

## ON THE GENERALIZED HYERS–ULAM STABILITY OF QUARTIC MAPPINGS IN NON-ARCHIMEDEAN BANACH SPACES

H. AZADI KENARY, H. KESHAVERZ, C. PARK AND D. Y. SHIN

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*Abstract.* Let  $X, Y$  are linear space. In this paper, we prove the generalized Hyers-Ulam stability of the following quartic equation

$$\sum_{k=2}^n \left( \sum_{i_1=2}^k \sum_{i_2=i_1+1}^{k+1} \dots \sum_{i_{n-k+1}=i_{n-k}+1}^n \right) f \left( \sum_{i=1, i \neq i_1, \dots, i_{n-k+1}}^n x_i - \sum_{r=1}^{n-k+1} x_{i_r} \right) + f \left( \sum_{i=1}^n x_i \right) \\ = 2^{n-2} \sum_{1 \leq i < j \leq n} (f(x_i + x_j) + f(x_i - x_j)) - 2^{n-5} (n-2) \sum_{i=1}^n f(2x_i)$$

( $n \in \mathbb{N}$ ,  $n \geq 3$ ) in non-Archimedean Banach spaces

### 1. Introduction and preliminaries

By a *non-Archimedean field* we mean a field  $\mathbb{K}$  equipped with a function (valuation)  $|\cdot| : \mathbb{K} \rightarrow [0, \infty)$  such that, for all  $r, s \in \mathbb{K}$ , the following conditions hold: (a)  $|r| = 0$  if and only if  $r = 0$ ; (b)  $|rs| = |r||s|$ ; (c)  $|r+s| \leq \max\{|r|, |s|\}$ .

Clearly, by (b),  $|1| = |-1| = 1$  and so, by induction, it follows from (c) that  $|n| \leq 1$  for all  $n \geq 1$ .

**DEFINITION 1.1.** Let  $X$  be a vector space over a scalar field  $\mathbb{K}$  with a non-Archimedean non-trivial valuation  $|\cdot|$ .

(1) A function  $\|\cdot\| : X \rightarrow \mathbb{R}$  is a *non-Archimedean norm* (valuation) if it satisfies the following conditions: (a)  $\|x\| = 0$  if and only if  $x = 0$  for all  $x \in X$ ; (b)  $\|rx\| = |r|\|x\|$  for all  $r \in \mathbb{K}$  and  $x \in X$ ; (c) the strong triangle inequality (ultra-metric) holds, that is,  $\|x+y\| \leq \max\{\|x\|, \|y\|\}$  for all  $x, y \in X$ .

(2) The space  $(X, \|\cdot\|)$  is called a *non-Archimedean normed space* (briefly, NAN-space).

Note that  $\|x_n - x_m\| \leq \max\{\|x_{j+1} - x_j\| : m \leq j \leq n-1\}$  for all  $m, n \in \mathbb{N}$  with  $n > m$ .

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DEFINITION 1.2. Let  $(X, \|\cdot\|)$  be a non-Archimedean normed space.

(a) A sequence  $\{x_n\}$  is a *Cauchy sequence* in  $X$  if  $\{x_{n+1} - x_n\}$  converges to zero in  $X$ .

(b) The non-Archimedean normed space  $(X, \|\cdot\|)$  is said to be *complete* if every Cauchy sequence in  $X$  is convergent.

The most important examples of non-Archimedean spaces are  $p$ -adic numbers. A key property of  $p$ -adic numbers is that they do not satisfy the Archimedean axiom: for all  $x, y > 0$ , there exists a positive integer  $n$  such that  $x < ny$ .

A basic question in the theory of functional equations is as follows: 'when is it true that a function, which approximately satisfies a functional equation must be close to an exact solution of the equation'?

If the problem accepts a solution, we say the equation is stable. The first stability problem concerning group homomorphisms was raised by Ulam [12] in 1940 and affirmatively solved by Hyers [5]. The result of Hyers was generalized by Aoki [1] for approximate additive function and by Rassias [10] for approximate linear functions by allowing the difference Cauchy equation  $\|f(x+y) - f(x) - f(y)\|$  to be controlled by  $\varepsilon(\|x\|^p + \|y\|^p)$ . Taking into consideration a lot of influence of Ulam, Hyers and Rassias on the development of stability problems of functional equations, the stability phenomenon that was proved by Rassias is called the Hyers–Ulam–Rassias stability. In 1994, a generalization of Rassias' theorem was obtained by Gǎvruta [4], who replaced  $\varepsilon(\|x\|^p + \|y\|^p)$  by a general control function  $\varphi(x, y)$ .

The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem ([1]–[5], [9], [14]–[17]).

The mapping  $f(x) = x^4$  satisfies equation:

$$f(2x_1 + x_2) + f(2x_1 - x_2) = 4f(x_1 + x_2) + 4f(x_1 - x_2) + 24f(x_1) - 6f(x_2) \tag{1.1}$$

every solution of Eq. (1.1) is called a quartic mapping. Equation (1.1) was solved by S. H. Lee, S. M. Im and I. S. Hwang [9].

Now, we introduce the new quartic equation in  $n$ -variables as follows:

$$\begin{aligned} &\sum_{k=2}^n \left( \sum_{i_1=2}^k \sum_{i_2=i_1+1}^{k+1} \dots \sum_{i_{n-k+1}=i_{n-k}+1}^n \right) f \left( \sum_{i=1, i \neq i_1, \dots, i_{n-k+1}}^n x_i - \sum_{r=1}^{n-k+1} x_{i_r} \right) + f \left( \sum_{i=1}^n x_i \right) \\ &= 2^{n-2} \sum_{1 \leq i < j \leq n} (f(x_i + x_j) + f(x_i - x_j)) - 2^{n-5}(n-2) \sum_{i=1}^n f(2x_i) \end{aligned} \tag{1.2}$$

where  $n \geq 3$ . As a special case, if  $n = 3$  in (1.2), then the equation (1.2) reduces to

$$\begin{aligned} &\sum_{i_1=2}^2 \sum_{i_2=i_1+1}^3 f \left( \sum_{i=1, i \neq i_1, i_2}^3 x_i - \sum_{r=1}^2 x_{i_r} \right) + \sum_{i_1=2}^3 f \left( \sum_{i=1, i \neq i_1}^3 x_i - x_{i_1} \right) + f \left( \sum_{i=1}^3 x_i \right) \\ &= 2 \sum_{1 \leq i < j \leq 3} (f(x_i + x_j) + f(x_i - x_j)) - 2^{-2} \sum_{i=1}^3 f(2x_i) \end{aligned}$$

that is,

$$\begin{aligned} & f(x_1 - x_2 - x_3) + f(x_1 - x_2 + x_3) + f(x_1 + x_2 - x_3) + f(x_1 + x_2 + x_3) \\ &= 2(f(x_1 + x_2) + f(x_1 - x_2) + f(x_1 + x_3) + f(x_1 - x_3) + f(x_2 + x_3) + f(x_2 - x_3)) \\ &\quad - 2^{-2}(f(2x_1) + f(2x_2) + f(2x_3)). \end{aligned}$$

The main purpose of this paper is to prove the generalized Hyers–Ulam stability for equation (1.2), in non-Archimedean normed spaces(briefly, NAN-spaces).

### 2. Solution

In this section, we prove the Hyers–Ulam–Rassias stability of quartic equation (1.2) in NAN-spaces. For convenience, we define the difference operator  $D_f$  for a given mapping  $f$  :

$$\begin{aligned} D_f(x_1, \dots, x_n) &= \sum_{k=2}^n \left( \sum_{i_1=2}^k \sum_{i_2=i_1+1}^{k+1} \dots \sum_{i_{n-k+1}=i_{n-k}+1}^n \right) f \left( \sum_{i=1, i \neq i_1, \dots, i_{n-k+1}}^n x_i - \sum_{r=1}^{n-k+1} x_{i_r} \right) \\ &\quad + f \left( \sum_{i=1}^n x_i \right) - 2^{n-2} \sum_{1 \leq i < j \leq n} (f(x_i + x_j) + f(x_i - x_j)) + 2^{n-5}(n-2) \sum_{i=1}^n f(2x_i). \end{aligned}$$

We will use the following lemma:

LEMMA 2.1. *A mapping  $f : X \rightarrow Y$  satisfies (1.2) if and only if the mapping  $f : X \rightarrow Y$  is quartic.*

*Proof.* Let  $f$  satisfies (1.2). Setting  $x_i = 0$  ( $i = 1, \dots, n$ ) in (1.2), we have

$$\sum_{k=2}^n \left( \sum_{i_1=2}^k \sum_{i_2=i_1+1}^{k+1} \dots \sum_{i_{n-k+1}=i_{n-k}+1}^n \right) f(0) + f(0) = 2^{n-2} \sum_{1 \leq i < j \leq n} 2f(0) - 2^{n-5}(n-2) \sum_{i=1}^n f(0),$$

that is,

$$\begin{aligned} & \sum_{i_1=2}^2 \sum_{i_2=i_1+1}^3 \dots \sum_{i_{n-1}=i_{n-2}+1}^n f(0) + \sum_{i_1=2}^3 \sum_{i_2=i_1+1}^4 \dots \sum_{i_{n-2}=i_{n-3}+1}^n f(0) + \dots + \sum_{i_1=2}^n f(0) + f(0) \\ &= 2^{n-2}n(n-1)f(0) - 2^{n-5}n(n-2)f(0), \end{aligned}$$

that is,

$$\left( \binom{n-1}{n-1} + \binom{n-1}{n-2} + \dots + \binom{n-1}{1} + 1 \right) f(0) = (2^{n-2}n(n-1) - 2^{n-5}n(n-2))f(0),$$

but,

$$1 + \sum_{\ell=1}^{n-1} \binom{n-1}{\ell} = \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} = 2^{n-1},$$

and also  $n \geq 3$  therefore  $f(0) = 0$ . By putting  $x_i = 0$  ( $i = 2, \dots, n$ ) in (1.2) and then using  $f(0) = 0$ , we get

$$\sum_{k=2}^n \left( \sum_{i_1=2}^k \sum_{i_2=i_1+1}^{k+1} \dots \sum_{i_{n-k+1}=i_{n-k}+1}^n \right) f(x_1) + f(x_1) = 2^{n-2} \sum_{j=2}^n 2f(x_1) - 2^{n-5}(n-2)f(2x_1),$$

for all  $x_1 \in X$ . Hence

$$2^{n-1}f(x_1) = 2^{n-1}(n-1)f(x_1) - 2^{n-5}(n-2)f(2x_1),$$

for all  $x_1 \in X$ . So  $f(2x_1) = 16f(x_1)$  for all  $x_1 \in X$ . Now, by using the identity  $f(2x_1) = 16f(x_1)$  and (1.2), we obtain that

$$\begin{aligned} & \sum_{k=2}^n \left( \sum_{i_1=2}^k \sum_{i_2=i_1+1}^{k+1} \dots \sum_{i_{n-k+1}=i_{n-k}+1}^n \right) f \left( \sum_{i=1, i \neq i_1, \dots, i_{n-k+1}}^n x_i - \sum_{r=1}^{n-k+1} x_{i_r} \right) + f \left( \sum_{i=1}^n x_i \right) \\ &= 2^{n-2} \sum_{1 \leq i < j \leq n} (f(x_i + x_j) + f(x_i - x_j)) - 2^{n-1}(n-2) \sum_{i=1}^n f(x_i), \end{aligned} \tag{2.1}$$

for all  $x_1, \dots, x_n \in X$ . Setting  $x_i = 0$  ( $i = 1, \dots, n-1$ ) in (2.1) and then using  $f(0) = 0$ , gives

$$\begin{aligned} & f(-x_n) + \left( \binom{n-2}{1} f(-x_n) + \binom{n-2}{n-2} f(x_n) \right) + \dots \\ &+ \left( \binom{n-2}{n-3} f(-x_n) + \binom{n-2}{2} f(x_n) \right) + \left( \binom{n-2}{n-2} f(-x_n) + \binom{n-2}{1} f(x_n) \right) + f(x_n) \\ &= 2^{n-2} \sum_{i=1}^{n-1} (f(x_n) + f(-x_n)) - 2^{n-1}(n-2)(f(x_n)), \end{aligned}$$

that is,

$$\left( 1 + \sum_{\ell=1}^{n-2} \binom{n-2}{\ell} \right) (f(x_n) + f(-x_n)) = 2^{n-2}(n-1)(f(x_n) + f(-x_n)) - 2^{n-1}(n-2)(f(x_n)), \tag{2.2}$$

for all  $x_n \in X$ . By using

$$\sum_{\ell=0}^{n-2} \binom{n-2}{\ell} = 2^{n-2},$$

and (2.2), we obtain  $f(-x_n) = f(x_n)$  for all  $x_n \in X$ . Putting  $x_2 = x_1$  and  $x_i = 0$  ( $i = 3, \dots, n-1$ ) in (2.1), hence similar to the above method, we infer that

$$\begin{aligned} & 2^{n-3}(f(2x_1 + x_n) + f(2x_1 - x_n) + f(x_n) + f(-x_n)) \\ &= 2^{n-2}f(2x_1) + 2^n(n-3)f(x_1) + 2^{n-1}(f(x_1 + x_n) + f(x_1 - x_n)) \\ &+ 2^{n-2}(n-3)(f(x_n) + f(-x_n)) - 2^n(n-2)f(x_1) - 2^{n-1}(n-2)f(x_n), \end{aligned}$$

for all  $x_1, x_n \in X$ . So

$$f(2x_1 + x_n) + f(2x_1 - x_n) = 4f(x_1 + x_n) + 4f(x_1 - x_n) + 24f(x_1) - 6f(x_n)$$

for all  $x_1, x_n \in X$ , which implies that  $f$  is quartic.

Conversely, suppose that  $f$  is quartic thus  $f$  satisfies (1.1). Hence we have  $f(0) = 0$ ,  $f$  is even and  $f(2x) = 16f(x)$  for all  $x \in X$ . Interchange  $x_1$  with  $x_2$  in (1.1), gives

$$f(2x_2 + x_1) + f(2x_2 - x_1) = 4f(x_2 + x_1) + 4f(x_2 - x_1) + 24f(x_2) - 6f(x_1) \quad (2.3)$$

for all  $x_1, x_2 \in X$ . By evenness of  $f$ , it follows from (2.3) that

$$f(x_1 + 2x_2) + f(x_1 - 2x_2) = 4f(x_1 + x_2) + 4f(x_1 - x_2) - 6f(x_1) + 24f(x_2) \quad (2.4)$$

for all  $x_1, x_2 \in X$ . Replacing  $x_2$  by  $2x_2$  in (2.4) and employing the fact that  $f(2x) = 16f(x)$  and then using (2.4), we obtain that

$$f(x_1 + 4x_2) + f(x_1 - 4x_2) = 16f(x_1 + x_2) + 16f(x_1 - x_2) - 30f(x_1) + 480f(x_2) \quad (2.5)$$

for all  $x_1, x_2 \in X$ . Putting  $x_1 = x_1 + x_2$  and  $x_2 = x_1 - x_2$  in (1.1) and then using the identity  $f(2x) = 4f(x)$ , we have

$$f(3x_1 + x_2) + f(x_1 + 3x_2) = 64(f(x_1) + f(x_2)) + 24f(x_1 + x_2) - 6f(x_1 - x_2) \quad (2.6)$$

for all  $x_1, x_2 \in X$ . Replacing  $x_1$  and  $x_2$  by  $x_1 + 2x_3$  and  $x_2 + 2x_3$  in (2.6), respectively, gives

$$\begin{aligned} f(3x_1 + x_2 + 8x_3) + f(x_1 + 3x_2 + 8x_3) \\ = 64(f(x_1 + 2x_3) + f(x_2 + 2x_3)) + 24f(x_1 + x_2 + 4x_3) - 6f(x_1 - x_2) \end{aligned} \quad (2.7)$$

for all  $x_1, x_2, x_3 \in X$ . Replacing  $x_1$  and  $x_2$  by  $x_1 - 2x_3$  and  $x_2 - 2x_3$  in (2.6), respectively, one gets that

$$\begin{aligned} f(3x_1 + x_2 - 8x_3) + f(x_1 + 3x_2 - 8x_3) \\ = 64(f(x_1 - 2x_3) + f(x_2 - 2x_3)) + 24f(x_1 + x_2 - 4x_3) - 6f(x_1 - x_2) \end{aligned} \quad (2.8)$$

for all  $x_1, x_2, x_3 \in X$ . Now, by adding (2.7) and (2.8), we arrive at

$$\begin{aligned} f(3x_1 + x_2 + 8x_3) + f(3x_1 + x_2 - 8x_3) + f(x_1 + 3x_2 + 8x_3) + f(x_1 + 3x_2 - 8x_3) \\ = 64(f(x_1 + 2x_3) + f(x_1 - 2x_3) + f(x_2 + 2x_3) + f(x_2 - 2x_3)) \\ + 24(f(x_1 + x_2 + 4x_3) + f(x_1 + x_2 - 4x_3)) - 12f(x_1 - x_2) \end{aligned} \quad (2.9)$$

for all  $x_1, x_2, x_3 \in X$ . On the other hand, we substitute  $x_1 = x_1 + 2x_3$  and  $x_2 = x_2 - 2x_3$  in (2.6), we obtain

$$\begin{aligned} f(3x_1 + x_2 + 4x_3) + f(x_1 + 3x_2 - 4x_3) \\ = 64(f(x_1 + 2x_3) + f(x_2 - 2x_3)) + 24f(x_1 + x_2) - 6f(x_1 - x_2 + 4x_3) \end{aligned} \quad (2.10)$$

for all  $x_1, x_2, x_3 \in X$ . And putting  $x_1 = x_1 - 2x_3$  and  $x_2 = x_2 + 2x_3$  in (2.6), we get

$$f(3x_1 + x_2 - 4x_3) + f(x_1 + 3x_2 + 4x_3) = 64(f(x_1 - 2x_3) + f(x_2 + 2x_3)) + 24f(x_1 + x_2) - 6f(x_1 - x_2 - 4x_3) \tag{2.11}$$

for all  $x_1, x_2, x_3 \in X$ . Adding (2.10) to (2.11), we lead to

$$f(3x_1 + x_2 + 4x_3) + f(3x_1 + x_2 - 4x_3) + f(x_1 + 3x_2 + 4x_3) + f(x_1 + 3x_2 - 4x_3) = 64(f(x_1 + 2x_3) + f(x_1 - 2x_3) + f(x_2 + 2x_3) + f(x_2 - 2x_3)) + 48f(x_1 + x_2) - 6(f(x_1 - x_2 + 4x_3) + f(x_1 - x_2 - 4x_3)) \tag{2.12}$$

for all  $x_1, x_2, x_3 \in X$ . Now, replacing  $x_3$  by  $\frac{x_3}{2}$  in (2.9), gives

$$f(3x_1 + x_2 + 4x_3) + f(3x_1 + x_2 - 4x_3) + f(x_1 + 3x_2 + 4x_3) + f(x_1 + 3x_2 - 4x_3) = 64(f(x_1 + x_3) + f(x_1 - x_3) + f(x_2 + x_3) + f(x_2 - x_3)) + 24(f(x_1 + x_2 + 2x_3) + f(x_1 + x_2 - 2x_3)) - 12f(x_1 - x_2) \tag{2.13}$$

for all  $x_1, x_2, x_3 \in X$ . If we compare (2.12) with (2.13), we conclude that

$$64(f(x_1 + 2x_3) + f(x_1 - 2x_3) + f(x_2 + 2x_3) + f(x_2 - 2x_3)) + 48f(x_1 + x_2) - 6(f(x_1 - x_2 + 4x_3) + f(x_1 - x_2 - 4x_3)) = 64(f(x_1 + x_3) + f(x_1 - x_3) + f(x_2 + x_3) + f(x_2 - x_3)) + 24(f(x_1 + x_2 + 2x_3) + f(x_1 + x_2 - 2x_3)) - 12f(x_1 - x_2) \tag{2.14}$$

for all  $x_1, x_2, x_3 \in X$ . It follows from (2.4), (2.5) and (2.14) that

$$f(x_1 - x_2 - x_3) + f(x_1 - x_2 + x_3) + f(x_1 + x_2 - x_3) + f(x_1 + x_2 + x_3) = 2(f(x_1 + x_2) + f(x_1 - x_2) + f(x_1 + x_3) + f(x_1 - x_3) + f(x_2 + x_3) + f(x_2 - x_3)) - 2^2(f(x_1) + f(x_2) + f(x_3))$$

for all  $x_1, x_2, x_3 \in X$ , which by considering  $f(2x) = 16f(x)$ , gives

$$f(x_1 - x_2 - x_3) + f(x_1 - x_2 + x_3) + f(x_1 + x_2 - x_3) + f(x_1 + x_2 + x_3) = 2(f(x_1 + x_2) + f(x_1 - x_2) + f(x_1 + x_3) + f(x_1 - x_3) + f(x_2 + x_3) + f(x_2 - x_3)) - 2^{-2}(f(2x_1) + f(2x_2) + f(2x_3)) \tag{2.15}$$

for all  $x_1, x_2, x_3 \in X$ . This means  $f$  satisfies (1.1) for  $n = 3$ . Assume that (1.1) holds on the case where  $n = p$ ; that is, we have

$$\sum_{k=2}^p \left( \sum_{i_1=2}^k \sum_{i_2=i_1+1}^{k+1} \dots \sum_{i_{p-k+1}=i_{p-k}+1}^p \right) f \left( \sum_{i=1, i \neq i_1, \dots, i_{p-k+1}}^p x_i - \sum_{r=1}^{p-k+1} x_{i_r} \right) + f \left( \sum_{i=1}^p x_i \right) = 2^{p-2} \sum_{1 \leq i < j \leq p} (f(x_i + x_j) + f(x_i - x_j)) - 2^{p-5}(p-2) \sum_{i=1}^p f(2x_i) \tag{2.16}$$

for all  $x_1, \dots, x_p \in X$ . Replacing  $x_1$  by  $x_1 + x_{p+1}$  in (2.16), we obtain

$$\begin{aligned} & \sum_{k=2}^p \left( \sum_{i_1=2}^k \sum_{i_2=i_1+1}^{k+1} \dots \sum_{i_{p-k+1}=i_{p-k}+1}^p \right) f \left( x_1 + x_{p+1} + \sum_{i=2, i \neq i_1, \dots, i_{p-k+1}}^p x_i - \sum_{r=1}^{p-k+1} x_{i_r} \right) + f \left( \sum_{i=1}^{p+1} x_i \right) \\ &= 2^{p-2} \left[ \sum_{j=2}^p (f(x_1 + x_{p+1} + x_j) + f(x_1 + x_{p+1} - x_j)) + \sum_{2 \leq i < j \leq p} (f(x_i + x_j) + f(x_i - x_j)) \right] \\ & \quad - 2^{p-5}(p-2)[f(2x_1 + 2x_{p+1}) + \sum_{i=2}^p f(2x_i)] \end{aligned} \tag{2.17}$$

for all  $x_1, \dots, x_p \in X$ . Replacing  $x_{p+1}$  by  $-x_{p+1}$  in (2.17), we obtain

$$\begin{aligned} & \sum_{k=2}^p \left( \sum_{i_1=2}^k \sum_{i_2=i_1+1}^{k+1} \dots \sum_{i_{p-k+1}=i_{p-k}+1}^p \right) f \left( x_1 - x_{p+1} + \sum_{i=2, i \neq i_1, \dots, i_{p-k+1}}^p x_i - \sum_{r=1}^{p-k+1} x_{i_r} \right) \\ & \quad + f \left( \sum_{i=1}^p x_i - x_{p+1} \right) = 2^{p-2} \left[ \sum_{j=2}^p (f(x_1 - x_{p+1} + x_j) + f(x_1 - x_{p+1} - x_j)) \right. \\ & \quad \left. + \sum_{2 \leq i < j \leq p} (f(x_i + x_j) + f(x_i - x_j)) \right] - 2^{p-5}(p-2)[f(2x_1 - 2x_{p+1}) + \sum_{i=2}^p f(2x_i)] \end{aligned} \tag{2.18}$$

for all  $x_1, \dots, x_{p+1} \in X$ . Adding (2.17) to (2.18), one gets

$$\begin{aligned} & \sum_{k=2}^{p+1} \left( \sum_{i_1=2}^k \sum_{i_2=i_1+1}^{k+1} \dots \sum_{i_{p-k+2}=i_{p-k+1}+1}^{p+1} \right) f \left( \sum_{i=1, i \neq i_1, \dots, i_{p-k+2}}^{p+1} x_i - \sum_{r=1}^{p-k+2} x_{i_r} \right) + f \left( \sum_{i=1}^{p+1} x_i \right) \\ &= 2^{p-2} \left[ \sum_{j=2}^p (f(x_1 + x_j + x_{p+1}) + f(x_1 + x_j - x_{p+1}) + f(x_1 - x_j + x_{p+1}) \right. \\ & \quad \left. + f(x_1 - x_j - x_{p+1})) + 2 \sum_{2 \leq i < j \leq p} (f(x_i + x_j) + f(x_i - x_j)) \right] \\ & \quad - 2^{p-5}(p-2)(f(2x_1 + 2x_{p+1}) + f(2x_1 - 2x_{p+1})) - 2^{p-4}(p-2) \sum_{i=2}^p f(2x_i) \end{aligned} \tag{2.19}$$

for all  $x_1, \dots, x_{p+1} \in X$ . Therefore, by the case  $n = 3$  and employing the fact that  $f(2x) = 16f(x)$ , we obtain that (1.2) holds for  $n = p + 1$ . This complete the proof of the lemma.  $\square$

### 3. Stability in non-Archimedean normed spaces

From now on, we deal with the stability problem for the generalized additive functional equation (1.1) in NAN-spaces. In the rest of the paper, let  $|16| \neq 1$ .

**THEOREM 3.1.** ([7]) *Let  $(X, d)$  be a complete generalized metric space and  $J : X \rightarrow X$  be a strictly contractive mapping with Lipschitz constant  $L < 1$ . Then, for all  $x \in X$ , either  $d(J^n x, J^{n+1} x) = \infty$  for all nonnegative integers  $n$  or there exists a positive integer  $n_0$  such that*

- (a)  $d(J^n x, J^{n+1} x) < \infty$  for all  $n_0 \geq n$ ;
- (b) the sequence  $\{J^n x\}$  converges to a fixed point  $y^*$  of  $J$ ;
- (c)  $y^*$  is the unique fixed point of  $J$  in the set  $Y = \{y \in X : d(J^{n_0} x, y) < \infty\}$ ;
- (d)  $d(y, y^*) \leq \frac{d(y, Jy)}{1-L}$  for all  $y \in Y$ .

**THEOREM 3.2.** *Let  $X$  be a non-Archimedean normed space and  $Y$  is a complete non-Archimedean space. Let  $\varphi : X^n \rightarrow [0, \infty)$  be a function such that there exists an  $\alpha < 1$  with*

$$\varphi\left(\frac{x_1}{2}, \frac{x_2}{2}, \dots, \frac{x_n}{2}\right) \leq \frac{\alpha\varphi(x_1, x_2, \dots, x_n)}{|16|}, \tag{3.1}$$

for all  $x_1, x_2, \dots, x_n \in X$ . Let  $f : X \rightarrow Y$  with  $f(0) = 0$  be a mapping satisfying

$$\|D_f(x_1, x_2, \dots, x_n)\|_Y \leq \varphi(x_1, x_2, \dots, x_n), \tag{3.2}$$

for all  $x_1, x_2, \dots, x_n \in X$ . Then there exists a unique additive mapping  $\mathfrak{S} : X \rightarrow Y$  such that

$$\|f(x) - \mathfrak{S}(x)\|_Y \leq \frac{\alpha\varphi\left(x, \overbrace{0, \dots, 0}^{n-1}\right)}{|2|^{n-1}|n-2| - |2|^{n-1}|n-2|\alpha}, \tag{3.3}$$

for all  $x \in X$ .

*Proof.* Putting  $x_1 = x$  and  $x_2 = \dots = x_n = 0$  in (3.2), we get

$$\left\| \sum_{k=2}^n \left( \sum_{i_1=2}^k \sum_{i_2=i_1+1}^{k+1} \dots \sum_{i_{n-k+1}=i_{n-k}+1}^n \right) f(x) + f(x) - 2^{n-2} \sum_{j=2}^n 2f(x) + 2^{n-5}(n-2)f(2x) \right\|_Y \leq \varphi\left(x, \underbrace{0, \dots, 0}_{n-1}\right) \tag{3.4}$$

for all  $x \in X$ . That is

$$\left\| \left( 1 + \sum_{\ell=1}^{n-1} \binom{n-1}{\ell} \right) f(x) - 2^{n-1}(n-1)f(x) + 2^{n-5}(n-2)f(2x) \right\|_Y \leq \alpha\left(x, \underbrace{0, \dots, 0}_{n-1}\right), \tag{3.5}$$



for all  $x \in X$ . So by using the equation

$$1 + \sum_{\ell=1}^{n-1} \binom{n-1}{\ell} = \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} = 2^{n-1},$$

gives

$$\left\| f(x) - \frac{1}{2^4} f(2x) \right\|_Y \leq \frac{1}{|2|^{n-1}|n-2|} \varphi\left(x, \underbrace{0, \dots, 0}_{n-1}\right). \tag{3.6}$$

Replacing  $x$  by  $\frac{x}{2}$  in (3.6), we have

$$\left\| f(x) - 2^4 f\left(\frac{x}{2}\right) \right\|_Y \leq \frac{|16|}{|2|^{n-1}|n-2|} \varphi\left(\frac{x}{2}, \underbrace{0, \dots, 0}_{n-1}\right) \leq \frac{\alpha}{|2|^{n-1}|n-2|} \varphi\left(x, \underbrace{0, \dots, 0}_{n-1}\right), \tag{3.7}$$

Consider the set  $S := \{h : X \rightarrow Y; h(0) = 0\}$  and introduce the generalized metric on  $S$ :

$$d(g, h) = \inf_{\mu \in (0, +\infty)} \|g(x) - h(x)\|_Y \leq \mu \varphi\left(x, \underbrace{0, \dots, 0}_{n-1}\right),$$

for all  $x \in X$ , where, as usual,  $\inf \phi = +\infty$ . It is easy to show that  $(S, d)$  is complete (see [12]). Now we consider the linear mapping  $J : S \rightarrow S$  such that

$$Jg(x) := 2^4 g\left(\frac{x}{2}\right)$$

for all  $x \in X$ . Let  $g, h \in S$  be given such that  $d(g, h) = \varepsilon$ . Then  $\|g(x) - h(x)\|_Y \leq \varepsilon \varphi\left(x, \underbrace{0, \dots, 0}_{n-1}\right)$  for all  $x \in X$ . Hence

$$\|Jg(x) - Jh(x)\|_Y = \left\| 2^4 g\left(\frac{x}{2}\right) - 2^4 h\left(\frac{x}{2}\right) \right\|_Y \leq \alpha \cdot \varepsilon \varphi\left(x, \underbrace{0, \dots, 0}_{n-1}\right),$$

for all  $x \in X$ . So  $d(g, h) = \varepsilon$  implies that  $d(Jg, Jh) \leq \alpha \varepsilon$ . This means that  $d(Jg, Jh) \leq \alpha d(g, h)$  for all  $g, h \in S$ . It follows from (3.7) that

$$d(f, Jf) \leq \frac{\alpha}{|2|^{n-1}|n-2|}.$$

By Theorem 3.1, there exists a mapping  $\mathfrak{S} : X \rightarrow Y$  satisfying the following:

- (1)  $\mathfrak{S}$  is a fixed point of  $J$ , i.e.,

$$\mathfrak{S}(x) = 2^4 \mathfrak{S}\left(\frac{x}{2}\right), \tag{3.8}$$

for all  $x \in X$ . The mapping  $\mathfrak{S}$  is a unique fixed point of  $J$  in the set  $M = \{g \in S : d(h, g) < \infty\}$ . This implies that  $\mathfrak{S}$  is a unique mapping satisfying (3.8) such that there exists a  $\mu \in (0, \infty)$  satisfying

$$\|f(x) - \mathfrak{S}(x)\|_Y \leq \mu \varphi\left(x, \underbrace{0, \dots, 0}_{n-1}\right)$$

for all  $x \in X$ ;

(2)  $d(J^p f, \mathfrak{S}) \rightarrow 0$  as  $p \rightarrow \infty$ . This implies the equality

$$\lim_{p \rightarrow \infty} 16^p f\left(\frac{x}{2^p}\right) = \mathfrak{S}(x), \tag{3.9}$$

for all  $x \in X$ ;

(3)  $d(f, \mathfrak{S}) \leq \frac{d(f, Jf)}{1-\alpha}$ , which implies the inequality

$$d(f, \mathfrak{S}) \leq \frac{\alpha}{|2|^{n-1}|n-2| - |2|^{n-1}|n-2|\alpha}.$$

This implies that the inequalities (3.3) holds. It follows from (3.1) and (3.2) that

$$\begin{aligned} & \|D_{\mathfrak{S}}(x_1, x_2, \dots, x_n)\|_Y \\ &= \lim_{p \rightarrow \infty} 16^p \left\| D_{\mathfrak{S}}\left(\frac{x_1}{2^p}, \frac{x_2}{2^p}, \dots, \frac{x_n}{2^p}\right) \right\|_Y \leq \lim_{p \rightarrow \infty} \alpha^p \varphi(x_1, x_2, \dots, x_n) = 0 \end{aligned}$$

for all  $x_1, x_2, \dots, x_n \in X$ . So

$$\begin{aligned} & \sum_{k=2}^n \left( \sum_{i_1=2}^k \sum_{i_2=i_1+1}^{k+1} \dots \sum_{i_{n-k+1}=i_{n-k}+1}^n \right) \mathfrak{S} \left( \sum_{i=1, i \neq i_1, \dots, i_{n-k+1}}^n x_i - \sum_{r=1}^{n-k+1} x_{i_r} \right) + \mathfrak{S} \left( \sum_{i=1}^n x_i \right) \\ &= 2^{n-2} \sum_{1 \leq i < j \leq n} (\mathfrak{S}(x_i + x_j) + \mathfrak{S}(x_i - x_j)) - 2^{n-5}(n-2) \sum_{i=1}^n \mathfrak{S}(2x_i) \end{aligned}$$

for all  $x_1, x_2, \dots, x_n \in X$ . Hence  $\mathfrak{S} : X \rightarrow Y$  is an additive mapping and we get desired results.  $\square$

**COROLLARY 3.3.** *Let  $\theta$  be a positive real number and  $r$  is a real number with  $0 < r < 1$ . Let  $f : X \rightarrow Y$  with  $f(0) = 0$  be a mapping satisfying*

$$\|D_f(x_1, x_2, \dots, x_n)\|_Y \leq \theta \left( \sum_{i=1}^n \|x_i\|^r \right), \tag{3.10}$$

for all  $x_1, x_2, \dots, x_n \in X$ . Then there exists a unique additive mapping  $\mathfrak{S} : X \rightarrow Y$  such that

$$\|f(x) - \mathfrak{S}(x)\|_Y \leq \frac{|16|\theta\|x\|^r}{|2|^{n+4r-1}|n-2| - |2|^{n+3}|n-2|}, \tag{3.11}$$

for all  $x \in X$ .

*Proof.* The proof follows from Theorem 3.2 by taking

$$\varphi(x_1, x_2, \dots, x_n) = \theta \left( \sum_{i=1}^n \|x_i\|^r \right)$$

for all  $x_1, x_2, \dots, x_n \in X$ . Then we can choose  $\alpha = |16|^{1-r}$  and we get the desired result.  $\square$

REMARK 3.4. Similar works have been done before. For example Ulam-Gavruta-Rassias product stability, (see [17]), used the control function  $\theta (\prod_{i=1}^n \|x_i\|^r)$  instead of  $\theta (\sum_{i=1}^n \|x_i\|^r)$ . But, since we put  $x_2 = \dots = x_n = 0$ , in this functional equation, the Ulam-Gavruta-Rassias product stability has the obvious approximation  $\mathfrak{S}(x) = f(x)$ . Also JMRassias mixed product-sum stability, (see [17]), used the control function  $\theta (\sum_{i=1}^n \|x_i\|^r + \prod_{i=1}^n \|x_i\|^r)$  instead of  $\theta (\sum_{i=1}^n \|x_i\|^r)$ . Again, since we put  $x_2 = \dots = x_n = 0$ , in this functional equation, the JMRassias mixed product-sum stability has the obvious approximation  $\mathfrak{S}(x) = f(x)$ .

THEOREM 3.5. Let  $X$  be a non-Archimedean normed space and  $Y$  is a complete non-Archimedean space. Let  $\varphi : X^3 \rightarrow [0, \infty)$  be a function such that there exists an  $\alpha < 1$  with  $\varphi (x_1, x_2, \dots, x_n) \leq |16|\alpha\varphi (\frac{x_1}{2}, \frac{x_2}{2}, \dots, \frac{x_n}{2})$ , for all  $x_1, x_2, \dots, x_n \in X$ . Let  $f : X \rightarrow Y$  with  $f(0) = 0$  be a mapping satisfying (3.2). Then there exists a unique additive mapping  $\mathfrak{S} : X \rightarrow Y$  such that

$$\|f(x) - \mathfrak{S}(x)\|_Y \leq \frac{\varphi(x, \overbrace{0, \dots, 0}^{n-1})}{|2|^{n-1}|n-2| - |2|^{n-1}|n-2|\alpha}, \tag{3.12}$$

for all  $x \in X$ .

Proof. Let  $(S, d)$  be the generalized metric space defined in the proof of Theorem 2.1. Now we consider the linear mapping  $J : S \rightarrow S$  such that

$$Jg(x) := \frac{g(2x)}{2^4}$$

for all  $x \in X$ . Let  $g, h \in S$  be given such that  $d(g, h) = \varepsilon$ . Then  $\|g(x) - h(x)\|_Y \leq \varepsilon\varphi(x, \overbrace{0, \dots, 0}^{n-1})$  for all  $x \in X$ . Hence

$$\|Jg(x) - Jh(x)\|_Y = \left\| \frac{g(2x)}{2^4} - \frac{h(2x)}{2^4} \right\|_Y \leq \frac{|16|\alpha \cdot \varepsilon\varphi(x, \overbrace{0, \dots, 0}^{n-1})}{|16|}$$

for all  $x \in X$ . So  $d(g, h) = \varepsilon$  implies that  $d(Jg, Jh) \leq \alpha\varepsilon$ . This means that  $d(Jg, Jh) \leq \alpha d(g, h)$  for all  $g, h \in S$ . It follows from (3.6) that

$$d(f, Jf) \leq \frac{1}{|2|^{n-1}|n-2|} \varphi(x, \overbrace{0, \dots, 0}^{n-1}).$$

By Theorem 3.1, there exists a mapping  $\mathfrak{S} : X \rightarrow Y$  satisfying the following:

(1)  $\mathfrak{S}$  is a fixed point of  $J$ , i.e.,

$$\frac{\mathfrak{S}(2x)}{2^4} = \mathfrak{S}(x) \tag{3.13}$$

for all  $x \in X$ . The mapping  $\mathfrak{S}$  is a unique fixed point of  $J$  in the set  $M = \{g \in S : d(h, g) < \infty\}$ . This implies that  $\mathfrak{S}$  is a unique mapping satisfying (3.13) such that there exists a  $\mu \in (0, \infty)$  satisfying

$$\|f(x) - \mathfrak{S}(x)\|_Y \leq \mu \varphi\left(x, \overbrace{0, \dots, 0}^{n-1}\right)$$

for all  $x \in X$ ;

(2)  $d(J^p f, \mathfrak{S}) \rightarrow 0$  as  $p \rightarrow \infty$ . This implies the equality

$$\lim_{p \rightarrow \infty} \frac{f(2^p x)}{2^{4p}} = \mathfrak{S}(x)$$

for all  $x \in X$ ;

(3)  $d(f, \mathfrak{S}) \leq \frac{d(f, Jf)}{1-\alpha}$ , which implies the inequality

$$d(f, \mathfrak{S}) \leq \frac{1}{|2|^{n-1}|n-2|}.$$

This implies that the inequalities (3.12) holds. The rest of the proof is similar to the proof of Theorem 2.1.  $\square$

**COROLLARY 3.6.** *Let  $\theta$  be a positive real number and  $r$  is a real number with  $r > 1$ . Let  $f : X \rightarrow Y$  be a mapping satisfying (3.10). Then there exists a unique additive mapping  $\mathfrak{S} : X \rightarrow Y$  such that*

$$\|f(x) - \mathfrak{S}(x)\|_Y \leq \frac{|16|\theta\|x\|^r}{|2|^{n+3}|n-2| - |2|^{n+4r-1}|n-2|} \tag{3.14}$$

for all  $x \in X$ .

*Proof.* The proof follows from Theorem 3.5 by taking

$$\varphi(x_1, x_2, \dots, x_n) = \theta \left( \sum_{i=1}^n \|x_i\|^r \right)$$

for all  $x_1, x_2, \dots, x_n \in X$ . Then we can choose  $\alpha = |16|^{r-1}$  and we get the desired result.  $\square$

**REMARK 3.7.** Similar works have been done before. For example Ulam-Gavruta-Rassias product stability, (see [17]), used the control function  $\theta(\prod_{i=1}^n \|x_i\|^r)$  instead of  $\theta(\sum_{i=1}^n \|x_i\|^r)$ . But, since we put  $x_2 = \dots = x_n = 0$ , in this functional equation, the Ulam-Gavruta-Rassias product stability has the obvious approximation  $\mathfrak{S}(x) = f(x)$ . Also JMRassias mixed product-sum stability, (see [17]), used the control function  $\theta(\sum_{i=1}^n \|x_i\|^r + \prod_{i=1}^n \|x_i\|^r)$  instead of  $\theta(\sum_{i=1}^n \|x_i\|^r)$ . Again, since we put  $x_2 = \dots = x_n = 0$ , in this functional equation, the JMRassias mixed product-sum stability has the obvious approximation  $\mathfrak{S}(x) = f(x)$ .

**THEOREM 3.8.** *Let  $G$  be an additive semigroup and  $X$  is a non-Archimedean Banach space. Assume that  $\gamma : G^n \rightarrow [0, +\infty)$  be a function such that*

$$\lim_{m \rightarrow \infty} |2|^{4m} \gamma \left( \frac{x_1}{2^m}, \frac{x_2}{2^m}, \dots, \frac{x_n}{2^m} \right) = 0, \tag{3.15}$$

for all  $x_1, x_2, \dots, x_n \in G$ . Suppose that, for any  $x \in G$ , the limit

$$\mathcal{L}(x) = \lim_{m \rightarrow \infty} \max_{0 \leq k < m} \frac{|2|^{4k+4}}{|2|^{n-1}|n-2|} \gamma \left( \frac{x}{2^{k+1}}, \underbrace{0, 0, \dots, 0}_{n-1} \right), \tag{3.16}$$

exists and  $f : G \rightarrow X$  with  $f(0) = 0$  be a mapping satisfying

$$\|D_f(x_1, x_2, \dots, x_n)\|_X \leq \gamma(x_1, x_2, \dots, x_n). \tag{3.17}$$

Then the limit

$$\mathfrak{S}(x) := \lim_{m \rightarrow \infty} 2^{4m} f \left( \frac{x}{2^m} \right)$$

exists for all  $x \in G$  and defines an additive mapping  $\mathfrak{S} : G \rightarrow X$  such that

$$\|f(x) - \mathfrak{S}(x)\|_X \leq \mathcal{L}(x). \tag{3.18}$$

Moreover, if

$$\lim_{j \rightarrow \infty} \lim_{m \rightarrow \infty} \max_{j \leq k < m+j} \frac{|2|^{4k+4}}{|2|^{n-1}|n-2|} \gamma \left( \frac{x}{2^{k+1}}, \underbrace{0, 0, \dots, 0}_{n-1} \right) = 0$$

then  $\mathfrak{S}$  is the unique additive mapping satisfying (3.18).

*Proof.* Putting  $x_1 = x$  and  $x_2 = x_3 = \dots = 0$  in (3.17) and replacing  $x$  by  $\frac{x}{2}$  in (3.17), we have

$$\left\| f(x) - 2^4 f \left( \frac{x}{2} \right) \right\|_X \leq \frac{|16|}{|2|^{n-1}|n-2|} \gamma \left( \frac{x}{2}, \underbrace{0, \dots, 0}_{n-1} \right), \tag{3.19}$$

for all  $x \in G$ . Replacing  $x$  by  $\frac{x}{2^m}$  in (3.19), we obtain

$$\left\| 2^{4m+4} f \left( \frac{x}{2^{m+1}} \right) - 2^{4m} f \left( \frac{x}{2^m} \right) \right\|_X \leq \frac{|2|^{4m+4} \gamma \left( \frac{x}{2^{m+1}}, \underbrace{0, \dots, 0}_{n-1} \right)}{|2|^{n-1}|n-2|}. \tag{3.20}$$

Thus, it follows from (3.15) and (3.20) that the sequence  $\left\{ 2^{4m} f \left( \frac{x}{2^m} \right) \right\}_{m \geq 1}$  is a Cauchy sequence. Since  $X$  is complete, it follows that  $\left\{ 2^{4m} f \left( \frac{x}{2^m} \right) \right\}_{m \geq 1}$  is convergent. Set  $\mathfrak{S}(x) := \lim_{m \rightarrow \infty} 2^{4m} f \left( \frac{x}{2^m} \right)$ . By induction on  $m$ , one can show that

$$\left\| 2^{4m} f \left( \frac{x}{2^m} \right) - f(x) \right\|_X \leq \max_{0 \leq k < m} \frac{|2|^{4k+4}}{|2|^{n-1}|n-2|} \gamma \left( \frac{x}{2^{k+1}}, \underbrace{0, \dots, 0}_{n-1} \right); \tag{3.21}$$

for all  $n \geq 1$  and  $x \in G$ . By taking  $m \rightarrow \infty$  in (3.21) and using (3.16), one obtains (3.18). By (3.15) and (3.17), we get

$$\begin{aligned} & \|D_{\mathfrak{S}}(x_1, x_2, \dots, x_n)\|_X \\ &= \lim_{m \rightarrow \infty} |2|^{4m} \left\| D_f \left( \frac{x_1}{2^m}, \frac{x_2}{2^m}, \dots, \frac{x_n}{2^m} \right) \right\|_X \leq \lim_{m \rightarrow \infty} |2|^{4m} \gamma \left( \frac{x_1}{2^m}, \frac{x_2}{2^m}, \dots, \frac{x_n}{2^m} \right) = 0 \end{aligned}$$

for all  $x_1, x_2, \dots, x_n \in X$ . So

$$\begin{aligned} & \sum_{k=2}^n \left( \sum_{i_1=2}^k \sum_{i_2=1}^{k+1} \dots \sum_{i_{n-k+1}=1}^n \right) \mathfrak{S} \left( \sum_{i=1, i \neq i_1, \dots, i_{n-k+1}}^n x_i - \sum_{r=1}^{n-k+1} x_{i_r} \right) + \mathfrak{S} \left( \sum_{i=1}^n x_i \right) \\ &= 2^{n-2} \sum_{1 \leq i < j \leq n} (\mathfrak{S}(x_i + x_j) + \mathfrak{S}(x_i - x_j)) - 2^{n-5} (n-2) \sum_{i=1}^n \mathfrak{S}(2x_i) \end{aligned}$$

for all  $x_1, x_2, \dots, x_n \in X$ . Hence  $\mathfrak{S} : G \rightarrow X$  is an additive mapping.

To prove the uniqueness property of  $\mathfrak{S}$ , let  $\mathfrak{R}$  be another mapping satisfying (3.18). Then we have

$$\begin{aligned} \left\| \mathfrak{S}(x) - \mathfrak{R}(x) \right\|_X &= \lim_{m \rightarrow \infty} |2|^{4m} \left\| \mathfrak{S} \left( \frac{x}{2^m} \right) - \mathfrak{R} \left( \frac{x}{2^m} \right) \right\|_X \\ &\leq \lim_{m \rightarrow \infty} |2|^{4m} \max \left\{ \left\| \mathfrak{S} \left( \frac{x}{2^m} \right) - f \left( \frac{x}{2^m} \right) \right\|_X, \left\| f \left( \frac{x}{2^m} \right) - \mathfrak{R} \left( \frac{x}{2^m} \right) \right\|_X \right\} \\ &\leq \lim_{j \rightarrow \infty} \lim_{m \rightarrow \infty} \max_{j \leq k < m+j} \frac{|2|^{4k+4}}{|2|^{n-1}|n-2|} \gamma \left( \frac{x}{2^{k+1}}, \underbrace{0, 0, \dots, 0}_{n-1} \right) = 0, \end{aligned}$$

for all  $x \in G$ . Therefore,  $\mathfrak{S} = \mathfrak{R}$ . This completes the proof.  $\square$

**COROLLARY 3.9.** *Let  $\xi : [0, \infty) \rightarrow [0, \infty)$  be a function satisfying  $\xi \left( \frac{t}{|2|} \right) \leq \xi \left( \frac{1}{|2|} \right) \xi(t)$ ,  $\xi \left( \frac{1}{|2|} \right) < \frac{1}{|2|^{\kappa}}$  for all  $t \geq 0$ . Assume that  $\kappa > 0$  and  $f : G \rightarrow X$  with  $f(0) = 0$  be a mapping such that*

$$\left\| D_f(x_1, x_2, \dots, x_n) \right\|_X \leq \kappa \left( \sum_{i=1}^n \xi(|x_i|) \right) \tag{3.22}$$

for all  $x_1, x_2, \dots, x_n \in G$ . Then there exists a unique additive mapping  $\mathfrak{S} : G \rightarrow X$  such that

$$\|f(x) - \mathfrak{S}(x)\| \leq \frac{\xi(|x|)}{|2|^{n-1}|n-2|}$$

*Proof.* If we define  $\gamma : G^n \rightarrow [0, \infty)$  by

$$\gamma(x_1, x_2, \dots, x_n) := \kappa \left( \sum_{i=1}^n \xi(|x_i|) \right),$$

then we have  $\lim_{m \rightarrow \infty} |2|^{4m} \gamma\left(\frac{x_1}{2^m}, \frac{x_2}{2^m}, \dots, \frac{x_n}{2^m}\right) = 0$ , for all  $x_1, x_2, \dots, x_n \in G$ . On the other hand, it follows that  $\mathfrak{L}(x) = \frac{\xi(|x|)}{|2|^{n-1}|n-2|}$  exists for all  $x \in G$ . Also, we have

$$\begin{aligned} & \lim_{j \rightarrow \infty} \lim_{m \rightarrow \infty} \max_{j \leq k < m+j} \frac{|2|^{4k+4}}{|2|^{n-1}|n-2|} \gamma\left(\frac{x}{2^{k+1}}, \underbrace{0, 0, \dots, 0}_{n-1}\right) \\ &= \lim_{j \rightarrow \infty} \frac{|2|^{4j+4}}{|2|^{n-1}|n-2|} \gamma\left(\frac{x}{2^{j+1}}, \underbrace{0, 0, \dots, 0}_{n-1}\right) = 0. \end{aligned}$$

Thus, applying Theorem 3.8, we have the conclusion. This completes the proof.  $\square$

**THEOREM 3.10.** *Let  $G$  be an additive semigroup and  $X$  is a non-Archimedean Banach space. Assume that  $\gamma : G^n \rightarrow [0, +\infty)$  be a function such that*

$$\lim_{m \rightarrow \infty} \frac{\gamma(2^m x_1, 2^m x_2, \dots, 2^m x_n)}{2^{4m}} = 0,$$

for all  $x_1, x_2, \dots, x_n \in G$ . Suppose that, for any  $x \in G$ , the limit

$$\mathfrak{L}(x) = \lim_{m \rightarrow \infty} \max_{0 \leq k < m} \frac{1}{|2|^{4k}} \gamma\left(2^k x, \underbrace{0, \dots, 0}_{n-1}\right) \tag{3.23}$$

exists and  $f : G \rightarrow X$  with  $f(0) = 0$  be a mapping satisfying (3.17). Then the limit

$$\mathfrak{S}(x) := \lim_{m \rightarrow \infty} \frac{f(2^m x)}{2^{4m}}$$

exists for all  $x \in G$  and

$$\|f(x) - \mathfrak{S}(x)\| \leq \frac{\mathfrak{L}(x)}{|2|^{n-1}|n-2|}, \tag{3.24}$$

for all  $x \in G$ . Moreover, if

$$\lim_{j \rightarrow \infty} \lim_{m \rightarrow \infty} \max_{j \leq k < m+j} \frac{1}{|2|^{4k}} \gamma\left(2^k x, \underbrace{0, \dots, 0}_{n-1}\right) = 0,$$

then  $\mathfrak{S}$  is the unique additive mapping satisfying (3.24).

*Proof.* It follows from (3.19), we get

$$\left\| f(x) - \frac{1}{2^4} f(2x) \right\|_X \leq \frac{1}{|2|^{n-1}|n-2|} \gamma\left(x, \underbrace{0, \dots, 0}_{n-1}\right), \tag{3.25}$$

for all  $x \in G$ . Replacing  $x$  by  $2^m x$  in (3.25), we obtain

$$\left\| \frac{f(2^m x)}{2^{4m}} - \frac{f(2^{m+1} x)}{2^{4m+4}} \right\|_X \leq \frac{1}{|2|^{n+4m-1}|n-2|} \gamma \left( 2^m x, \underbrace{0, \dots, 0}_{n-1} \right). \quad (3.26)$$

Thus it follows from (3.26) that the sequence  $\left\{ \frac{f(2^m x)}{2^{4m}} \right\}_{m \geq 1}$  is convergent. Set  $\mathfrak{S}(x) := \lim_{m \rightarrow \infty} \frac{f(2^m x)}{2^{4m}}$ . On the other hand, it follows from (3.26) that

$$\begin{aligned} \left\| \frac{f(2^p x)}{2^{4p}} - \frac{f(2^q x)}{2^{4q}} \right\|_X &= \left\| \sum_{k=p}^{q-1} \frac{f(2^{k+1} x)}{2^{4k+4}} - \frac{f(2^k x)}{2^{4k}} \right\|_X \leq \max_{p \leq k < q} \left\{ \left\| \frac{f(2^{k+1} x)}{2^{4k+4}} - \frac{f(2^k x)}{2^{4k}} \right\|_X \right\} \\ &\leq \frac{1}{|2|^{n-1}|n-2|} \max_{p \leq k < q} \frac{1}{|2|^{4k}} \gamma \left( 2^k x, \underbrace{0, \dots, 0}_{n-1} \right) \end{aligned}$$

for all  $x \in G$  and  $p, q \geq 0$  with  $q > p \geq 0$ . Letting  $p = 0$ , taking  $q \rightarrow \infty$  in the last inequality and using (3.23), we obtain (3.24).

The rest of the proof is similar to the proof of Theorem 3.8. This completes the proof.  $\square$

Similarly, we have the following corollary and we will omit the proof.

**COROLLARY 3.11.** *Let  $\xi : [0, \infty) \rightarrow [0, \infty)$  be a function satisfying*

$$\xi(|2|t) \leq \xi(|2|)\xi(t), \quad \xi(|2|) < |2|^4$$

for all  $t \geq 0$ . Let  $\kappa > 0$  and  $f : G \rightarrow X$  with  $f(0) = 0$  be a mapping satisfying (3.22). Then there exists a unique additive mapping  $\mathfrak{S} : G \rightarrow X$  such that

$$\|f(x) - \mathfrak{S}(x)\| \leq \frac{\kappa \xi(|x|)}{|2|^{n-1}|n-2|}.$$

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H. Azadi Kenary  
 Department of Mathematics  
 College of Sciences, Yasouj University  
 Yasouj 75918-74831, Iran  
 e-mail: azadi@yu.ac.ir

H. Keshavarz  
 Department of Mathematics  
 College of Sciences, Yasouj University  
 Yasouj 75918-74831, Iran

C. Park  
 Department of Mathematics, Hanyang University  
 Seoul 133-791, Korea  
 e-mail: baak@hanyang.ac.kr

D. Y. Shin  
 Department of Mathematics, University of Seoul  
 Seoul 130-743, Korea  
 e-mail: dyshin@uos.ac.kr