||f||_{p,\varphi}, \tag{5.10}$$

where, the constant factor $k_{\lambda,1}^{(1)}(\frac{\lambda}{2})k_\lambda^{(2)}(\frac{\lambda}{2})$ *is the best possible. In particular, for* $g(y) \geqslant 0$, $g \in L_{q,\Psi}(\mathbf{R}_+)$, $||g||_{q,\Psi} > 0$, *and*

$$G(x) = G_\lambda(x) := \begin{cases} x^{\lambda-1}\int_0^\infty k_\lambda^{(3)}(x,y)g(y)dy, & x \in \{x \in \mathbf{R}_+; g(x) > 0\}, \\ 0, & x \in \{x \in \mathbf{R}_+; g(x) = 0\}, \end{cases}$$

we have the following inequality

$$\int_0^\infty G_\lambda(x)\int_0^{\frac{1}{x}} k_\lambda^{(1)}(xy,1)\overline{F}_\lambda(y)dydx < k_{\lambda,1}^{(1)}\left(\frac{\lambda}{2}\right)\prod_{i=2}^{3} k_\lambda^{(i)}\left(\frac{\lambda}{2}\right)||f||_{p,\Phi}||g||_{q,\Psi}, \tag{5.11}$$

where, the constant factor $k_{\lambda,1}^{(1)}(\frac{\lambda}{2})\prod_{i=2}^{3} k_\lambda^{(i)}(\frac{\lambda}{2})$ *is still the best possible.*

As the assumptions of Theorem 4.2, if $0 < p < 1$, $k_{\lambda,1}^{(1)}(\frac{\lambda}{2}) \in \mathbf{R}_+$, $f(x), G(y) \geqslant 0$, $f \in L_{p,\Phi}(\mathbf{R}_+)$, $G \in L_{q,\Psi}(\mathbf{R}_+)$, $||f||_{p,\Phi}, ||G||_{q,\Psi} > 0$, and

$$F_\lambda(y) := \begin{cases} y^{\lambda-1}\int_0^\infty k_\lambda^{(2)}(xy,1)f(x)dx, & y \in \{y \in \mathbf{R}_+; f(\frac{1}{y}) > 0\}, \\ 0, & y \in \{y \in \mathbf{R}_+; f(\frac{1}{y}) = 0\}, \end{cases}$$

then we have the equivalent reverses of (5.9) and (5.10), where, the constant $k_{\lambda,1}^{(1)}(\frac{\lambda}{2})k_{\lambda}^{(2)}(\frac{\lambda}{2})$ is the best possible. In particular, for $g(y) \geqslant 0$, $g \in L_{q,\Psi}(\mathbf{R}_+)$, $||g||_{q,\Psi} > 0$, and

$$G(x) = G_\lambda(x) := \begin{cases} x^{\lambda-1}\int_0^\infty k_\lambda^{(3)}(x,y)g(y)dy, & x \in \{x \in \mathbf{R}_+; g(x) > 0\}, \\ 0, & x \in \{x \in \mathbf{R}_+; g(x) = 0\}, \end{cases}$$

we have the reverse of (5.11) with the best possible constant factor $k_{\lambda,1}^{(1)}(\frac{\lambda}{2})\prod_{i=2}^{3}k_{\lambda}^{(i)}(\frac{\lambda}{2})$.

Assuming that $k_\lambda^{(1)}(xy,1) = 0$ $(\frac{1}{x} \geqslant y > 0)$, then $k_\lambda^{(1)}(u,1) = 0$ $(0 < u \leqslant 1)$, and $k_\lambda^{(1)}(\frac{\lambda}{2}) = k_{\lambda,2}^{(1)}(\frac{\lambda}{2})$. By Theorem 3 and Theorem 4, we have

COROLLARY 5.4. *As the assumptions of Theorem 4.1, if* $p > 1$, $k_{\lambda,2}^{(1)}(\frac{\lambda}{2}) \in \mathbf{R}_+$, $f(x), G(y) \geqslant 0$, $f \in L_{p,\Phi}(\mathbf{R}_+)$, $G \in L_{q,\Psi}(\mathbf{R}_+)$, $||f||_{p,\Phi}, ||G||_{q,\Psi} > 0$, *and*

$$\overline{F}_\lambda(y) := \begin{cases} y^{\lambda-1}\int_0^\infty k_\lambda^{(2)}(xy,1)f(x)dx, & y \in \{y \in \mathbf{R}_+; f(\frac{1}{y}) > 0\}, \\ 0, & y \in \{y \in \mathbf{R}_+; f(\frac{1}{y}) = 0\}, \end{cases}$$

then we have the following equivalent inequalities:

$$\int_0^\infty G(x)\int_{\frac{1}{x}}^\infty k_\lambda^{(1)}(xy,1)\overline{F}_\lambda(y)dydx < k_{\lambda,2}^{(1)}\left(\frac{\lambda}{2}\right)k_\lambda^{(2)}\left(\frac{\lambda}{2}\right)||f||_{p,\Phi}||G||_{q,\Psi} \tag{5.12}$$

$$\left[\int_0^\infty x^{\frac{p\lambda}{2}-1}\left(\int_{\frac{1}{x}}^\infty k_\lambda^{(1)}(xy,1)\overline{F}_\lambda(y)dy\right)^p dx\right]^{\frac{1}{2}} < k_{\lambda,2}^{(1)}\left(\frac{\lambda}{2}\right)k_\lambda^{(2)}\left(\frac{\lambda}{2}\right)||f||_{p,\Phi}, \tag{5.13}$$

where, the constant factor $k_{\lambda,2}^{(1)}(\frac{\lambda}{2})k_\lambda^{(2)}(\frac{\lambda}{2})$ *is the best possible. In particular, for* $g(y) \geqslant 0$, $g \in L_{q,\psi}(\mathbf{R}_+)$, $||g||_{q,\psi} > 0$, *and*

$$G(x) = G_\lambda(x) := \begin{cases} x^{\lambda-1}\int_0^\infty k_\lambda^{(3)}(x,y)g(y)dy, & x \in \{x \in \mathbf{R}_+; g(x) > 0\}, \\ 0, & x \in \{x \in \mathbf{R}_+; g(x) = 0\}, \end{cases}$$

we have the following inequality

$$\int_0^\infty G_\lambda(x)\int_{\frac{1}{x}}^\infty k_\lambda^{(1)}(xy,1)\overline{F}_\lambda(y)dydx < k_{\lambda,2}^{(1)}\left(\frac{\lambda}{2}\right)\prod_{i=2}^{3}k_\lambda^{(i)}\left(\frac{\lambda}{2}\right)||f||_{p,\Phi}||g||_{q,\Psi}, \tag{5.14}$$

where, the constant factor $k_{\lambda,2}^{(1)}(\frac{\lambda}{2})\prod_{i=2}^{3}k_\lambda^{(i)}(\frac{\lambda}{2})$ *is still the best possible.*

As the assumptions of Theorem 4.2, if $0 < p < 1$, $k_{\lambda,2}^{(1)}(\frac{\lambda}{2}) \in \mathbf{R}_+$, $f(x), G(y) \geqslant 0$, $f \in L_{p,\Phi}(\mathbf{R}_+)$, $G \in L_{q,\Psi}(\mathbf{R}_+)$, $||f||_{p,\Phi}, ||G||_{q,\Psi} > 0$, and

$$F_\lambda(y) := \begin{cases} y^{\lambda-1}\int_0^\infty k_\lambda^{(2)}(xy,1)f(x)dx, & y \in \{y \in \mathbf{R}_+; f(\frac{1}{y}) > 0\}, \\ 0, & y \in \{y \in \mathbf{R}_+; f(\frac{1}{y}) = 0\}, \end{cases}$$

then we have the equivalent reverses of (5.12) and (5.13), where, the constant $k^{(1)}_{\lambda,2}(\frac{\lambda}{2})k^{(2)}_{\lambda}(\frac{\lambda}{2})$ is the best possible. In particular, for $g(y) \geqslant 0$, $g \in L_{q,\Psi}(\mathbf{R}_+)$, $||g||_{q,\Psi} > 0$, and

$$G(x) = G_\lambda(x) := \begin{cases} x^{\lambda-1}\int_0^\infty k^{(3)}_\lambda(x,y)g(y)dy, & x \in \{x \in \mathbf{R}_+; g(x) > 0\}, \\ 0, & x \in \{x \in \mathbf{R}_+; g(x) = 0\}, \end{cases}$$

we have the reverse of (5.14) with the best possible constant factor $k^{(1)}_{\lambda,2}(\frac{\lambda}{2})\prod_{i=2}^{3} k^{(i)}_{\lambda}(\frac{\lambda}{2})$.

EXAMPLE 5.5. (i) If $k^{(1)}_\lambda(xy,1) = \frac{|\ln xy|^{\beta-1}}{(\max\{xy,1\})^\lambda}$ $(\lambda > 0, \beta \geqslant 1)$, then we find

$$\begin{aligned} k^{(1)}_\lambda\left(\frac{\lambda}{2}\right) &= \int_0^\infty k^{(1)}_\lambda(u,1)u^{\frac{\lambda}{2}-1}du = \int_0^\infty \frac{|\ln u|^{\beta-1}}{(\max\{u,1\})^\lambda}u^{\frac{\lambda}{2}-1}du \\ &= 2\int_0^1(-\ln u)^{\beta-1}u^{\frac{\lambda}{2}-1}du \overset{v=-\ln u}{=} 2\int_0^\infty e^{-\frac{\lambda}{2}v}v^{\beta-1}dv \\ &= 2\left(\frac{2}{\lambda}\right)^\beta\Gamma(\beta), k^{(1)}_{\lambda,1}\left(\frac{\lambda}{2}\right) = k^{(1)}_{\lambda,2}(\mu) = \left(\frac{2}{\lambda}\right)^\beta\Gamma(\beta). \end{aligned}$$

(ii) If $k^{(1)}_\lambda(xy,1) = \frac{1}{|xy-1|^\lambda}$ $(0 < \lambda < 1)$, then we find

$$\begin{aligned} k^{(1)}_\lambda\left(\frac{\lambda}{2}\right) &= \int_0^\infty k^{(1)}_\lambda(u,1)u^{\frac{\lambda}{2}-1}du = \int_0^\infty \frac{1}{|u-1|^\lambda}u^{\frac{\lambda}{2}-1}du \\ &= 2\int_0^1 \frac{1}{(1-u)^\lambda}u^{\frac{\lambda}{2}-1}du = 2B\left(1-\lambda,\frac{\lambda}{2}\right), \\ k^{(1)}_{\lambda,1}\left(\frac{\lambda}{2}\right) &= k^{(1)}_{\lambda,2}\left(\frac{\lambda}{2}\right) = B\left(1-\lambda,\frac{\lambda}{2}\right). \end{aligned}$$

REMARK 5.6. For $x > 0$, setting $A_{x,0} := (0,\infty)$, $A_{x,1} := (x,\infty)$, $A_{x,2} := (0,x)$, by (3.2), (5.3) and (5.7), setting $k^{(1)}_{\lambda,0}(\frac{\lambda}{2}) := k^{(1)}_\lambda(\frac{\lambda}{2})$, we have the following united expression of Hilbert-Hardy-type inequalities:

$$\left[\int_0^\infty x^{p\sigma-1}\left(\int_{A_{x,i}} k^{(1)}_\lambda(x,y)F_\lambda(y)dy\right)^p dx\right]^{\frac{1}{2}} < k^{(1)}_{\lambda,i}(\mu)k^{(2)}_\lambda(\mu)||f||_{p,\varphi}, \tag{5.15}$$

where, the constant factor $k^{(1)}_{\lambda,i}(\mu)k^{(2)}_\lambda(\mu)$ $(i = 0,1,2)$ is the best possible.

If $k^{(2)}_\lambda(x,y) = 0$ $(y \in \mathbf{R}_+ \backslash A_{x,1})$, then $k^{(2)}_\lambda(u,1) = 0$ $(u \geqslant 1)$,

$$k^{(2)}_\lambda(\mu) = k^{(2)}_{\lambda,1}(\mu) := \int_0^1 k^{(2)}_\lambda(u,1)u^{\mu-1}du;$$

if $k_\lambda^{(2)}(x,y)=0 \ (y\in\mathbf{R}_+\backslash A_{x,2})$, then $k_\lambda^{(2)}(u,1)=0 \ (0<u\leqslant 1)$,

$$k_\lambda^{(2)}(\mu)=k_{\lambda,2}^{(2)}(\mu):=\int_1^\infty k_\lambda^{(2)}(u,1)u^{\mu-1}du.$$

Assuming that $k_{\lambda,0}^{(2)}(\mu):=k_\lambda^{(2)}(\mu),\ k_{\lambda,i}^{(1)}(\mu),k_{\lambda,j}^{(2)}(\mu)\in\mathbf{R}_+$, for $i,j=0,1,2,$ setting

$$F_{\lambda,j}(y):=\begin{cases} y^{\lambda-1}\int_{A_{x,j}}k_\lambda^{(2)}(x,y)f(x)dx, & y\in\{y\in\mathbf{R}_+;f(y)>0\},\\ 0, & y\in\{y\in\mathbf{R}_+;f(y)=0\},\end{cases}$$

then it follows that $F_{\lambda,0}(y)=F_\lambda(y)$, and by (5.15), we have

$$\left[\int_0^\infty x^{p\sigma-1}\left(\int_{A_{x,i}}k_\lambda^{(1)}(x,y)F_{\lambda,j}(y)dy\right)^p dx\right]^{\frac{1}{2}}<k_{\lambda,i}^{(1)}(\mu)k_{\lambda,j}^{(2)}(\mu)||f||_{p,\varphi}, \tag{5.16}$$

where, the constant factor $k_{\lambda,i}^{(1)}(\mu)k_{\lambda,j}^{(2)}(\mu)\ (i,j=0,1,2)$ is the best possible.

We still can find similar to (3.4) and (3.5) that

$$\left[\int_0^\infty x^{p\sigma-1}\left(\int_{A_{x,i}}k_\lambda^{(1)}(x,y)F_{\lambda,j}(y)dy\right)^p dx\right]^{\frac{1}{p}}\leqslant k_{\lambda,i}^{(1)}(\mu)||F_{\lambda,j}||_{p,\varphi}, \tag{5.17}$$

$$||F_{\lambda,j}||_{p,\varphi}\leqslant k_{\lambda,j}^{(2)}(\mu)||f||_{p,\varphi}\quad (j=0,1,2), \tag{5.18}$$

where, the constant factors $k_{\lambda,i}^{(1)}(\mu)$ and $k_{\lambda,j}^{(2)}(\mu)$ are the best possible.

REMARK 5.7. For $x>0$, setting $B_{x,0}:=(0,\infty)$, $B_{x,1}:=(0,\frac{1}{x})$, $B_{x,2}:=(\frac{1}{x},\infty)$, by (4.2), (5.10) and (5.13), for $i=0,1,2,$ we have the following united expression of Hilbert-Hardy-type inequalities:

$$\left[\int_0^\infty x^{\frac{p\lambda}{2}-1}\left(\int_{B_{x,i}}^\infty k_\lambda^{(1)}(xy,1)\overline{F}_\lambda(y)dy\right)^p dx\right]^{\frac{1}{2}}<k_{\lambda,i}^{(1)}\left(\frac{\lambda}{2}\right)k_\lambda^{(2)}\left(\frac{\lambda}{2}\right)||f||_{p,\Phi}, \tag{5.19}$$

where, the constant factor $k_{\lambda,i}^{(1)}(\frac{\lambda}{2})k_\lambda^{(2)}(\frac{\lambda}{2})$ is the best possible.

If $k_\lambda^{(2)}(xy,1)=0\ (y\in\mathbf{R}_+\backslash B_{x,1})$, then $k_\lambda^{(2)}(u,1)=0\ (u\geqslant 1)$, $k_\lambda^{(2)}(\frac{\lambda}{2})=k_{\lambda,1}^{(2)}(\frac{\lambda}{2})$; if $k_\lambda^{(2)}(xy,1)=0\ (y\in\mathbf{R}_+\backslash B_{x,2})$, then $k_\lambda^{(2)}(u,1)=0\ (0<u\leqslant 1)$, $k_\lambda^{(2)}(\frac{\lambda}{2})=k_{\lambda,2}^{(2)}(\frac{\lambda}{2})$. Assuming that $k_{\lambda,i}^{(1)}(\frac{\lambda}{2})$, $k_{\lambda,j}^{(2)}(\frac{\lambda}{2})\in\mathbf{R}_+$, for $i,j=0,1,2,$ setting

$$\overline{F}_{\lambda,j}(y):=\begin{cases} y^{\lambda-1}\int_{B_{x,j}}k_\lambda^{(2)}(xy,1)f(x)dx, & y\in\{y\in\mathbf{R}_+;f(\frac{1}{y})>0\},\\ 0, & y\in\{y\in\mathbf{R}_+;f(\frac{1}{y})=0\},\end{cases}$$

then it follows that $\overline{F}_{\lambda,0}(y) = \overline{F}_{\lambda}(y)$, and by (5.19), we have

$$\left[\int_0^\infty x^{\frac{p\lambda}{2}-1}\left(\int_{B_{x,i}} k_\lambda^{(1)}(xy,1)\overline{F}_{\lambda,j}(y)dy\right)^p dx\right]^{\frac{1}{2}} < k_{\lambda,i}^{(1)}\left(\frac{\lambda}{2}\right)k_{\lambda,j}^{(2)}\left(\frac{\lambda}{2}\right)||f||_{p,\Phi}, \quad (5.20)$$

where, the constant factor $k_{\lambda,i}^{(1)}(\frac{\lambda}{2})k_{\lambda,j}^{(2)}(\frac{\lambda}{2})$ $(i,j=0,1,2)$ is the best possible.

We still can find that

$$\left[\int_0^\infty x^{\frac{p\lambda}{2}-1}\left(\int_{B_{x,i}} k_\lambda^{(1)}(xy,1)\overline{F}_{\lambda,j}(y)dy\right)^p dx\right]^{\frac{1}{p}} \leqslant k_{\lambda,i}^{(1)}\left(\frac{\lambda}{2}\right)||\overline{F}_{\lambda,j}||_{p,\Phi}, \quad (5.21)$$

$$||\overline{F}_{\lambda,j}||_{p,\Phi} \leqslant k_{\lambda,j}^{(2)}\left(\frac{\lambda}{2}\right)||f||_{p,\Phi} \quad (j=0,1,2), \quad (5.22)$$

where, the constant factors $k_{\lambda,i}^{(1)}(\frac{\lambda}{2})$ and $k_{\lambda,j}^{(2)}(\frac{\lambda}{2})$ are the best possible.

6. Two kinds of compositions of Hilbert-Hardy-type integral operators

For $F \in L_{p,\varphi}(\mathbf{R}_+)$, we set $h(x) := x^{\lambda-1}\int_{A_{x,i}} k_\lambda^{(1)}(x,y)F(y)dy$ $(x \in \mathbf{R}_+)$. Then by (5.17), we have

$$||h||_{p,\varphi} \leqslant k_{\lambda,i}^{(1)}(\mu)||F||_{p,\varphi}. \quad (6.1)$$

DEFINITION 6.1. As the assumptions of Theorem 3.1, define a Hilbert-Hardy-type operator $T_1^{(i)} : L_{p,\varphi}(\mathbf{R}_+) \to L_{p,\varphi}(\mathbf{R}_+)$ as follows: For any $F \in L_{p,\varphi}(\mathbf{R}_+)$, there exists a unified expression $T_1^{(i)}F = h \in L_{p,\varphi}(\mathbf{R}_+)$, such that for any $x \in \mathbf{R}_+$, $T_1^{(i)}F(x) = h(x)$.

By (6.1), we have $||T_1^{(i)}F||_{p,\varphi} \leqslant k_{\lambda,i}^{(1)}(\mu)||F||_{p,\varphi}$. Hence $T_1^{(i)}$ is a bounded linear operator with

$$||T_1^{(i)}|| := \sup_{F(\neq\theta)\in L_{p,\varphi}(\mathbf{R}_+)} \frac{||T_1^{(i)}F||_{p,\varphi}}{||F||_{p,\varphi}} \leqslant k_{\lambda,i}^{(1)}(\mu).$$

Since the constant factor in (6.1) is the best possible, we have $||T_1^{(i)}|| = k_{\lambda,i}^{(1)}(\mu)$.

DEFINITION 6.2. As the assumptions of Theorem 3.1, define a Hilbert-Hardy-type operator $T_2^{(j)} : L_{p,\varphi}(\mathbf{R}_+) \to L_{p,\varphi}(\mathbf{R}_+)$ as follows: For any $f \in L_{p,\varphi}(\mathbf{R}_+)$, there exists a unified expression $T_2^{(j)}f = F_{\lambda,j} \in L_{p,\varphi}(\mathbf{R}_+)$, such that for any $y \in \mathbf{R}_+$, $T_2^{(j)}f(y) = F_{\lambda,j}(y)$.

By (5.18), we have $||T_2^{(j)}f||_{p,\varphi} = ||F_{\lambda,j}||_{p,\varphi} \leqslant k_{\lambda,j}^{(2)}(\mu)||f||_{p,\varphi}$. Hence $T_2^{(j)}$ is a bounded linear operator with

$$||T_2^{(j)}|| := \sup_{f(\neq\theta)\in L_{p,\varphi}(\mathbf{R}_+)} \frac{||T_2^{(j)}f||_{p,\varphi}}{||f||_{p,\varphi}} \leqslant k_{\lambda,j}^{(2)}(\mu).$$

Since the constant factor in (5.18) is the best possible, we have $||T_2^{(j)}|| = k_{\lambda,j}^{(2)}(\mu)$.

DEFINITION 6.3. As the assumptions of Theorem 3.1, define a First kind of Hilbert-Hardy-type operator $T_{i,j} : L_{p,\varphi}(\mathbf{R}_+) \to L_{p,\varphi}(\mathbf{R}_+)$ as follows: For any $f \in L_{p,\varphi}(\mathbf{R}_+)$, there exists a unified expression $T_{i,j}f = T_1^{(i)}F_{\lambda,j} \in L_{p,\varphi}(\mathbf{R}_+)$, such that for any $x \in \mathbf{R}_+$,

$$T_{i,j}f(x) = T_1^{(i)}F_{\lambda,j}(x) = x^{\lambda-1}\int_{A_{x,i}} k_\lambda^{(1)}(x,y)F_{\lambda,j}(y)dy.$$

It is evident that $T_{i,j}f = T_1^{(i)}F_{\lambda,j} = T_1^{(i)}(T_2^{(j)}f) = (T_1^{(i)}T_2^{(j)})f$, and then $T_{i,j} = T_1^{(i)} \cdot T_2^{(j)}$. Hence, $T_{i,j}$ is a composition of $T_1^{(i)}$ and $T_2^{(j)}$, and (cf. [20])

$$||T_{i,j}|| = ||T_1^{(i)} \cdot T_2^{(j)}|| \leqslant ||T_1^{(i)}|| \cdot ||T_2^{(j)}|| = k_{\lambda,i}^{(1)}(\mu)k_{\lambda,j}^{(2)}(\mu).$$

By (5.16), we have

$$||T_{i,j}f||_{p,\varphi} = ||T_1^{(i)}F_{\lambda,j}||_{p,\varphi} \leqslant k_{\lambda,i}^{(1)}(\mu)k_{\lambda,j}^{(2)}(\mu)||f||_{p,\varphi}.$$

Since the constant factor in (5.16) is the best possible, we have

THEOREM 6.4. *As the assumptions of Theorem 3.1, if* $k_{\lambda,i}^{(1)}(\mu), k_{\lambda,j}^{(2)}(\mu) \in \mathbf{R}_+$ $(i,j = 0,1,2)$*, then we have*

$$||T_{i,j}|| = ||T_1^{(i)} \cdot T_2^{(j)}|| = ||T_1^{(i)}|| \cdot ||T_2^{(j)}|| = k_{\lambda,i}^{(1)}(\mu)k_{\lambda,j}^{(2)}(\mu) \quad (i,j = 0,1,2). \tag{6.2}$$

For $F \in L_{p,\Phi}(\mathbf{R}_+)$, we set $\widetilde{h}(x) := x^{\lambda-1}\int_{B_{x,i}} k_\lambda^{(1)}(xy,1)F(y)dy \ (x \in \mathbf{R}_+)$. Then by (5.21), we have

$$||\widetilde{h}||_{p,\Phi} \leqslant k_{\lambda,i}^{(1)}\left(\frac{\lambda}{2}\right)||F||_{p,\Phi}. \tag{6.3}$$

DEFINITION 6.5. As the assumptions of Theorem 4.1, define a Hilbert-Hardy-type operator $\widetilde{T}_1^{(i)} : L_{p,\Phi}(\mathbf{R}_+) \to L_{p,\Phi}(\mathbf{R}_+)$ as follows: For any $F \in L_{p,\Phi}(\mathbf{R}_+)$, there exists a unified expression $\widetilde{T}_1^{(i)}F = \widetilde{h} \in L_{p,\Phi}(\mathbf{R}_+)$, such that for any $x \in \mathbf{R}_+, \widetilde{T}_1^{(i)}F(x) = \widetilde{h}(x)$.

By (6.3), we have $||\widetilde{T}_1^{(i)}F||_{p,\Phi} \leqslant k_{\lambda,i}^{(1)}(\frac{\lambda}{2})||F||_{p,\Phi}$. Hence $\widetilde{T}_1^{(i)}$ is a bounded linear operator with

$$||\widetilde{T}_1^{(i)}|| := \sup_{\overline{F}_{\lambda,j}(\neq\theta)\in L_{p,\Phi}(\mathbf{R}_+)} \frac{||\widetilde{T}_1^{(i)}F||_{p,\Phi}}{||F||_{p,\Phi}} \leqslant k_{\lambda,i}^{(1)}\left(\frac{\lambda}{2}\right).$$

Since the constant factor in (5.21) is the best possible, we have $||\widetilde{T}_1^{(i)}|| = k_{\lambda,i}^{(1)}(\frac{\lambda}{2})$.

DEFINITION 6.6. As the assumptions of Theorem 4.1, define a Hilbert-Hardy-type operator $\widetilde{T}_2^{(j)}: L_{p,\Phi}(\mathbf{R}_+) \to L_{p,\Phi}(\mathbf{R}_+)$ as follows: For any $f \in L_{p,\Phi}(\mathbf{R}_+)$, there exists a unified expression $\widetilde{T}_2^{(j)} f = \overline{F}_{\lambda,j} \in L_{p,\Phi}(\mathbf{R}_+)$, such that for any $y \in \mathbf{R}_+, \widetilde{T}_2^{(j)} f(y) = \overline{F}_{\lambda,j}(y)$.

By (5.22), we have $||\widetilde{T}_2^{(j)} f||_{p,\Phi} = ||\overline{F}_{\lambda,j}||_{p,\Phi} \leqslant k_{\lambda,j}^{(2)}(\frac{\lambda}{2})||f||_{p,\Phi}$. Hence $\widetilde{T}_2^{(j)}$ is a bounded linear operator with

$$||\widetilde{T}_2^{(j)}|| := \sup_{f(\neq\theta)\in L_{p,\Phi}(\mathbf{R}_+)} \frac{||\widetilde{T}_2^{(j)} f||_{p,\Phi}}{||f||_{p,\Phi}} \leqslant k_{\lambda,j}^{(2)}\left(\frac{\lambda}{2}\right).$$

Since the constant factor in (5.22) is the best possible, we have $||\widetilde{T}_2^{(j)}|| = k_{\lambda,j}^{(2)}(\frac{\lambda}{2})$.

DEFINITION 6.7. As the assumptions of Theorem 4.1, define a Second kind of Hilbert-Hardy-type operator $\widetilde{T}_{i,j} : L_{p,\Phi}(\mathbf{R}_+) \to L_{p,\Phi}(\mathbf{R}_+)$ as follows: For any $f \in L_{p,\Phi}(\mathbf{R}_+)$, there exists a unified expression $\widetilde{T}_{i,j} f = \widetilde{T}_1^{(i)} \overline{F}_{\lambda,j} \in L_{p,\Phi}(\mathbf{R}_+)$, such that for any $x \in \mathbf{R}_+$,

$$\widetilde{T}_{i,j} f(x) = \widetilde{T}_1^{(i)} \overline{F}_{\lambda,j}(x) = x^{\lambda-1} \int_{B_{x,i}} k_\lambda^{(1)}(xy,1) \overline{F}_{\lambda,j}(y) dy.$$

It is evident that

$$\widetilde{T}_{i,j} f = \widetilde{T}_1^{(i)} \overline{F}_{\lambda,j} = \widetilde{T}_1^{(i)} (\widetilde{T}_2^{(j)} f) = (\widetilde{T}_1^{(i)} \widetilde{T}_2^{(j)}) f,$$

and then $\widetilde{T}_{i,j} = \widetilde{T}_1^{(i)} \widetilde{T}_2^{(j)}$. Hence, $\widetilde{T}_{i,j}$ is the composition of $\widetilde{T}_1^{(i)}$ and $\widetilde{T}_2^{(j)}$, and (cf. [20])

$$||\widetilde{T}_{i,j}|| = ||\widetilde{T}_1^{(i)} \widetilde{T}_2^{(j)}|| \leqslant ||\widetilde{T}_1^{(i)}|| \cdot ||\widetilde{T}_2^{(j)}|| = k_{\lambda,i}^{(1)}\left(\frac{\lambda}{2}\right) k_{\lambda,j}^{(2)}(\frac{\lambda}{2}).$$

By (5.20), we have

$$||\widetilde{T}_{i,j} f||_{p,\Phi} = ||\widetilde{T}_1^{(i)} \overline{F}_{\lambda,j}||_{p,\Phi} \leqslant k_{\lambda,i}^{(1)}\left(\frac{\lambda}{2}\right) k_{\lambda,j}^{(2)}\left(\frac{\lambda}{2}\right) ||f||_{p,\Phi}.$$

Since the constant factor in (5.20) is the best possible, then we have

THEOREM 6.8. *As the assumptions of Theorem 4.1, if* $k_{\lambda,i}^{(1)}(\frac{\lambda}{2}),\ k_{\lambda,j}^{(2)}(\frac{\lambda}{2}) \in \mathbf{R}_+$ $(i,j = 0,1,2)$, *then we have*

$$||\widetilde{T}_{i,j}|| = ||\widetilde{T}_1^{(i)} \widetilde{T}_2^{(j)}|| = ||\widetilde{T}_1^{(i)}|| \cdot ||\widetilde{T}_2^{(j)}|| = k_{\lambda,i}^{(1)}\left(\frac{\lambda}{2}\right) k_{\lambda,j}^{(2)}\left(\frac{\lambda}{2}\right) \quad (i,j = 0,1,2). \quad (6.4)$$

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