Journal of
Mathematical
Inequalities
Volume 9, Number 2 (2015), 587–596 doi:10.7153/jmi-09-50

WEIGHTED SHARP FUNCTION ESTIMATE AND BOUNDEDNESS FOR COMMUTATOR ASSOCIATED WITH SINGULAR INTEGRAL OPERATOR SATISFYING A VARIANT OF HÖRMANDER'S CONDITION

XIAOSHA ZHOU

(*Communicated by R. Oinarov*)

Abstract. In this paper, we establish a weighted sharp maximal function estimate for the commutator associated with the singular integral operator satisfying a variant of Hörmander's condition. As an application, we obtain the weighted boundedness of the commutators on Lebesgue and Morrey spaces.

1. Introduction and preliminaries

As the development of singular integral operators (see [8], [19]), their commutators have been well studied. In [4], [17], [18], the authors prove that the commutators generated by the singular integral operators and BMO functions are bounded on $L^p(R^n)$ for $1 < p < \infty$. Chanillo (see [2]) proves a similar result when singular integral operators are replaced by the fractional integral operators. In [11], [14], the boundedness for the commutators generated by the singular integral operators and Lipschitz functions on Triebel-Lizorkin and $L^p(R^n)$ $(1 < p < \infty)$ spaces are obtained. In [1], [10], the boundedness for the commutators generated by the singular integral operators and the weighted BMO and Lipschitz functions on $L^p(R^n)$ $(1 < p < \infty)$ spaces are obtained. In [9], some singular integral operators satisfying a variant of Hörmander's condition are introduced (see [20]). The purpose of this paper is to prove a weighted sharp maximal function inequality for the commutators related to the singular integral operators satisfying a variant of Hörmander's condition and weighted Lipschitz functions. As an application, we obtain the weighted boundedness of the commutator on Lebesgue and Morrey spaces.

First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f, the sharp maximal function of f is defined by

$$M^{\#}(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

Mathematics subject classification (2010): 42B20, 42B25.
Keywords and phrases: commutator; singular integral operator; sharp maximal function; Morrey space.

© ELEMENT, Zagreb
Paper JMI-09-50

where, and in what follows, $f_Q = |Q|^{-1}\int_Q f(x)dx$. It is well-known that (see [8], [19])

$$M^{\#}(f)(x) \approx \sup_{Q\ni x}\inf_{c\in C}\frac{1}{|Q|}\int_Q |f(y)-c|dy.$$

Let

$$M(f)(x) = \sup_{Q\ni x}\frac{1}{|Q|}\int_Q |f(y)|dy.$$

For $\eta > 0$, let $M^{\#}_{\eta}(f) = M^{\#}(|f|^{\eta})^{1/\eta}$ and $M_{\eta}(f) = M(|f|^{\eta})^{1/\eta}$.

For $0<\eta<n$, $1\leqslant q<\infty$ and the non-negative weight function w, set

$$M_{\eta,w,q}(f)(x) = \sup_{Q\ni x}\left(\frac{1}{w(Q)^{1-q\eta/n}}\int_Q |f(y)|^q w(y)dy\right)^{1/q}.$$

The A_p weight is defined by (see [8])

$$A_p = \left\{w\in L^1_{loc}(R^n) : \sup_Q\left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1}<\infty\right\},$$
$$1<p<\infty,$$

and

$$A_1 = \{w\in L^p_{loc}(R^n) : M(w)(x)\leqslant Cw(x), a.e.\}.$$

Given a weight function w. For $1\leqslant p<\infty$, the weighted Lebesgue space $L^p(R^n,w)$ is the space of functions f such that

$$||f||_{L^p(w)} = \left(\int_{R^n}|f(x)|^p w(x)dx\right)^{1/p}<\infty.$$

For the non-negative weight function w and $0<\beta<1$, the weighted Lipschitz space $Lip_{\beta}(w)$ is the space of functions b such that

$$||b||_{Lip_{\beta}(w)} = \sup_Q \frac{1}{w(Q)^{1+\beta/n}}\int_Q |b(y)-b_Q|dy<\infty,$$

where, and in what follows, $w(Q)=\int_Q w(x)dx$.

REMARK.

(1). It has been known that, for $b\in Lip_{\beta}(w)$, $w\in A_1$ and $x\in Q$,

$$|b_Q - b_{2^kQ}| \leqslant Ck||b||_{Lip_{\beta}(w)}w(x)w(2^kQ)^{\beta/n}.$$

(2). Let $b\in Lip_{\beta}(w)$ and $w\in A_1$. By [7], we know that spaces $Lip_{\beta}(w)$ coincide and the norms $||b||_{Lip_{\beta}(w)}$ are equivalent with respect to different values $1\leqslant p\leqslant\infty$ (see [10]).

DEFINITION 1. Let $\Phi = \{\phi_1, \ldots, \phi_m\}$ be a finite family of bounded functions in R^n. For any locally integrable function f, the Φ sharp maximal function of f is defined by

$$M_\Phi^\#(f)(x) = \sup_{Q \ni x} \inf_{\{c_1,\ldots,c_m\}} \frac{1}{|Q|} \int_Q |f(y) - \sum_{j=1}^m c_j \phi_j(x_Q - y)| dy,$$

where the infimum is taken over all m-tuples $\{c_1, \ldots, c_m\}$ of complex numbers and x_Q is the center of Q. For $\eta > 0$, let

$$M_{\Phi,\eta}^\#(f)(x) = \sup_{Q \ni x} \inf_{\{c_1,\ldots,c_m\}} \left(\frac{1}{|Q|} \int_Q |f(y) - \sum_{j=1}^m c_j \phi_j(x_Q - y)|^\eta dy \right)^{1/\eta}.$$

REMARK. Note that $M_\Phi^\# \approx M^\#(f)$ if $m = 1$ and $\phi_1 = 1$.

DEFINITION 2. Given a positive and locally integrable function f in R^n, we say that f satisfies the reverse Hölder's condition (write this as $f \in RH_\infty(R^n)$), if for any cube Q centered at the origin we have

$$0 < \sup_{x \in Q} f(x) \leqslant C \frac{1}{|Q|} \int_Q f(y) dy.$$

In this paper, we will study some singular integral operators as following (see [9], [20]).

DEFINITION 3. Let $K \in L^2(R^n)$ and satisfy

$$||\hat{K}||_{L^\infty} \leqslant C$$

and

$$|K(x)| \leqslant C|x|^{-n}.$$

There exist functions $B_1, \ldots B_m \in L^1_{loc}(R^n - \{0\})$ and $\Phi = \{\phi_1, \ldots, \phi_m\} \subset L^\infty(R^n)$ such that $|det[\phi_j(y_i)]|^2 \in RH_\infty(R^{nm})$, and for a fixed $\delta > 0$ and any $|x| > 2|y| > 0$,

$$|K(x-y) - \sum_{j=1}^m B_j(x)\phi_j(y)| \leqslant C \frac{|y|^\delta}{|x-y|^{n+\delta}}.$$

For $f \in C_0^\infty$, we define the singular integral operator related to the kernel K by

$$T(f)(x) = \int_{R^n} K(x-y) f(y) dy.$$

Let b be a locally integrable function on R^n. The commutator related to T is defined by

$$T_b(f)(x) = \int_{R^n} (b(x) - b(y)) K(x-y) f(y) dy.$$

DEFINITION 4. Let φ be a positive, increasing function on R^+ and there exists a constant $D > 0$ such that

$$\varphi(2t) \leqslant D\varphi(t) \text{ for } t \geqslant 0.$$

Let w be a non-negative weight function on R^n and f be a locally integrable function on R^n. Set, for $0 \leqslant \eta < n$ and $1 \leqslant p < n/\eta$,

$$||f||_{L^{p,\eta,\varphi}(w)} = \sup_{x\in R^n,\ d>0} \left(\frac{1}{\varphi(d)^{1-p\eta/n}} \int_{Q(x,d)} |f(y)|^p w(y)dy \right)^{1/p},$$

where $Q(x,d) = \{y \in R^n : |x-y| < d\}$. The generalized fractional weighted Morrey space is defined by

$$L^{p,\eta,\varphi}(R^n,w) = \{f \in L^1_{loc}(R^n) : ||f||_{L^{p,\eta,\varphi}(w)} < \infty\}.$$

We write $L^{p,\eta,\varphi}(R^n) = L^{p,\varphi}(R^n)$ if $\eta = 0$, which is the generalized Morrey space. If $\varphi(d) = d^\delta$, $\delta > 0$, then $L^{p,\varphi}(R^n,w) = L^{p,\delta}(R^n,w)$, which is the classical Morrey spaces (see [15], [16]). If $\varphi(d) = 1$, then $L^{p,\varphi}(R^n,w) = L^p(R^n,w)$, which is the weighted Lebesgue spaces (see [10]).

As the Morrey space may be considered as an extension of the Lebesgue space, it is natural and important to study the boundedness of the operator on the Morrey spaces (see [5], [6], [12], [13]).

It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [17–18]). In [18], Pérez and Trujillo-Gonzalez prove a sharp estimate for the multilinear commutator. The main purpose of this paper is to prove a weighted sharp inequality for the commutator. As the application, we obtain the weighted L^p-norm inequality and Morrey space boundedness for the commutator.

2. Theorems and proofs

We shall prove the following theorems.

THEOREM 1. *Let T be the singular integral operator as Definition 3, $w \in A_1$, $0<\beta<1$, $0<\eta<1$, $1<s<\infty$ and $b \in Lip_\beta(w)$. Then there exists a constant $C>0$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,*

$$M^\#_{\Phi,\eta}(T_b(f))(\tilde{x}) \leqslant C||b||_{Lip_\beta(w)} w(\tilde{x}) \left(M_{\beta,w,s}(f)(\tilde{x}) + M_{\beta,w,s}(T(f))(\tilde{x})\right).$$

THEOREM 2. *Let T be the singular integral operator as Definition 3, $w \in A_1$, $0<\beta<1$, $1<p<n/\beta$, $1/q=1/p-\beta/n$ and $b \in Lip_\beta(w)$. Then commutator T_b is bounded from $L^p(R^n,w)$ to $L^q(R^n,w^{1-q})$.*

THEOREM 3. *Let T be the singular integral operator as Definition 3, $w \in A_1$, $0<\beta<1$, $0<D<2^n$, $1<p<n/\beta$, $1/q=1/p-\beta/n$ and $b \in Lip_\beta(w)$. Then commutator T_b is bounded from $L^{p,\beta,\varphi}(R^n,w)$ to $L^{q,\varphi}(R^n,w^{1-q})$.*

To prove the theorems, we need the following lemma.

LEMMA 1. (see [9], [20]) *Let T be the singular integral operator as Definition 3. Then T is bounded on $L^p(R^n,w)$ for $w \in A_p$ with $1<p<\infty$, and weak (L^1,L^1) bounded.*

LEMMA 2. (see [7], [8]) *Suppose that* $1 \leqslant s < p < n/\eta$, $1/q = 1/p - \eta/n$ *and* $w \in A_1$. *Then*

$$||M_{\eta,w,s}(f)||_{L^q(w)} \leqslant C||f||_{L^p(w)}.$$

LEMMA 3. (see [7], [10]) *For any cube* Q, $b \in Lip_\beta(w)$, $0 < \beta < 1$ *and* $w \in A_1$, *we have*

$$\sup_{x\in Q}|b(x) - b_Q| \leqslant C||b||_{Lip_\beta(w)} w(Q)^{1+\beta/n}|Q|^{-1}.$$

LEMMA 4. (see [8]) *Let* $1 < p < \infty$, $0 < \eta < \infty$, $w \in A_\infty$ *and* $\Phi = \{\phi_1, \ldots, \phi_m\} \subset L^\infty(R^n)$ *such that* $|det[\phi_j(y_i)]|^2 \in RH_\infty(R^{nm})$. *Then*

$$\int_{R^n} M_\eta(f)(x)^p w(x)dx \leqslant C\int_{R^n} M^\#_{\Phi,\eta}(f)(x)^p w(x)dx$$

for any smooth function f *for which the left-hand side is finite.*

LEMMA 5. *Let* $0 < D < 2^n$, $1 < p < \infty$, $0 < \eta < \infty$, $w \in A_1$ *and* $\Phi = \{\phi_1, \ldots, \phi_m\} \subset L^\infty(R^n)$ *such that* $|det[\phi_j(y_i)]|^2 \in RH_\infty(R^{nm})$. *Then, for any smooth function* f *for which the left-hand side is finite,*

$$||M_\eta(f)||_{L^{p,\varphi}(w)} \leqslant C||M^\#_{\Phi,\eta}(f)||_{L^{p,\varphi}(w)}.$$

Proof. For any cube $Q = Q(x_0, d)$ in R^n, we know $M(w\chi_Q) \in A_1$ (see [3]). By Lemma 4, we have, for $f \in L^{p,\varphi}(R^n)$,

$$\begin{aligned}
&\int_Q |M_\eta(f)(x)|^p w(x)dx \\
&\leqslant \int_{R^n} |M_\eta(f)(x)|^p M(w\chi_Q)(x)dx \\
&\leqslant C\int_{R^n} |M^\#_{\Phi,\eta}(f)(x)|^p M(w\chi_Q)(x)dx \\
&= C\left[\int_Q |M^\#_{\Phi,\eta}(f)(x)|^p M(w\chi_Q)(x)dx + \sum_{k=0}^\infty \int_{2^{k+1}Q\setminus 2^kQ} |M^\#_{\Phi,\eta}(f)(x)|^p M(w\chi_Q)(x)dx\right] \\
&\leqslant C\left[\int_Q |M^\#_{\Phi,\eta}(f)(x)|^p w(x)dx + \sum_{k=0}^\infty \int_{2^{k+1}Q\setminus 2^kQ} |M^\#_{\Phi,\eta}(f)(x)|^p \frac{w(x)}{|2^{k+1}Q|}dx\right] \\
&\leqslant C\left[\int_Q |M^\#_{\Phi,\eta}(f)(x)|^p w(x)dx + \sum_{k=0}^\infty \int_{2^{k+1}Q} |M^\#_{\Phi,\eta}(f)(x)|^p \frac{M(w)(x)}{2^{n(k+1)}}dx\right] \\
&\leqslant C\left[\int_Q |M^\#_{\Phi,\eta}(f)(x)|^p w(x)dx + \sum_{k=0}^\infty \int_{2^{k+1}Q} |M^\#_{\Phi,\eta}(f)(x)|^p \frac{w(x)}{2^{nk}}dx\right] \\
&\leqslant C||M^\#_{\Phi,\eta}(f)||^p_{L^{p,\varphi}(w)} \sum_{k=0}^\infty 2^{-nk}\varphi(2^{k+1}d) \\
&\leqslant C||M^\#_{\Phi,\eta}(f)||^p_{L^{p,\varphi}(w)} \sum_{k=0}^\infty (2^{-n}D)^k\varphi(d) \\
&\leqslant C||M^\#_{\Phi,\eta}(f)||^p_{L^{p,\varphi}(w)}\varphi(d),
\end{aligned}$$

thus

$$||M_\eta(f)||_{L^{p,\varphi}(w)} \leqslant C||M^{\#}_{\Phi,\eta}(f)(x)||_{L^{p,\varphi}(w)}.$$

This finishes the proof. □

LEMMA 6. *Let T be the singular integral operator as Definition 3, $0<D<2^n$, $1<p<\infty$ and $w\in A_1$. Then*

$$||T(f)||_{L^{p,\varphi}(w)} \leqslant C||f||_{L^{p,\varphi}(w)}.$$

LEMMA 7. *Let $0<D<2^n$, $1\leqslant s<p<n/\eta$, $1/q=1/p-\eta/n$ and $w\in A_1$. Then*

$$||M_{\eta,w,s}(f)||_{L^{q,\varphi}(w)} \leqslant C||f||_{L^{p,\eta,\varphi}(w)}.$$

The proofs of two Lemmas are similar to that of Lemma 5 by Lemma 1 and 2, we omit the details.

Proof of Theorem 1. It suffices to prove for $f\in C_0^\infty(R^n)$, the following inequality holds:

$$\left(\frac{1}{|Q|}\int_Q |T_b(f)(x)-C_0|^\eta\,dx\right)^{1/\eta} \leqslant C||b||_{Lip_\beta(w)}w(\tilde{x})\left(M_{\beta,w,s}(f)(\tilde{x})+M_{\beta,w,s}(T(f))(\tilde{x})\right),$$

where Q is any a cube centered at x_0, $C_0=\sum_{j=1}^m g_j\phi_j(x_0-x)$ and $g_j=\int_{R^n}B_j(x_0-y)(b(y)-b_{2Q})f_2(y)dy$.

Fix a cube $Q=Q(x_0,d)$ and $\tilde{x}\in Q$. Write, for $f_1=f\chi_{2Q}$ and $f_2=f\chi_{(2Q)^c}$,

$$T_b(f)(x)=(b(x)-b_{2Q})T(f)(x)-T((b-b_{2Q})f_1)(x)-T((b-b_{2Q})f_2)(x).$$

Then

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q |T_b(f)(x)-C_0|^\eta dx\right)^{1/\eta}\\
&\leqslant C\left(\frac{1}{|Q|}\int_Q |(b(x)-b_{2Q})T(f)(x)|^\eta dx\right)^{1/\eta}+C\left(\frac{1}{|Q|}\int_Q |T((b-b_{2Q})f_1)(x)|^\eta dx\right)^{1/\eta}\\
&\quad +C\left(\frac{1}{|Q|}\int_Q |T((b-b_{2Q})f_2)(x)-C_0|^\eta dx\right)^{1/\eta}\\
&=I_1+I_2+I_3.
\end{aligned}$$

For I_1, by Hölder's inequality, we obtain

$$\begin{aligned}
I_1 &\leqslant \frac{C}{|Q|}\int_Q |(b(x)-b_{2Q})|w(x)^{-1/s}|T(f)(x)|w(x)^{1/s}dx\\
&\leqslant C\left(\frac{1}{|2Q|}\int_{2Q}|b(x)-b_{2Q}|^{s'}w(x)^{1-s'}dx\right)^{1/s'}\left(\frac{1}{|2Q|}\int_{2Q}|T(f)(x)|^s w(x)dx\right)^{1/s}\\
&\leqslant C||b||_{Lip_\beta(w)}\frac{w(2Q)}{|2Q|}M_{\beta,w,s}(T(f))(\tilde{x})\\
&\leqslant C||b||_{Lip_\beta(w)}w(\tilde{x})M_{\beta,w,s}(T(f))(\tilde{x}).
\end{aligned}$$

For I_2, by the Kolmogoro's inequality and weak (L^1, L^1) boundedness of T, we get, similar to the proof of I_1,

$$I_2 \leqslant \frac{C}{|Q|}\int_{2Q}|(b(x)-b_{2Q})f(x)|dx \leqslant C||b||_{Lip_\beta(w)}w(\tilde{x})M_{\beta,w,s}(f)(\tilde{x}).$$

For I_3, we have

$$\begin{aligned}
I_3 &\leqslant \frac{C}{|Q|}\int_Q\left|\int_{R^n}(K(x-y)-\sum_{j=1}^m B_j(x_0-y)\phi_j(x_0-x))(b(y)-b_{2Q})f_2(y)dy\right|dx\\
&\leqslant \frac{C}{|Q|}\int_Q\sum_{k=1}^\infty\int_{2^kd\leqslant|y-x_0|<2^{k+1}d}|K(x-y)-\sum_{j=1}^m B_j(x_0-y)\phi_j(x_0-x)||b(y)-b_{2Q}||f(y)|dydx\\
&\leqslant \frac{C}{|Q|}\int_Q\sum_{k=1}^\infty\int_{2^kd\leqslant|y-x_0|<2^{k+1}d}|K(x-y)-\sum_{j=1}^m B_j(x_0-y)\phi_j(x_0-x)||b(y)-b_{2^{k+1}Q}||f(y)|dydx\\
&\quad+\frac{C}{|Q|}\int_Q\sum_{k=1}^\infty\int_{2^kd\leqslant|y-x_0|<2^{k+1}d}|K(x-y)-\sum_{j=1}^m B_j(x_0-y)\phi_j(x_0-x)||b_{2^{k+1}Q}-b_{2Q}||f(y)|dydx\\
&= I_3^{(1)}+I_3^{(2)}.
\end{aligned}$$

For $I_3^{(1)}$, noting that $w\in A_1\subset A_s$ for $s>1$, then by Hölder's inequality and the condition of A_s, we obtain

$$\begin{aligned}
I_3^{(1)} &\leqslant \frac{C}{|Q|}\int_Q\sum_{k=1}^\infty\sup_{y\in 2^{k+1}Q}|b(y)-b_{2^{k+1}Q}|\\
&\qquad\times\int_{2^kd\leqslant|y-x_0|<2^{k+1}d}|K(x-y)-\sum_{j=1}^m B_j(x_0-y)\phi_j(x_0-x)||f(y)|dydx\\
&\leqslant \frac{C}{|Q|}\int_Q\sum_{k=1}^\infty k||b||_{Lip_\beta(w)}w(\tilde{x})w(2^{k+1}Q)^{\beta/n}\\
&\qquad\times\left(\int_{2^kd\leqslant|y-x_0|<2^{k+1}d}\frac{|x-x_0|^\delta}{|y-x_0|^{n+\delta}}|f(y)|w(y)^{1/s}w(y)^{-1/s}dy\right)dx\\
&\leqslant C||b||_{Lip_\beta(w)}w(\tilde{x})\sum_{k=1}^\infty kw(2^{k+1}Q)^{\beta/n}\frac{d^\delta}{(2^kd)^{n+\delta}}\\
&\qquad\times\left(\int_{2^{k+1}Q}|f(y)|^sw(y)dy\right)^{1/s}\left(\int_{2^{k+1}Q}w(y)^{-s'/s}dy\right)^{1/s'}\\
&\leqslant C||b||_{Lip_\beta(w)}w(\tilde{x})\sum_{k=1}^\infty kw(2^{k+1}Q)^{\beta/n}\frac{d^\delta}{(2^kd)^{n+\delta}}\\
&\qquad\times w(2^{k+1}Q)^{1/s-\beta/n}M_{\beta,w,s}(f)(\tilde{x})\left(\frac{|2^{k+1}Q|}{w(2^{k+1}Q)}\right)^{1/s}|2^{k+1}Q|^{1-1/s}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C||b||_{Lip_\beta(w)} w(\tilde{x}) M_{\beta,w,s}(f)(\tilde{x}) \sum_{k=1}^{\infty} k 2^{-k\delta} \\
&\leqslant C||b||_{Lip_\beta(w)} w(\tilde{x}) M_{\beta,w,s}(f)(\tilde{x}).
\end{aligned}
$$

For $I_3^{(2)}$, similar to the proof of $I_3^{(1)}$, we get

$$
\begin{aligned}
I_3^{(2)} &\leqslant \frac{C}{|Q|} \int_Q \sum_{k=1}^{\infty} |b_{2^{k+1}Q} - b_{2Q}| \\
&\quad \times \int_{2^k d \leqslant |y-x_0| < 2^{k+1} d} |K(x-y) - \sum_{j=1}^{m} B_j(x_0 - y) \phi_j(x_0 - x)||f(y)| dy dx \\
&\leqslant \frac{C}{|Q|} \int_Q \sum_{k=1}^{\infty} k||b||_{Lip_\beta(w)} w(\tilde{x}) w(2^{k+1}Q)^{\beta/n} \\
&\quad \times \left(\int_{2^k d \leqslant |y-x_0| < 2^{k+1} d} \frac{|x-x_0|^\delta}{|y-x_0|^{n+\delta}} |f(y)| w(y)^{1/s} w(y)^{-1/s} dy \right) dx \\
&\leqslant C||b||_{Lip_\beta(w)} w(\tilde{x}) \sum_{k=1}^{\infty} k w(2^{k+1}Q)^{\beta/n} \frac{d^\delta}{(2^k d)^{n+\delta}} \\
&\quad \times \left(\int_{2^{k+1}Q} |f(y)|^s w(y) dy \right)^{1/s} \left(\int_{2^{k+1}Q} w(y)^{-s'/s} dy \right)^{1/s'} \\
&\leqslant C||b||_{Lip_\beta(w)} w(\tilde{x}) \sum_{k=1}^{\infty} k w(2^{k+1}Q)^{\beta/n} \frac{d^\delta}{(2^k d)^{n+\delta}} \\
&\quad \times w(2^{k+1}Q)^{1/s - \beta/n} M_{\beta,w,s}(f)(\tilde{x}) \left(\frac{|2^{k+1}Q|}{w(2^{k+1}Q)} \right)^{1/s} |2^{k+1}Q|^{1-1/s} \\
&\leqslant C||b||_{Lip_\beta(w)} w(\tilde{x}) M_{\beta,w,s}(f)(\tilde{x}) \sum_{k=1}^{\infty} k 2^{-k\delta} \\
&\leqslant C||b||_{Lip_\beta(w)} w(\tilde{x}) M_{\beta,w,s}(f)(\tilde{x}).
\end{aligned}
$$

These complete the proof of Theorem 1. □

Proof of Theorem 2. Choose $1 < s < p$ in Theorem 1 and notice $w^{1-q} \in A_1$, we have, by Lemmas 1,2 and 4,

$$
\begin{aligned}
||T_b(f)||_{L^q(w^{1-q})} &\leqslant \|M_\eta(T_b(f))\|_{L^q(w^{1-q})} \leqslant C\|M^{\#}_{\Phi,\eta}(T_b(f))\|_{L^q(w^{1-q})} \\
&\leqslant C||b||_{Lip_\beta(w)} [\|M_{\beta,w,s}(T(f))w\|_{L^q(w^{1-q})} + \|M_{\beta,w,s}(f)w\|_{L^q(w^{1-q})}] \\
&\leqslant C||b||_{Lip_\beta(w)} [\|M_{\beta,w,s}(T(f))\|_{L^q(w)} + \|M_{\beta,w,s}(f)\|_{L^q(w)}] \\
&\leqslant C||b||_{Lip_\beta(w)} [\|T(f)\|_{L^p(w)} + \|f\|_{L^p(w)}] \\
&\leqslant C||b||_{Lip_\beta(w)} \|f\|_{L^p(w)}.
\end{aligned}
$$

This completes the proof of Theorem 2. □

Proof of Theorem 3. Choose $1 < s < p$ in Theorem 1 and notice $w^{1-q} \in A_1$, we have, by Lemmas 5-7,

$$\begin{aligned}
||T_b(f)||_{L^{q,\varphi}(w^{1-q})} &\leqslant \|M_\eta(T_b(f))\|_{L^{q,\varphi}(w^{1-q})} \leqslant C\|M^{\#}_{\Phi,\eta}(T_b(f))\|_{L^{q,\varphi}(w^{1-q})} \\
&\leqslant C||b||_{Lip_\beta(w)}[\|M_{\beta,w,s}(T(f))w\|_{L^{q,\varphi}(w^{1-q})} + \|M_{\beta,w,s}(f)w\|_{L^{q,\varphi}(w^{1-q})}] \\
&\leqslant C||b||_{Lip_\beta(w)}[\|M_{\beta,w,s}(T(f))\|_{L^{q,\varphi}(w)} + \|M_{\beta,w,s}(f)\|_{L^{q,\varphi}(w)}] \\
&\leqslant C||b||_{Lip_\beta(w)}[\|T(f)\|_{L^{p,\beta,\varphi}(w)} + \|f\|_{L^{p,\beta,\varphi}(w)}] \\
&\leqslant C||b||_{Lip_\beta(w)}\|f\|_{L^{p,\beta,\varphi}(w)}.
\end{aligned}$$

This completes the proof of Theorem 3. □

REFERENCES

[1] S. BLOOM, *A commutator theorem and weighted BMO*, Trans. Amer. Math. Soc., **292** (1985), 103–122.

[2] S. CHANILLO, *A note on commutators*, Indiana Univ. Math. J., **31** (1982), 7–16.

[3] R. COIFMAN AND R. ROCHBERG, *Another characterization of BMO*, Proc. Amer. Math. Soc., **79**, 249–254 (1980).

[4] R. R. COIFMAN, R. ROCHBERG AND G. WEISS, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math., **103** (1976), 611–635.

[5] G. DI FAZIO AND M. A. RAGUSA, *Commutators and Morrey spaces*, Boll. Un. Mat. Ital., **5-A** (7)(1991), 323–332.

[6] G. DI FAZIO AND M. A. RAGUSA, *Interior estimates in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients*, J. Func. Anal., **112** (1993), 241–256.

[7] J. GARCIA-CUERVA, *Weighted H^p spaces*, Dissert. Math., **162** (1979).

[8] J. GARCIA-CUERVA AND J. L. RUBIO DE FRANCIA, *Weighted norm inequalities and related topics*, North-Holland Math., **16**, Amsterdam, 1985.

[9] D. J. GRUBB AND C. N. MOORE, *A variant of Hörmander's condition for singular integrals*, Colloq. Math., **73**, 165–172 (1997).

[10] B. HU AND J. GU, *Necessary and sufficient conditions for boundedness of some commutators with weighted Lipschitz spaces*, J. of Math. Anal. and Appl., **340** (2008), 598–605.

[11] S. JANSON, *Mean oscillation and commutators of singular integral operators*, Ark. Math., **16** (1978), 263–270.

[12] L. Z. LIU, *Interior estimates in Morrey spaces for solutions of elliptic equations and weighted boundedness for commutators of singular integral operators*, Acta Math. Scientia, **25(B)** (1), 89–94 (2005).

[13] T. MIZUHARA, *Boundedness of some classical operators on generalized Morrey spaces*, in "Harmonic Analysis", Proceedings of a conference held in Sendai, Japan, 1990, 183–189.

[14] M. PALUSZYNSKI, *Characterization of the Besov spaces via the commutator operator of Coifman*, Rochberg and Weiss, Indiana Univ. Math. J., **44** (1995), 1–17.

[15] J. PEETRE, *On convolution operators leaving $L^{p,\lambda}$-spaces invariant*, Ann. Mat. Pura. Appl., **72** (1966), 295–304.

[16] J. PEETRE, *On the theory of $L^{p,\lambda}$-spaces*, J. Func. Anal., **4** (1969), 71–87.

[17] C. PÉREZ, *Endpoint estimate for commutators of singular integral operators*, J. Func. Anal., **128** (1995), 163–185.

[18] C. PÉREZ AND R. TRUJILLO-GONZALEZ, *Sharp weighted estimates for multilinear commutators*, J. London Math. Soc., **65** (2002), 672–692.

[19] E. M. STEIN, *Harmonic analysis: real variable methods, orthogonality and oscillatory integrals*, Princeton Univ. Press, Princeton NJ, 1993.

[20] R. TRUJILLO-GONZALEZ, *Weighted norm inequalities for singular integral operators satisfying a variant of Hörmander's condition*, Comment. Math. Univ. Carolin., **44**, 137–152 (2003).

(Received February 24, 2013)

Xiaosha Zhou
Department of Mathematics
Changsha University of Science and Technology
Changsha 410077, P. R. of China
e-mail: zhouxiaosha57@126.com