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SHARP INEQUALITIES INVOLVING NEUMAN MEANS OF THE SECOND KIND

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Abstract. In the article, we present several sharp inequalities involving the Neuman means of the second kind, and the logarithmic, Seiffert, arithmetic, Neuman-Sándor, quadratic and contra-harmonic means.

1. Introduction

Let $a,b,c \geqslant 0$ with $ab+ac+bc \neq 0$. Then the symmetric integral $R_F(a,b,c)$ [1] of the first kind is defined as

$$R_F(a,b,c) = \frac{1}{2}\int_0^\infty [(t+a)(t+b)(t+c)]^{-1/2}dt.$$

The degenerate case $R_C(a,b) = R_F(a,b,b)$ of R_F plays an important role in the theory of special functions [1, 2].

For $a,b>0$ with $a \neq b$, the Schwab-Borchardt mean $SB(a,b)$ [3–5] of a and b is given by

$$SB(a,b) = \begin{cases} \frac{\sqrt{b^2-a^2}}{\cos^{-1}(a/b)}, & a<b, \\ \frac{\sqrt{a^2-b^2}}{\cosh^{-1}(a/b)}, & a>b, \end{cases}$$

where $\cos^{-1}(x)$ and $\cosh^{-1}(x) = \log(x+\sqrt{x^2-1})$ are the inverse cosine and inverse hyperbolic cosine functions, respectively.

Carlson [6] (see also [7, (3.21)]) proved that

$$SB(a,b) = \left[R_C\left(a^2,b^2\right)\right]^{-1}.$$

It is well known that the Schwab-Borchardt mean $SB(a,b)$ is strictly increasing in both a and b, nonsymmetric and homogeneous of degree 1 with respect to a and b. Many symmetric bivariate means are special cases of the Schwab-Borchardt mean. For example, $P(a,b) = (a-b)/[2\arcsin((a-b)/(a+b))] = SB[G(a,b),A(a,b)]$ is the first

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Seiffert mean, $T(a,b)=(a-b)/[2\arctan((a-b)/(a+b))]=SB[A(a,b),Q(a,b)]$ is the second Seiffert mean, $M(a,b)=(a-b)/[2\sinh^{-1}((a-b)/(a+b))]=SB[Q(a,b),A(a,b)]$ is the Neuman-Sándor mean, $L(a,b)=(a-b)/[2\tanh^{-1}((a-b)/(a+b))]=SB[A(a,b),G(a,b)]$ is the logarithmic mean, where $G(a,b)=\sqrt{ab}$, $A(a,b)=(a+b)/2$ and $Q(a,b)=\sqrt{(a^2+b^2)/2}$ are the geometric, arithmetic and quadratic means of a and b, respectively. Recently, the Schwab-Borchardt mean and its special cases have attracted the attention of many researchers. In particular, many remarkable inequalities for these means can be found in the literatrue [8–21].

Let $H(a,b)=2ab/(a+b)$ and $C(a,b)=(a^2+b^2)/(a+b)$ be the harmonic and contraharmonic means of a and b, respectively. Then it is well known that the inequalities

$$H(a,b)<G(a,b)<L(a,b)<P(a,b)<A(a,b)<M(a,b)<T(a,b)<Q(a,b)<C(a,b)$$

hold for all $a,b>0$ with $a\neq b$.

Let $X(a,b)$ and $Y(a,b)$ be the symmetric bivariate means of a and b. Then the Neuman means $S_{XY}(a,b)$ [22, 23] of the first kind and the Neuman means $N_{XY}(a,b)$ [24] of the second kind of are defined by

$$S_{XY}(a,b)=SB[X(a,b),Y(a,b)],\ \ N_{XY}(a,b)=\frac{1}{2}\left[X(a,b)+\frac{Y^2(a,b)}{S_{XY}(a,b)}\right].$$

Let $a>b>0$, $v=(a-b)/(a+b)\in(0,1)$, $p\in(0,\infty)$, $q\in(0,\pi/2)$, $r\in(0,\log(2+\sqrt{3}))$ and $s\in(0,\pi/3)$ be the parameters such that $1/\cosh(p)=\cos(q)=1-v^2$, $\cosh(r)=1/\cos(s)=1+v^2$. Then the following explicit formulas and inequalities can be found in the literature [22–26].

$$S_{AH}(a,b)=A(a,b)\frac{\tanh(p)}{p},\ \ S_{HA}(a,b)=A(a,b)\frac{\sin(q)}{q}, \tag{1.1}$$

$$S_{CA}(a,b)=A(a,b)\frac{\sinh(r)}{r},\ \ S_{AC}(a,b)=A(a,b)\frac{\tan(s)}{s}, \tag{1.2}$$

$$N_{AG}(a,b)=\frac{A(a,b)}{2}\left[1+(1-v^2)\frac{\tanh^{-1}(v)}{v}\right], \tag{1.3}$$

$$N_{GA}(a,b)=\frac{A(a,b)}{2}\left[\sqrt{1-v^2}+\frac{\sin^{-1}(v)}{v}\right], \tag{1.4}$$

$$N_{AQ}(a,b)=\frac{A(a,b)}{2}\left[1+(1+v^2)\frac{\tan^{-1}(v)}{v}\right], \tag{1.5}$$

$$N_{QA}(a,b)=\frac{A(a,b)}{2}\left[\sqrt{1+v^2}+\frac{\sinh^{-1}(v)}{v}\right], \tag{1.6}$$

$$N_{AH}(a,b)=\frac{A(a,b)}{2}\left[1+\frac{2p}{\sinh(2p)}\right], \tag{1.7}$$

$$N_{HA}(a,b) = \frac{A(a,b)}{2}\left[\cos(q) + \frac{q}{\sin(q)}\right], \tag{1.8}$$

$$N_{AC}(a,b) = \frac{A(a,b)}{2}\left[1 + \frac{2s}{\sin(2s)}\right], \tag{1.9}$$

$$N_{CA}(a,b) = \frac{A(a,b)}{2}\left[\cosh(r) + \frac{r}{\sinh(r)}\right], \tag{1.10}$$

$$\begin{aligned} &L(a,b) < N_{AG}(a,b) < P(a,b) < N_{GA}(a,b) < A(a,b)\\ &< M(a,b) < N_{QA}(a,b) < T(a,b) < N_{AQ}(a,b) < Q(a,b),\\ &S_{AH}(a,b) < N_{AH}(a,b) < S_{HA}(a,b) < N_{HA}(a,b) < A(a,b)\\ &< S_{CA}(a,b) < N_{CA}(a,b) < S_{AC}(a,b) < N_{AC}(a,b) < C(a,b). \end{aligned}$$

Neuman [24] proved that the double inequalities

$$\alpha_1 A(a,b) + (1-\alpha_1)G(a,b) < N_{GA}(a,b) < \beta_1 A(a,b) + (1-\beta_1)G(a,b),$$

$$\alpha_2 Q(a,b) + (1-\alpha_2)A(a,b) < N_{AQ}(a,b) < \beta_2 Q(a,b) + (1-\beta_2)A(a,b),$$

$$\alpha_3 A(a,b) + (1-\alpha_3)G(a,b) < N_{AG}(a,b) < \beta_3 A(a,b) + (1-\beta_3)G(a,b),$$

$$\alpha_4 Q(a,b) + (1-\alpha_4)A(a,b) < N_{QA}(a,b) < \beta_4 Q(a,b) + (1-\beta_4)A(a,b)$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\alpha_1 \leqslant 2/3$, $\beta_1 \geqslant \pi/4$, $\alpha_2 \leqslant 2/3$, $\beta_2 \geqslant (\pi-2)/[4(\sqrt{2}-1)] = 0.689\ldots$, $\alpha_3 \leqslant 1/3$, $\beta_3 \geqslant 1/2$, $\alpha_4 \leqslant 1/3$ and $\beta_4 \geqslant [\log(1+\sqrt{2})+\sqrt{2}-2]/[2(\sqrt{2}-1)] = 0.356\ldots$.

In [25], The authors proved that the double inequalities

$$\alpha_1 A(a,b) + (1-\alpha_1)H(a,b) < N_{AH}(a,b) < \beta_1 A(a,b) + (1-\beta_1)H(a,b),$$

$$\alpha_2 A(a,b) + (1-\alpha_2)H(a,b) < N_{HA}(a,b) < \beta_2 A(a,b) + (1-\beta_2)H(a,b),$$

$$\alpha_3 C(a,b) + (1-\alpha_3)A(a,b) < N_{CA}(a,b) < \beta_3 C(a,b) + (1-\beta_3)A(a,b),$$

$$\alpha_4 C(a,b) + (1-\alpha_4)A(a,b) < N_{AC}(a,b) < \beta_4 C(a,b) + (1-\beta_4)A(a,b)$$

$$\frac{\alpha_5}{H(a,b)} + \frac{1-\alpha_5}{A(a,b)} < \frac{1}{N_{AH}(a,b)} < \frac{\beta_5}{H(a,b)} + \frac{1-\beta_5}{A(a,b)},$$

$$\frac{\alpha_6}{H(a,b)} + \frac{1-\alpha_6}{A(a,b)} < \frac{1}{N_{HA}(a,b)} < \frac{\beta_6}{H(a,b)} + \frac{1-\beta_6}{A(a,b)},$$

$$\frac{\alpha_7}{A(a,b)} + \frac{1-\alpha_7}{C(a,b)} < \frac{1}{N_{CA}(a,b)} < \frac{\beta_7}{A(a,b)} + \frac{1-\beta_7}{C(a,b)},$$

$$\frac{\alpha_8}{A(a,b)} + \frac{1-\alpha_8}{C(a,b)} < \frac{1}{N_{AC}(a,b)} < \frac{\beta_8}{A(a,b)} + \frac{1-\beta_8}{C(a,b)},$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\alpha_1 \leqslant 1/3$, $\beta_1 \geqslant 1/2$, $\alpha_2 \leqslant 2/3$, $\beta_2 \geqslant \pi/4 = 0.7853\ldots$, $\alpha_3 \leqslant 1/3$, $\beta_3 \geqslant \sqrt{3}\log(2+\sqrt{3})/6 = 0.3801\ldots$, $\alpha_4 \leqslant 2/3$, $\beta_4 \geqslant (4\sqrt{3}\pi-9)/18 = 0.7901\ldots$, $\alpha_5 \leqslant 0$, $\beta_5 \geqslant 2/3$, $\alpha_6 \leqslant 0$, $\beta_6 \geqslant 1/3$, $\alpha_7 \leqslant [2\sqrt{3}-$

$\log(2+\sqrt{3})]/[2\sqrt{3}+\log(2+\sqrt{3})] = 0.4490\ldots$, $\beta_7 \geqslant 2/3$, $\alpha_8 \leqslant (9\sqrt{3}-4\pi)/(3\sqrt{3}+4\pi) = 0.1701\ldots$ and $\beta_8 \geqslant 1/3$.

Zhang et. al. [27] presented the best possible parameters $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0, 1/2]$ and $\alpha_3, \alpha_4, \beta_3, \beta_4 \in [1/2, 1]$ such that the double inequalities

$$G(\alpha_1 a+(1-\alpha_1)b, \alpha_1 b+(1-\alpha_1)a) < N_{AG}(a,b) < G(\beta_1 a+(1-\beta_1)b, \beta_1 b+(1-\beta_1)a),$$

$$G(\alpha_2 a+(1-\alpha_2)b, \alpha_2 b+(1-\alpha_2)a) < N_{GA}(a,b) < G(\beta_2 a+(1-\beta_2)b, \beta_2 b+(1-\beta_2)a),$$

$$Q(\alpha_3 a+(1-\alpha_3)b, \alpha_3 b+(1-\alpha_3)a) < N_{QA}(a,b) < Q(\beta_3 a+(1-\beta_3)b, \beta_3 b+(1-\beta_3)a),$$

$$Q(\alpha_4 a+(1-\alpha_4)b, \alpha_4 b+(1-\alpha_4)a) < N_{AQ}(a,b) < Q(\beta_4 a+(1-\beta_4)b, \beta_4 b+(1-\beta_4)a)$$

hold for all $a, b > 0$ with $a \neq b$.

The main purpose of this paper is to present some new sharp bounds for the second kind of Neuman means N_{AG}, N_{GA}, N_{AH}, N_{HA}, N_{AQ}, N_{QA}, N_{AC} and N_{CA} in terms of certain combinations of the first kind of Neuman means S_{AH}, S_{HA}, S_{AC}, S_{CA} and the logarithmic mean L, first Seiffert mean P, arithmetic mean A, Neuman-Sándor mean M, second Seiffert mean T, quadratic mean Q and contraharmonic mean C.

2. Lemmas

In order to prove our main results we need several lemmas, which we present in this section.

LEMMA 2.1. (See [28, 29]) *For $-\infty < a < b < \infty$, let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and be differentiable on (a,b), let $g'(x) \neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are*

$$\frac{f(x)-f(a)}{g(x)-g(a)}, \quad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

LEMMA 2.2. (See [30]) *Suppose that the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ have the radius of convergence $r > 0$ with $a_n, b_n > 0$ for all $n = 0, 1, 2, \cdots$. Let $h(x) = f(x)/g(x)$, if the sequence $\{a_n/b_n\}_{n=0}^{\infty}$ is (strictly) increasing (decreasing), then $h(x)$ is also (strictly) increasing (decreasing) on $(0, r)$.*

LEMMA 2.3. *The function*

$$\phi(x) = \frac{x^2 + x\sinh(x)\cosh(x) - 2\sinh^2(x)}{\sinh(x)[x\cosh(x) - \sinh(x)]}$$

is strictly increasing from $(0, \infty)$ onto $(0, 1)$.

Proof. Let $\phi_1(x) = x^2 + x\sinh(x)\cosh(x) - 2\sinh^2(x)$ and $\phi_2(x) = \sinh(x)[x\cosh(x) - \sinh(x)]$. Then simple computations lead to

$$\phi_1(0) = \phi_2(0) = 0, \tag{2.1}$$

$$\frac{\phi_1'(x)}{\phi_2'(x)} = \frac{4x+2x\cosh(2x)-3\sinh(2x)}{2x\cosh(2x)-\sinh(2x)} \tag{2.2}$$
$$= \frac{\sum_{n=1}^{\infty}\frac{(n-1)2^{2n+2}}{(2n+1)!}x^{2n+1}}{\sum_{n=1}^{\infty}\frac{n\times 2^{2n+2}}{(2n+1)!}x^{2n+1}} = \frac{\sum_{n=0}^{\infty}\frac{n\times 2^{2n+4}}{(2n+3)!}x^{2n+3}}{\sum_{n=0}^{\infty}\frac{(n+1)2^{2n+4}}{(2n+3)!}x^{2n+3}} =: \frac{\sum_{n=0}^{\infty}a_n x^{2n}}{\sum_{n=0}^{\infty}b_n x^{2n}},$$

where

$$a_n = \frac{n\times 2^{2n+4}}{(2n+3)!},\quad b_n = \frac{(n+1)2^{2n+4}}{(2n+3)!}. \tag{2.3}$$

From (2.3) we clearly see that

$$a_n > 0,\quad b_n > 0 \tag{2.4}$$

and

$$\frac{a_n}{b_n} = 1 - \frac{1}{n+1}$$

is strictly increasing for all $n \geqslant 0$.

Note that

$$\phi(0^+) = \frac{a_0}{b_0} = 0,\quad \lim_{x\to\infty}\phi(x) = 1. \tag{2.5}$$

Therefore, Lemma 2.3 follows easily from Lemmas 2.1 and 2.2, (2.1), (2.2), (2.4) and (2.5) together with the monotonicity of the sequence $\{a_n/b_n\}$. □

LEMMA 2.4. *The function*

$$\varphi(x) = \frac{x^2 + x\sin(x)\cos(x) - 2\sin^2(x)}{\sin(x)[x-\sin(x)]}$$

is strictly increasing from $(0,\pi/2)$ *onto* $(0,(\pi^2-8)/2\pi-4)$.

Proof. Let

$$\varphi_1(x) = 1 + \frac{x}{\sin(x)}, \tag{2.6}$$

$$\varphi_2(x) = \frac{x\cos(x)-\sin(x)}{x-\sin(x)}. \tag{2.7}$$

Then

$$\varphi(x) = \varphi_1(x) + \varphi_2(x). \tag{2.8}$$

Let $\varphi_3(x) = x\cos(x) - \sin(x)$ and $\varphi_4(x) = x - \sin(x)$. Then

$$\varphi_3(0) = \varphi_4(0) = 0, \tag{2.9}$$

$$\frac{\varphi_3'(x)}{\varphi_4'(x)} = -\frac{x}{\tan(x/2)}. \tag{2.10}$$

It is not difficult to verify that both the functions $x \to x/\sin(x)$ and $x \to -x/\tan(x/2)$ are strictly increasing on $(0, \pi/2)$. Note that

$$\varphi(0^+) = 0, \quad \varphi\left(\frac{\pi}{2}\right) = \frac{\pi^2 - 8}{2\pi - 4}. \tag{2.11}$$

Therefore, Lemma 2.4 follows from (2.6)–(2.11) and Lemma 2.1 together with the monotonicity of the functions $x \to x/\sin(x)$ and $x \to -x/\tan(x/2)$ on $(0, \pi/2)$. □

3. Main results

THEOREM 3.1. *The double inequalities*

$$\alpha_1 A(a,b) + (1-\alpha_1)L(a,b) < N_{AG}(a,b) < \beta_1 A(a,b) + (1-\beta_1)L(a,b),$$

$$\alpha_2 A(a,b) + (1-\alpha_2)S_{AH}(a,b) < N_{AH}(a,b) < \beta_2 A(a,b) + (1-\beta_2)S_{AH}(a,b),$$

$$\alpha_3 Q(a,b) + (1-\alpha_3)M(a,b) < N_{QA}(a,b) < \beta_3 Q(a,b) + (1-\beta_3)M(a,b),$$

$$\alpha_4 C(a,b) + (1-\alpha_4)S_{CA}(a,b) < N_{CA}(a,b) < \beta_4 C(a,b) + (1-\beta_4)S_{CA}(a,b)$$

hold for all $a, b > 0$ *with* $a \neq b$ *if and only if* $\alpha_1 \leqslant 0$, $\beta_1 \geqslant 1/2$, $\alpha_2 \leqslant 0$, $\beta_2 \geqslant 1/2$, $\alpha_3 \leqslant 0$, $\beta_3 \geqslant [\sqrt{2}\log^2(1+\sqrt{2}) + 2\log(1+\sqrt{2}) - 2\sqrt{2}]/[4\log(1+\sqrt{2}) - 2\sqrt{2}] = 0.04721\cdots$, $\alpha_4 \leqslant 0$ *and* $\beta_4 \geqslant [6\log(2+\sqrt{3}) + \sqrt{3}\log^2(2+\sqrt{3}) - 6\sqrt{3}]/[12\log(2+\sqrt{3}) - 6\sqrt{3}] = 0.0948\cdots$.

Proof. Since all the bivariate means concerned in Theorem 3.1 are symmetric and homogeneous of degree one, we assume that $a > b > 0$. Let $v = (a-b)/(a+b) \in (0,1)$, $x = \tanh^{-1}(v) \in (0, \infty)$, $t = \sinh^{-1}(v) \in (0, \log(1+\sqrt{2}))$, $p \in (0, \infty)$ and $r \in (0, \log(2+\sqrt{3}))$ be the parameters such that $1/\cosh(p) = 1 - v^2$ and $\cosh(r) = 1 + v^2$. Then (1.1)–(1.3), (1.6), (1.7) and (1.10) together with the fact that

$$L(a,b) = A(a,b)\frac{v}{\tanh^{-1}(v)}, \quad M(a,b) = A(a,b)\frac{v}{\sinh^{-1}(v)}$$

and

$$Q(a,b) = A(a,b)\sqrt{1+v^2}, \quad C(a,b) = A(a,b)(1+v^2) \tag{3.1}$$

lead to

$$\frac{N_{AG}(a,b) - L(a,b)}{A(a,b) - L(a,b)} = \frac{\frac{1}{2}\left[1 + (1-v^2)\frac{\tanh^{-1}(v)}{v})\right] - \frac{v}{\tanh^{-1}(v)}}{1 - \frac{v}{\tanh^{-1}(v)}} \tag{3.2}$$

$$= \frac{x^2 + x\sinh(x)\cosh(x) - 2\sinh^2(x)}{2\sinh(x)[x\cosh(x) - \sinh(x)]} = \frac{1}{2}\phi(x),$$

$$\frac{N_{AH}(a,b) - S_{AH}(a,b)}{A(a,b) - S_{AH}(a,b)} = \frac{\frac{1}{2}\left[1 + \frac{2p}{\sinh(2p)})\right] - \frac{\tanh(p)}{p}}{1 - \frac{\tanh(p)}{p}} \tag{3.3}$$

$$= \frac{p^2 + p\sinh(p)\cosh(p) - 2\sinh^2(p)}{2\sinh(p)[p\cosh(p) - \sinh(p)]} = \frac{1}{2}\phi(p),$$

$$\frac{N_{QA}(a,b)-M(a,b)}{Q(a,b)-M(a,b)} = \frac{\frac{1}{2}\left[\sqrt{1+v^2}+\frac{\sinh^{-1}(v)}{v})\right]-\frac{v}{\sinh^{-1}(v)}}{\sqrt{1+v^2}-\frac{v}{\sinh^{-1}(v)}} \tag{3.4}$$
$$= \frac{t^2+t\sinh(t)\cosh(t)-2\sinh^2(t)}{2\sinh(t)[t\cosh(t)-\sinh(t)]} = \frac{1}{2}\phi(t),$$

$$\frac{N_{CA}(a,b)-S_{CA}(a,b)}{C(a,b)-S_{CA}(a,b)} = \frac{\frac{1}{2}\left[\cosh(r)+\frac{r}{\sinh(r)})\right]-\frac{\sinh(r)}{r}}{\cosh(r)-\frac{\sinh(r)}{r}} \tag{3.5}$$
$$= \frac{r^2+r\sinh(r)\cosh(r)-2\sinh^2(r)}{2\sinh(r)[r\cosh(r)-\sinh(r)]} = \frac{1}{2}\phi(r),$$

where the function $\phi(\cdot)$ is defined as in Lemma 2.3. Note that

$$\phi[\log(1+\sqrt{2})] = \frac{\sqrt{2}\log^2(1+\sqrt{2})+2\log(1+\sqrt{2})-2\sqrt{2}}{2\log(1+\sqrt{2})-\sqrt{2}}, \tag{3.6}$$

$$\phi[\log(2+\sqrt{3})] = \frac{6\log(2+\sqrt{3})+\sqrt{3}\log^2(2+\sqrt{3})-6\sqrt{3}}{6\log(2+\sqrt{3})-3\sqrt{3}}. \tag{3.7}$$

Therefore, Theorem 3.1 follows easily from Lemma 2.3 and (3.2)–(3.7). □

THEOREM 3.2. *The double inequalities*

$$\alpha_5 A(a,b)+(1-\alpha_5)P(a,b) < N_{GA}(a,b) < \beta_5 A(a,b)+(1-\beta_5)P(a,b),$$

$$\alpha_6 A(a,b)+(1-\alpha_6)S_{HA}(a,b) < N_{HA}(a,b) < \beta_6 A(a,b)+(1-\beta_6)S_{HA}(a,b),$$

$$\alpha_7 Q(a,b)+(1-\alpha_7)T(a,b) < N_{AQ}(a,b) < \beta_7 Q(a,b)+(1-\beta_7)T(a,b),$$

$$\alpha_8 C(a,b)+(1-\alpha_8)S_{AC}(a,b) < N_{AC}(a,b) < \beta_8 C(a,b)+(1-\beta_8)S_{AC}(a,b)$$

hold for all $a,b>0$ *with* $a\neq b$ *if and only if* $\alpha_5 \leqslant 0$, $\beta_5 \geqslant (\pi^2-8)/(4\pi-8) = 0.4094\cdots$, $\alpha_6 \leqslant 0$, $\beta_6 \geqslant (\pi^2-8)/(4\pi-8) = 0.4094\cdots$, $\alpha_7 \leqslant 0$, $\beta_7 \geqslant (\pi^2+2\pi-16)/(4\sqrt{2}\pi-16) = 0.08624\cdots$, $\alpha_8 \leqslant 0$ *and* $\beta_8 \geqslant (4\sqrt{3}\pi^2+9\pi-54\sqrt{3})/(36\pi-54\sqrt{3}) = 0.1595\cdots$.

Proof. Since all the bivariate means concerned in Theorem 3.2 are symmetric and homogeneous of degree one, we assume that $a>b>0$. Let $v=(a-b)/(a+b)\in(0,1)$, $x=\sin^{-1}(v)\in(0,\pi/2)$, $t=\tan^{-1}(v)\in(0,\pi/4)$, $q\in(0,\pi/2)$ and $s\in(0,\pi/3)$ be the parameters such that $\cos(q)=1-v^2$ and $1/\cos(s)=1+v^2$. Then from (1.1), (1.2), (1.4), (1.5), (1.8), (1.9), (3.1) and the fact that

$$P(a,b)=A(a,b)\frac{v}{\sin^{-1}(v)}, \quad T(a,b)=A(a,b)\frac{v}{\tan^{-1}(v)}$$

we get

$$\frac{N_{GA}(a,b)-P(a,b)}{A(a,b)-P(a,b)} = \frac{\frac{1}{2}\left[\sqrt{1-v^2}+\frac{\sin^{-1}(v)}{v}\right]-\frac{v}{\sin^{-1}(v)}}{1-\frac{v}{\sin^{-1}(v)}} = \frac{x^2+x\sin(x)\cos(x)-2\sin^2(x)}{2\sin(x)[x-\sin(x)]} = \frac{1}{2}\varphi(x), \tag{3.8}$$

$$\frac{N_{HA}(a,b)-S_{HA}(a,b)}{A(a,b)-S_{HA}(a,b)} = \frac{\frac{1}{2}\left[\cos(q)+\frac{q}{\sin(q)}\right]-\frac{\sin(q)}{q}}{1-\frac{\sin(q)}{q}} = \frac{q^2+q\sin(q)\cos(q)-2\sin^2(q)}{2\sin(q)[q-\sin(q)]} = \frac{1}{2}\varphi(q), \tag{3.9}$$

$$\frac{N_{AQ}(a,b)-T(a,b)}{Q(a,b)-T(a,b)} = \frac{\frac{1}{2}\left[1+(1+v^2)\frac{\tan^{-1}(v)}{v})\right]-\frac{v}{\tan^{-1}(v)}}{\sqrt{1+v^2}-\frac{v}{\tan^{-1}(v)}} = \frac{t^2+t\sin(t)\cos(t)-2\sin^2(t)}{2\sin(t)[t-\sin(t)]} = \frac{1}{2}\varphi(t), \tag{3.10}$$

$$\frac{N_{AC}(a,b)-S_{AC}(a,b)}{C(a,b)-S_{AC}(a,b)} = \frac{\frac{1}{2}\left[1+\frac{2s}{\sin(2s)}\right]-\frac{\tan(s)}{s}}{\frac{1}{\cos(s)}-\frac{\tan(s)}{s}} = \frac{s^2+s\sin(s)\cos(s)-2\sin^2(s)}{2\sin(s)[s-\sin(s)]} = \frac{1}{2}\varphi(s), \tag{3.11}$$

where the function $\varphi(\cdot)$ is defined as in Lemma 2.4. Note that

$$\varphi\left(\frac{\pi}{4}\right) = \frac{\pi^2+2\pi-16}{2\sqrt{2}\pi-8}, \tag{3.12}$$

$$\varphi\left(\frac{\pi}{3}\right) = \frac{4\sqrt{3}\pi^2+9\pi-54\sqrt{3}}{18\pi-27\sqrt{3}}. \tag{3.13}$$

Therefore, Theorem 3.2 follows from Lemma 2.4 and (3.8)–(3.13). □

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