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THE GENERALIZED L_p–WINTERNITZ PROBLEM

TONGYI MA

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Abstract. This article introduced the notion of the (i,j) type L_p-affine surface area of convex body in $\mathbb{R}^n$, and discussed its some proposition. In addition, we consider the more general L_p-Winterniz monotonicity problem about the $(i,0)$ type L_p-affine surface area and ith L_p-projection body in $\mathbb{R}^n$, and get a positive answer in all dimensions.

1. Introduction

Let $\mathscr{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors)in Euclidean space $\mathbb{R}^n$, $\mathscr{K}_o^n$ and $\mathscr{K}_c^n$ denote the set of convex bodies containing origin in their interiors and the set of origin-symmetric convex bodies in $\mathscr{K}^n$, respectively. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$, $V(K)$ denote the n-dimensional volume of body K. If K is the standard unit ball B in $\mathbb{R}^n$, then it is denoted as $\omega_n = V(B)$.

The classical curvature function of convex body is defined as follows (see [5]): A convex body $K \in \mathscr{K}^n$ is said to have a classical curvature function $f(K,\cdot) : S^{n-1} \to \mathbb{R}$, if its surface area measure $S(K,\cdot)$ is absolutely continuous with respect to spherical Lebesgue measure S, and

$$\frac{dS(K,\cdot)}{dS} = f(K,\cdot). \tag{1}$$

We write $\mathscr{F}^n$ to denote the set of all bodies in $\mathscr{K}^n$ that has a positive continuous curvature function. Let $\mathscr{F}_o^n$ and $\mathscr{F}_c^n$ denote the set of all bodies in $\mathscr{K}_o^n$ and $\mathscr{K}_c^n$, respectively, and both of them have a positive continuous curvature function.

Let $K \in \mathscr{F}^n$, then the affine surface area $\Omega(K)$, of K is defined by (see [4, 6, 9])

$$\Omega(K) = \int_{S^{n-1}} f(K,u)^{\frac{n}{n+1}} dS(u), \tag{2}$$

where the integration is with respect to spherical Lebesgue measure on S^{n-1}.

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Winternitz in [1] proved the following results: If $K \in \mathscr{F}^n, E$ is an ellipsoid in $\mathbb{R}^n$, and satisfying $K \subset E$, then $\Omega(K) \leqslant \Omega(E)$. Subsequently, Petty in [18] defines a special convex sets of elliptic type:

$$\mathscr{V}^n = \{K \in \mathscr{F}_o^n : \exists Q \in \mathscr{K}_c^n \text{ s.t. } f(K,\cdot) = h(Q,\cdot)^{-(n+1)}\},$$

and popularized Winternitz's monotonicity results: Let $K \in \mathscr{F}^n$ and $L \in \mathscr{V}^n$, if $K \subseteq L$, then $\Omega(K) \leqslant \Omega(L)$.

In [5], Lutwak defines a special convex sets:

$$\mathscr{W}^n = \{Q \in \mathscr{F}_o^n : \exists Z \in \Pi^n, \text{s.t. } f(Q,\cdot) = h(Z,\cdot)^{-(n+1)}\},$$

where $\Pi^n = \{\Pi K : K \in \mathscr{K}^n\}$ is set of the classical projection bodies. He proved that between the projection body and the affine surface area have the similar monotonicity: Let $K \in \mathscr{K}^n$, $L \in \mathscr{W}^n$ and $\Pi K \subseteq \Pi L$, then $\Omega(K) \leqslant \Omega(L)$.

Based on the classical affine surface area, Lutwak in [9] introduced the notion of mixed affine surface area and obtained some isoperimetric inequalities for this notion. In 1996, Lutwak in [6] showed the following notion of L_p-affine surface area: For $K \in \mathscr{F}_o^n$, the L_p-affine surface area, $\Omega_p(K)$, of K is defined by

$$\Omega_p(K) = \int_{S^{n-1}} f_p(K,u)^{\frac{n}{n+p}} dS(u), \tag{3}$$

where $f_p(K)$ is the L_p-curvature function of the convex body K.

Further, Lutwak is given the extension of the concept of L_p-affine surface area as follows (see [6]): For $p \geqslant 1$ and $K \in \mathscr{K}_o^n$, the L_p-affine surface area, $\Omega_p(K)$, of K can be defined by

$$n^{-\frac{p}{n}} \Omega_p(K)^{\frac{n+p}{n}} = \inf\{nV_p(K,Q^*)V(Q)^{\frac{p}{n}} : Q \in \mathscr{S}_o^n\}, \tag{4}$$

where $V_p(K,L)$ is the L_p-mixed volume of body K and L, $\mathscr{S}_o^n$ denote the set of star bodies (about the origin).

Lutwak, Yang and Zhang posed the notion of L_p-projection body as follows (see [10, 14]): For each $K \in \mathscr{K}^n$ and $p \geqslant 1$, the L_p-projection body, $\Pi_p K$, of K is an origin-symmetric convex body whose support function is given by

$$h^p_{\Pi_p K}(u) = \frac{1}{n\omega_n c_{n-2,p}} \int_{S^{n-1}} |\langle u,v\rangle|^p dS_p(K,v), \tag{5}$$

for all $u \in S^{n-1}$, where $S_p(K,\cdot)$ is a positive Borel measure on S^{n-1}, called the L_p-surface area measure of K, and

$$c_{n,p} = \frac{\omega_{n+p}}{\omega_2 \omega_n \omega_{p-1}}.$$

The well-known L_p-Winterniz monotonicity problem can be expressed as follows: If K and L are two origin-symmetric convex bodies in $\mathbb{R}^n$, and both of them have a positive continuous curvature function. Suppose that

$$\Pi_p K \subseteq \Pi_p L$$

for $p \geqslant 1$. Does it follow that

$$\Omega_p(K) \leqslant \Omega_p(L)?$$

The L_p-Winterniz monotonicity problem was solved independently by Yuan, Lv and Leng ([23]), as well as Ma and Wang ([13]). Yuan, Lv and Leng defines a special convex sets:

$$\mathscr{W}_p^n = \{Q \in \mathscr{F}_o^n : \exists Z \in \Pi_p^n, \text{s.t. } f_p(Q,\cdot) = h(Z,\cdot)^{-(n+p)}\},$$

where $\Pi_p^n = \{\Pi_p K : K \in \mathscr{K}^n\}$ is set of the L_p-projection bodies. And they proved the following result:

THEOREM A. *Let $K \in \mathscr{K}_o^n, L \in \mathscr{W}_p^n$ and $n \neq p \geqslant 1$. If $\Pi_p K \subseteq \Pi_p L$, then*

$$\Omega_p(K) \leqslant \Omega_p(L).$$

Ma and Wang in [13] proved that the L_p-Winterniz monotonicity problem has a positive answer if and only if for every $Q \in \mathscr{F}_c^n$ such that $(\mathbb{R}^n, \|\cdot\|_{\Lambda_p Q})$ is isometric embedding to a subspace of L_p, where $\Lambda_p L$ denotes L_p-curvature image of $L \in \mathscr{F}_o^n$.

Wang and Leng in [22] shown the notion of ith L_p-projection body as follows: For each $K \in \mathscr{K}^n$, real $p \geqslant 1$ and $i = 0, 1, \cdots, n-1$, the ith L_p-projection body, $\Pi_{p,i}K$, of K is an origin-symmetric convex body whose support function is given by

$$h^p_{\Pi_{p,i}K}(u) = \frac{1}{n\omega_n c_{n-2,p}} \int_{S^{n-1}} |\langle u, v\rangle|^p dS_{p,i}(K,v), \tag{6}$$

for all $u \in S^{n-1}$. Where $S_{p,i}(K,\cdot)$ $(i = 0,1,\cdots,n-1)$ is the ith L_p-surface area measure with $n-i-1$ copies of K and i copies of B. More precisely, the Borel measure $S_{p,i}(K,\cdot)$ on S^{n-1}, is defined by ([7])

$$S_{p,i}(K,\omega) = \int_\omega h_K^{1-p}(u) dS_i(K,u),$$

for each Borel $\omega \subset S^{n-1}$. If $i = 0$, $S_{p,i}(K,\cdot)$ is just L_p-surface area measure $S_p(K,\cdot)$. A convex body M is called the ith L_p-projection body if there is a convex body K such that $M = \Pi_{p,i}K$. Obviously, $\Pi_{p,0}K = \Pi_p K$. For the standard unit ball B, we have $\Pi_{p,i}B = B$.

Recently, Liu, Wang and He ([11]), Lu and Wang ([12]), Ma and Liu ([15,16]) independently proposed the following concept of ith L_p-curvature function of convex body: Let $p \geqslant 1$, $i = 0,1,\cdots,n-1$. A convex body $K \in \mathscr{K}_o^n$ is said to have an ith L_p-curvature function $f_{p,i}(K,\cdot) : S^{n-1} \to \mathbb{R}$, if its ith L_p-surface area measure $S_{p,i}(K,\cdot)$ is absolutely continuous with respect to spherical Lebesgue measure S, and has the Radon-Nikodym derivative

$$\frac{dS_{p,i}(K,\cdot)}{dS} = f_{p,i}(K,\cdot). \tag{7}$$

If the ith surface area measure $S_i(K,\cdot)$ is absolutely continuous with respect to spherical Lebesgue measure S, we have

$$f_{p,i}(K,u) = h(K,u)^{1-p} f_i(K,u), \tag{8}$$

where $f_i(K,\cdot)$ is called the ith curvature function of a body $K \in \mathscr{K}_o^n$ (see [7]). Obviously, $f_0(K,\cdot) = f(K,\cdot)$ and $f_{p,0}(K,\cdot) = f_p(K,\cdot)$.

We write $\mathscr{F}_i^n$ to denote the subset of $\mathscr{K}^n$ that has a positive continuous ith curvature function. Let $\mathscr{F}_{i,o}^n, \mathscr{F}_{i,c}^n$ to denote the subset of all bodies in $\mathscr{K}_o^n, \mathscr{K}_c^n$, respectively, and both of them have a positive continuous ith curvature function. In particular, $\mathscr{F}_0^n := \mathscr{F}^n$, $\mathscr{F}_{0,o}^n := \mathscr{F}_o^n$, $\mathscr{F}_{0,c}^n := \mathscr{F}_c^n$.

According to the ith L_p-curvature function $f_{p,i}(K,\cdot)$, we have defined the concept of (i,j)type L_p-affine surface area $\Omega_{p,j}^{(i)}(K)$ of convex body $K \in \mathscr{K}_o^n$ (see Section 3). In particular, the $(i,0)$type L_p-affine surface area $\Omega_p^{(i)}(K) = \Omega_{p,0}^{(i)}(K)$ defined as follows:

DEFINITION 1.1. For $K \in \mathscr{F}_{i,o}^n$ and $p \geqslant 1$, the $(i,0)$type L_p-affine surface area, $\Omega_p^{(i)}(K)$, of K is defined by

$$\Omega_p^{(i)}(K) = \int_{S^{n-1}} f_{p,i}(K,u)^{\frac{n-i}{n+p-i}} dS(u). \tag{9}$$

Obviously, $\Omega_p^{(0)}(K)$ is just L_p-affine surface area $\Omega_p(K)$.

Together with the ith L_p-projection body $\Pi_{p,i}K$ and the $(i,0)$type L_p-affine surface area $\Omega_p^{(i)}(K)$, a very natural the generalized L_p-Winterniz monotonicity problem is: Let K and L are two origin-symmetric convex bodies in $\mathbb{R}^n$, and both of them have a positive continuous ith curvature function. Suppose that

$$\Pi_{p,i}K \subseteq \Pi_{p,i}L$$

for $i = 0,1,\cdots,n-1$ and $p \geqslant 1$. Does it follow that

$$\Omega_p^{(i)}(K) \leqslant \Omega_p^{(i)}(L)?$$

The main purpose of this article is study specific affirmative answers to the generalized L_p-Winterniz monotonicity problem for the ith L_p-projection body $\Pi_{p,i}K$. At the same time, some properties of the (i,j)type L_p-affine surface area will be discussed.

We denote

$$\mathscr{W}_{p,i}^n = \{Q \in \mathscr{F}_{i,c}^n : \exists Z \in \Pi_{p,i}^n, \ \text{s.t. } f_{p,i}(Q,\cdot) = h(Z,\cdot)^{-(n+p-i)}\},$$

and

$$\mathscr{P}_{p,i}^n = \{M \in \mathscr{F}_{i,c}^n : \exists Q \in \Pi_{p,i}^n, \ \text{s.t. } Q^* = \Lambda_{p,i}M\},$$

where $\Pi_{p,i}^n = \{\Pi_{p,i}K : K \in \mathscr{K}^n\}$ is the set of ith L_p-projection bodies, $\Lambda_{p,i}M$ is the ith L_p-curvature image of convex body M (see Section 3).

From the definition (27) of the i-th L_p-curvature image, we easily see that

$$K \in \mathscr{W}_{p,i}^n \text{ if and only if } K \in \mathscr{P}_{p,i}^n.$$

Our main result is the following two Theorems.

THEOREM 1.1. *Let* $n-i\neq p\geqslant 1,\ i=0,1,\cdots,n-1,\ K\in\mathscr{K}_o^n$ *and* $L\in\mathscr{W}_{p,i}^n$. *If*

$$\Pi_{p,i}K\subseteq\Pi_{p,i}L,$$

then

$$\Omega_p^{(i)}(K)\leqslant\Omega_p^{(i)}(L).$$

THEOREM 1.2. *Let* $n-i\neq p\geqslant 1,\ i=0,1,\cdots,n-1,\ K\in\mathscr{K}_o^n$ *and* $L\in\mathscr{W}_{p,i}^n$. *If for all* $Q\in\mathscr{K}_o^n$ *such that*

$$W_{p,i}(K,Q)\leqslant W_{p,i}(L,Q),$$

then

$$\Omega_p^{(i)}(K)\leqslant\Omega_p^{(i)}(L).$$

Contents of the paper. In Section 2 we will introduce some preparatory knowledge of convex body geometry; In Section 3 we propose two new concepts of the $(i,0)$ type L_p-affine surface area and (i,j) type L_p-affine surface area. In addition, some properties for the ith L_p-curvature image and (i,j) type L_p-affine surface area have been discussed; In Section 4 we will study the answers to the generalized L_p-Winterniz monotonicity problem, that is to complete the proof of Theorem 1.1 and Theorem 1.2.

2. Preliminaries

2.1. Support function, radial function and polar body of convex body

If $K\in\mathscr{K}^n$, then its support function $h_K=h(K,\cdot)$ is defined by $h(K,x)=\max\{\langle x,y\rangle : y\in K\},\ x\in\mathbb{R}^n$, where $\langle x,y\rangle$ denotes the standard inner product of x and y. Obviously, if $K\in\mathscr{K}^n$, λ is a positive constant and $x\in\mathbb{R}^n$, then $h(\lambda K,x)=\lambda h(K,x)$.

If K is a compact star-shaped (about the origin) in $\mathbb{R}^n$, its radial function $\rho_K=\rho(K,\cdot)$ is defined by $\rho(K,x)=\max\{\lambda\geqslant 0:\lambda x\in K\},x\in\mathbb{R}^n\backslash\{0\}$. When ρ_K is positive and continuous, K is called a star body (about the origin). Let $\mathscr{S}_o^n$ denote the set of star bodies (about the origin) in $\mathbb{R}^n$, and let $\mathscr{S}_c^n$ denote the set of origin-symmetric star bodies in $\mathscr{S}_o^n$. Two star bodied K and L are said to be dilates each other if $\rho_K(u)/\rho_L(u)$ is independent on $u\in S^{n-1}$. Obviously, for $K\in\mathscr{S}_o^n$, $\alpha>0$ and $x\in\mathbb{R}^n$, we have $\rho(\alpha K,x)=\alpha\rho(K,x)$ and $\rho(K,\alpha x)=\alpha^{-1}\rho(K,x)$; If $K\in\mathscr{S}_o^n$, $\phi\in GL(n)$, $x\in\mathbb{R}^n$, then $\rho(\phi K,x)=\rho(K,\phi^{-1}x)$.

For $K\in\mathscr{K}_o^n$, the polar body, K^*, of K is defined by $K^*=\{x\in\mathbb{R}^n:\langle x,y\rangle\leqslant 1,y\in K\}$. Obviously, we have $(K^*)^*=K$. If $\lambda>0$, then $(\lambda K)^*=\lambda^{-1}K^*$; If $\phi\in GL(n)$, then

$$(\phi K)^*=\phi^{-t}K^*. \tag{10}$$

For $K\in\mathscr{K}_o^n$, the support and radial function of the polar body, K^*, of K are defined respectively by (see [2, 17])

$$h_{K^*}(u)=\frac{1}{\rho_K(u)}\quad\text{and}\quad\rho_{K^*}(u)=\frac{1}{h_K(u)}, \tag{11}$$

for all $u\in S^{n-1}$.

2.2. The L_p-mixed volume and L_p-mixed quermassintegrals

For $p \geqslant 1$, $K, L \in \mathscr{K}_o^n$ and $\varepsilon > 0$, the Firey L_p-combination $K +_p \varepsilon \cdot L \in \mathscr{K}_o^n$ is defined by (see [7])

$$h(K +_p \varepsilon \cdot L, \cdot)^p = h(K, \cdot)^p + \varepsilon h(L, \cdot)^p,$$

where "$\cdot$" in $\varepsilon \cdot L$ denotes the Firey scalar multiplication, i.e., $\varepsilon \cdot L = \varepsilon^{\frac{1}{p}} L$.

Associated with the Firey L_p-combination, the L_p-mixed volume $V_p(K, L)$ of K and L is defined (see [7])

$$\frac{n}{p} V_p(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon}. \tag{12}$$

Corresponding to each $K \in \mathscr{K}_o^n$, there is a positive Borel measure $S_p(K, \cdot)$ on S^{n-1} such that (see [14])

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L^p(v) dS_p(K, v), \tag{13}$$

for each $L \in \mathscr{K}_o^n$. the measure $S_p(K, \cdot)$ is just the L_p-surface area measure of K, which is absolutely continuous with respect to classical surface area measure $S(K, \cdot)$, and has the Radon-Nikodym derivative

$$\frac{dS_p(K, \cdot)}{dS(K, \cdot)} = h^{1-p}(K, \cdot). \tag{14}$$

The mixed quermassintegrals $W_i(K, L)$ with $n - i - 1$ copies of K, i copies of $L(0, 1, \cdots, n-1)$ is defined by (see [7])

$$(n - i) W_i(K, L) = \lim_{\varepsilon \to 0^+} \frac{W_i(K + \varepsilon \cdot L) - W_i(K)}{\varepsilon}. \tag{15}$$

If $L = B$, then $W_i(K, B)$ is just ith quermassintegrals $W_i(K)$.

For $K \in \mathscr{K}^n$ and $i = 0, 1, \cdots, n-1$, there exists a regular Borel measure $S_i(K, \cdot)$ on S^{n-1}, such that the mixed quermassintegrals $W_i(K, L)$ has the following integral representation (see [7, 19]):

$$W_i(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L(v) dS_i(K, v) \tag{16}$$

for all $L \in \mathscr{K}^n$. As a general reference for the mixed surface area measure we recommend the article by Lutwak ([7]). From the fact that $S_i(K, \cdot)$ is generated only by i copies of B and $(n - 1 - i)$ copies of K, we know that the measure $S_{n-1}(K, \cdot)$ is independent of the body K, and is just ordinary Lebesgue measure S on S^{n-1}. In fact, the ith surface area measure of the unit ball, $S_i(B, \cdot) = S$ for all i. The surface area measure $S_0(K, \cdot)$ will frequently be written simply as $S(K, \cdot)$.

For $K,L \in \mathscr{K}_o^n$, $\varepsilon > 0$ and real $p \geqslant 1$, the L_p-mixed quermassintegrals, $W_{p,i}(K,L)$, of K and L $(i = 0,1,\ldots,n-1)$ are defined by (see [7])

$$\frac{n-i}{p} W_{p,i}(K,L) = \lim_{\varepsilon \to 0^+} \frac{W_i(K +_p \varepsilon \cdot L) - W_i(K)}{\varepsilon}. \tag{17}$$

Further, Lutwak ([7]) has shown that, for $p \geqslant 1$, $i = 0,1,\ldots,n-1$ and each $K \in \mathscr{K}_o^n$, there exists a positive Borel measure $S_{p,i}(K,\cdot)$ on S^{n-1}, such that the L_p-mixed quermassintegral $W_{p,i}(K,L)$ has the following integral representation:

$$W_{p,i}(K,L) = \frac{1}{n} \int_{S^{n-1}} h_L^p(v) dS_{p,i}(K,v), \tag{18}$$

for all $L \in \mathscr{K}_o^n$. It turns out that the measure $S_{p,i}(K,\cdot)$ $(i = 0,1,\ldots,n-1)$ on S^{n-1} is absolutely continuous with respect to $S_i(K,\cdot)$, and has the Radon-Nikodym derivative

$$\frac{dS_{p,i}(K,\cdot)}{dS_i(K,\cdot)} = h^{1-p}(K,\cdot). \tag{19}$$

From the definition (18) of L_p-mixed quermassintegrals and the definition (13) of L_p-mixed volume, it follows immediately that, for $K,L \in \mathscr{K}_o^n$ and for all $p \geqslant 1$, $W_{p,i}(K,K) = W_i(K)$, $W_{p,0}(K,L) = V_p(K,L)$.

If $K \in \mathscr{F}_{i,o}^n$, $L \in \mathscr{K}_o^n$, $p \geqslant 1$, by definition (7), then the formula (18) of the L_p-mixed quermassintegral can be rewritten as follows:

$$W_{p,i}(K,L) = \frac{1}{n} \int_{S^{n-1}} h(L,u)^p f_{p,i}(K,u) dS(u). \tag{20}$$

2.3. Dual quermassintegrals and L_p-dual mixed quermassintegrals

For $K \in \mathscr{S}_o^n$ and any real i, the dual quermassintegrals, $\widetilde{W}_i(K)$, of K are defined by (see [8])

$$\widetilde{W}_i(K) = \frac{1}{n} \int_{S^{n-1}} \rho_K^{n-i}(u) dS(u). \tag{21}$$

Obviously, $\widetilde{W}_0(K) = V(K)$.

For $K,L \in \mathscr{S}_o^n$, $p \geqslant 1$ and $\varepsilon > 0$, the L_p-harmonic radial combination $K \widetilde{+}_{-p} \varepsilon \cdot L \in S_o^n$ is defined by (see [7])

$$\rho(K \widetilde{+}_{-p} \varepsilon \cdot L, \cdot)^{-p} = \rho(K,\cdot)^{-p} + \varepsilon \rho(L,\cdot)^{-p}.$$

Note that here "$\varepsilon \cdot L$" is different from "$\varepsilon \cdot L$" in L_p-combination.

For $K,L \in \mathscr{S}_o^n$, $\varepsilon > 0$, $p \geqslant 1$ and real $i \neq n$, the L_p-dual mixed quermassintegrals, $\widetilde{W}_{-p,i}(K,L)$, of K and L are defined by (see [21])

$$\frac{n-i}{-p} \widetilde{W}_{-p,i}(K,L) = \lim_{\varepsilon \to 0^+} \frac{\widetilde{W}_i(K \widetilde{+}_{-p} \varepsilon \cdot L) - \widetilde{W}_i(K)}{\varepsilon}. \tag{22}$$

If $i=0$, we easily see that (22) is just definition of L_p-dual mixed volume, i.e., $\widetilde{W}_{-p,0}(K,L)=\widetilde{V}_{-p}(K,L)$.

From (22), the integral representation of the L_p-dual mixed quermassintegrals is given by (see [21]): If $K,L\in\mathscr{S}_o^n$, $p\geqslant 1$, and real $i\neq n$, $i\neq n+p$, then

$$\widetilde{W}_{-p,i}(K,L)=\frac{1}{n}\int_{S^{n-1}}\rho_K^{n+p-i}(u)\rho_L^{-p}(u)dS(u). \tag{23}$$

Together with (21) and (23), for $K\in\mathscr{S}_o^n$, $p\geqslant 1$, and $i\neq n,n+p$, we have $\widetilde{W}_{-p,i}(K,K)=\widetilde{W}_i(K)$.

2.4. The ith curvature function, L_p-curvature function, L_p-curvature image and ith L_p-curvature image

A body $K\in\mathscr{K}^n$ is said to have a continuous ith curvature function $f_i(K,\cdot)):S^{n-1}\to[0,\infty)$ if and only if $S_i(K,\cdot)$ is absolutely continuous with respect to S and has the Radon-Nikodym derivative (see [7])

$$\frac{dS_i(K,\cdot)}{dS}=f_i(K,\cdot). \tag{24}$$

A convex body $K\in\mathscr{K}_o^n$ is said to have a L_p-curvature function $f_p(K,\cdot):S^{n-1}\to\mathbb{R}$, if its L_p-surface area measure $S_p(K,\cdot)$ is absolutely continuous with respect to spherical Lebesgue measure S, and has the Radon-Nikodym derivative (see [6])

$$\frac{dS_p(K,\cdot)}{dS}=f_p(K,\cdot). \tag{25}$$

In addition, Lutwak in [6] showed the notion of L_p-curvature image as follows: For each $K\in\mathscr{F}_o^n$ and $p\geqslant 1$, define $\Lambda_pK\in\mathscr{S}_o^n$, the L_p-curvature image of K, by

$$\rho(\Lambda_pK,\cdot)^{n+p}=\frac{V(\Lambda_pK)}{\omega_n}f_p(K,\cdot). \tag{26}$$

Note that for $p=1$, this definition is different from the classical curvature image (see [6]).

According to the concept of ith L_p-curvature function of convex body, we introduce the concept of ith L_p-curvature image of convex body as follows:

DEFINITION 2.1. For each $K\in\mathscr{F}_{i,o}^n$ $(i=0,1,\cdots,n-1)$ and real $p\geqslant 1$, define $\Lambda_{p,i}K\in\mathscr{S}_o^n$, the ith L_p-curvature image of K, by

$$\rho(\Lambda_{p,i}K,\cdot)^{n+p-i}=\frac{\widetilde{W}_i(\Lambda_{p,i}K)}{\omega_n}f_{p,i}(K,\cdot). \tag{27}$$

Taking $i=0$ in (27), using the formula $\widetilde{W}_0(K)=V(K)$ and $f_{p,0}(K,\cdot)=f_p(K,\cdot)$, we have $\Lambda_{p,0}K=\Lambda_pK$.

2.5. The ith L_p-mixed affine surface area and ith L_p-affine surface area

Luwak in [6] introduced the notion of L_p-mixed affine surface area: For $p \geqslant 1$, the L_p-mixed affine surface area, $\Omega_p(K_1,\cdots,K_n)$, of $K_1,\cdots,K_n \in \mathscr{F}_o^n$ is defined by

$$\Omega_p(K_1,\cdots,K_n) = \int_{S^{n-1}} [f_p(K_1,u)\cdots f_p(K_n,u)]^{\frac{1}{n+p}} dS(u). \tag{28}$$

Taking $K_1 = \cdots = K_{n-i} = K$ and $K_{n-i+1} = \cdots = K_n = L$ $(i = 0,\cdots,n)$ in (28), we denote $\Omega_{p,i}(K,L) = \Omega_p(K,\cdots,L,\cdots,L)$, with $n-i$ copies of K, and i copies of L. From this, if i is any real, Wang and Leng (see [20]) introduced the concept of the ith L_p-mixed affine surface area as follows: For $K,L \in \mathscr{F}_o^n$, $p \geqslant 1$, $i \in \mathbb{R}$, the ith L_p-mixed affine surface area, $\Omega_{p,i}(K,L)$, of K,L is defined by

$$\Omega_{p,i}(K,L) = \int_{S^{n-1}} f_p(K,u)^{\frac{n-i}{n+p}} f_p(L,u)^{\frac{i}{n+p}} dS(u). \tag{29}$$

Specially, for the case $i = -p$, it follows that

$$\Omega_{p,-p}(K,L) = \int_{S^{n-1}} f_p(K,u) f_p(L,u)^{\frac{-p}{n+p}} dS(u). \tag{30}$$

If $p = 1$, then $\Omega_{1,-1}(K,L)$ is just $\Omega_{-1}(K,L)$ (see [4]).

Let $L = B$ in (29) and write $\Omega_{p,i}(K,B) = \Omega_{p,i}(K)$. Then ith L_p-affine surface area of $K \in \mathscr{F}_o^n$ is expressed as follows:

$$\Omega_{p,i}(K) = \int_{S^{n-1}} f_p(K,u)^{\frac{n-i}{n+p}} dS(u). \tag{31}$$

3. The (i,j) type L_p-affine surface area

In this section, based on the concept of ith L_p-curvature function of convex body, we introduced the notion of the i-type L_p-mixed affine surface area of convex bodies $K_1,K_2,\cdots,K_{n-i}$ $(i = 0,1,\cdots,n-1)$ as follows:

DEFINITION 3.1. For $p \geqslant 1$, $i = 0,1,\cdots,n-1$, the i-type L_p-mixed affine surface area, $\Omega_p^{(i)}(K_1,\cdots,K_{n-i})$, of $K_1,\cdots,K_{n-i} \in \mathscr{F}_{i,o}^n$ is defined by

$$\Omega_p^{(i)}(K_1,\cdots,K_{n-i}) = \int_{S^{n-1}} [f_{p,i}(K_1,u)\cdots f_{p,i}(K_{n-i},u)]^{\frac{1}{n+p-i}} dS(u). \tag{32}$$

From (32), let $K_1 = \cdots = K_{n-i-j} = K$ and $K_{n-i-j+1} = \cdots = K_{n-i} = L$ $(j = 0,\cdots,n-i)$, we denote $\Omega_{p,j}^{(i)}(K,L) = \Omega_p^{(i)}(K,\cdots,K,L,\cdots,L)$, with $n-i-j$ copies of K, and j copies of L. From this, if j is any real, we can define that:

DEFINITION 3.2. For $K,L \in \mathscr{F}_{i,o}^n$, $i = 0,\cdots,n-1$, $p \geqslant 1$, $j \in \mathbb{R}$, the (i,j) type L_p-mixed affine surface area, $\Omega_{p,j}^{(i)}(K,L)$, of K,L is defined by

$$\Omega_{p,j}^{(i)}(K,L) = \int_{S^{n-1}} f_{p,i}(K,u)^{\frac{n-i-j}{n+p-i}} f_{p,i}(L,u)^{\frac{j}{n+p-i}} dS(u). \tag{33}$$

Specially, for the case $j=-p$ in (33), it follows that

$$\Omega_{p,-p}^{(i)}(K,L)=\int_{S^{n-1}} f_{p,i}(K,u)f_{p,i}(L,u)^{\frac{-p}{n+p-i}}dS(u). \tag{34}$$

Take $L=B$ in (33), and write

$$\Omega_{p,j}^{(i)}(K):=\Omega_{p,j}^{(i)}(K,B). \tag{35}$$

Because for $u\in S^{n-1}$, $S_i(B,u)=S$, $h(B,u)=1$, using this, (19) and (7) we get $f_{p,i}(B,u)=1$. This, together with (33) and (35) yield

$$\Omega_{p,j}^{(i)}(K)=\int_{S^{n-1}} f_{p,i}(K,u)^{\frac{n-i-j}{n+p-i}}dS(u), \tag{36}$$

$\Omega_{p,j}^{(i)}(K)$ is called the (i,j) type L_p-affine surface area of $K\in\mathscr{F}_{i,o}^n$.

Obviously, from (29), (33), (9) and (36), we have that

$$\Omega_{p,j}^{(0)}(K,L)=\Omega_{p,j}(K,L), \tag{37}$$

$$\Omega_{p,0}^{(i)}(K)=\Omega_p^{(i)}(K), \tag{38}$$

$$\Omega_{p,0}^{(i)}(K,L)=\Omega_p^{(i)}(K), \tag{39}$$

$$\Omega_{p,j}^{(i)}(K,K)=\Omega_p^{(i)}(K), \tag{40}$$

$$\Omega_{p,n-i}^{(i)}(K,L)=\Omega_p^{(i)}(L). \tag{41}$$

PROPOSITION 3.1. *Let $K\in\mathscr{F}_{i,o}^n$, $i=0,1,\cdots,n-1$, $j\in\mathbb{R}$ and $p\geqslant 1$, then*

$$\Omega_{p,j}^{(i)}(K)=n\left(\frac{\omega_n}{\widetilde{W}_i(\Lambda_{p,i}K)}\right)^{\frac{n-i-j}{n+p-i}}\widetilde{W}_{i+j}(\Lambda_{p,i}K). \tag{42}$$

In particular, take $j=0$ in (42), then

$$\Omega_p^{(i)}(K)=n\omega_n^{\frac{n-i}{n+p-i}}\widetilde{W}_i(\Lambda_{p,i}K)^{\frac{p}{n+p-i}}. \tag{43}$$

Proof. From (36), (27) and (21), we have

$$\begin{aligned}\Omega_{p,j}^{(i)}(K)&=\int_{S^{n-1}} f_{p,i}(K,u)^{\frac{n-i-j}{n+p-i}}dS(u)\\&=\left(\frac{\omega_n}{\widetilde{W}_i(\Lambda_{p,i}K)}\right)^{\frac{n-i-j}{n+p-i}}\int_{S^{n-1}}\rho(\Lambda_{p,i}K,u)^{n-i-j}dS(u)\\&=\left(\frac{\omega_n}{\widetilde{W}_i(\Lambda_{p,i}K)}\right)^{\frac{n-i-j}{n+p-i}}\widetilde{W}_{i+j}(\Lambda_{p,i}K).\quad\square\end{aligned}$$

PROPOSITION 3.2. *Let* $p \geqslant 1$, $K \in \mathscr{F}^n_{i,o}$ *and* $i = 0, 1, \cdots, n-1$, *then*

$$W_{p,i}(K, Q^*) = \frac{\omega_n}{\widetilde{W}_i(\Lambda_{p,i}K)} \widetilde{W}_{-p,i}(\Lambda_{p,i}K, Q), \tag{44}$$

for each $Q \in \mathscr{K}^n_o$.

Proof. For each $Q \in \mathscr{K}^n_o$, from (20), (11), (27), (21) and (23), we have

$$\begin{aligned} W_{p,i}(K, Q^*) &= \frac{1}{n} \int_{S^{n-1}} \rho(Q, u)^{-p} f_{p,i}(K, u) dS(u) \\ &= \frac{\omega_n}{n\widetilde{W}_i(\Lambda_{p,i}K)} \int_{S^{n-1}} \rho(Q, u)^{-p} \rho(\Lambda_{p,i}K, u)^{n+p-i} dS(u) \\ &= \frac{\omega_n}{\widetilde{W}_i(\Lambda_{p,i}K)} \widetilde{W}_{-p,i}(\Lambda_{p,i}K, Q). \quad \square \end{aligned}$$

PROPOSITION 3.3. *If* $K \in \mathscr{F}^n_{i,o}$ *and* $p \geqslant 1$, *then*

(i) For $\phi \in O(n)$,

$$\Lambda_{p,i}\phi K = \phi^{-t} \Lambda_{p,i} K, \tag{45}$$

where $O(n)$ *denotes orthogonal transformation group in* $\mathbb{R}^n$, ϕ^{-t} *denotes the inverse of the transpose of* ϕ.

(ii) For $n - i \neq p \geqslant 1$, $\lambda > 0$,

$$\Lambda_{p,i}\lambda K = \lambda^{\frac{n-p-i}{p}} \Lambda_{p,i} K. \tag{46}$$

(iii) For the standard unit ball B *in* $\mathbb{R}^n$,

$$\Lambda_{p,i} B = B. \tag{47}$$

To prove Proposition 3.3, we first give several lemmas.

LEMMA 3.1. (see [7]) *Suppose* $K, L \in \mathscr{K}^n_o$, $p \geqslant 1$ *and* $i = 0, 1, \cdots, n-1$. *If* $\phi \in O(n)$, *then*

$$W_{p,i}(\phi K, \phi L) = W_{p,i}(K, L). \tag{48}$$

LEMMA 3.2. (see [7]) *Suppose* $K, L \in \mathscr{K}^n_o$, $p \geqslant 1$ *and* $i = 0, 1, \cdots, n-1$. *Then for any real* $\alpha, \beta > 0$,

$$W_{p,i}(\alpha K, \beta L) = \alpha^{n-i-p} \beta^p W_{p,i}(K, L). \tag{49}$$

LEMMA 3.3. (see [14]) *Suppose* $K, L \in \mathscr{S}^n_o$, $p \geqslant 1$, *and real* $i \in \mathbb{R}$, $i \neq n, n+p$. *Then for any* $\phi \in O(n)$,

$$\widetilde{W}_{-p,i}(\phi K, \phi L) = \widetilde{W}_{-p,i}(K, L). \tag{50}$$

From (50), we immediately have that

$$\widetilde{W}_{-p,i}(\phi K, L) = \widetilde{W}_{-p,i}(K, \phi^{-1} L), \quad \widetilde{W}_i(\phi K) = \widetilde{W}_i(K). \tag{51}$$

LEMMA 3.4. *Suppose $K,L \in \mathscr{S}_o^n$, $p \geqslant 1$, and real $i \in \mathbb{R}$, $i \neq n, n+p$. Then for any real $\alpha, \beta > 0$,*

$$\widetilde{W}_{-p,i}(\alpha K, \beta L) = \alpha^{n+p-i}\beta^{-p}\widetilde{W}_{-p,i}(K,L). \tag{52}$$

Thus, when $\beta = 1$,

$$\widetilde{W}_{-p,i}(\alpha K, L) = \alpha^{n+p-i}\widetilde{W}_{-p,i}(K,L). \tag{53}$$

Proof. Because for $K \in \mathscr{S}_o^n$ and $\mu > 0$, we know that $\rho_{\mu K}(x) = \mu\rho_K(x)$. This, together with the formula (23) of L_p-dual mixed quermassintegrals, we easily obtain (52). □

LEMMA 3.5. (see [14]) *Suppose $K,L \in \mathscr{S}_o^n$, $p \geqslant 1$, $i \in \mathbb{R}$ and $i \neq n$, $i \neq n+p$. Then for all $Q \in \mathscr{S}_o^n$, either*

$$\widetilde{W}_{-p,i}(K,Q) = \widetilde{W}_{-p,i}(L,Q) \ \text{ or } \ \widetilde{W}_{-p,i}(Q,K) = \widetilde{W}_{-p,i}(Q,L) \tag{54}$$

is true if and only if $K = L$.

Proof of Proposition 3.3. (i) Since $\phi \in O(n)$, then from (44), (48), (10), (50) and (51), we have

$$\begin{aligned}
\frac{\widetilde{W}_{-p,i}(\Lambda_{p,i}\phi K, Q)}{\widetilde{W}_i(\Lambda_{p,i}\phi K)} &= \frac{W_{p,i}(\phi K, Q^*)}{\omega_n} = \frac{W_{p,i}(\phi K, \phi\phi^{-1}Q^*)}{\omega_n} \\
&= \frac{W_{p,i}(K, \phi^{-1}Q^*)}{\omega_n} = \frac{W_{p,i}(K, (\phi^t Q)^*)}{\omega_n} = \frac{\widetilde{W}_{-p,i}(\Lambda_{p,i}K, \phi^t Q)}{\widetilde{W}_i(\Lambda_{p,i}K)} \\
&= \frac{\widetilde{W}_{-p,i}(\phi^{-t}\Lambda_{p,i}K, \phi^{-t}\phi^t Q)}{\widetilde{W}_i(\phi^{-t}\Lambda_{p,i}K)} = \frac{\widetilde{W}_{-p,i}(\phi^{-t}\Lambda_{p,i}K, Q)}{\widetilde{W}_i(\phi^{-t}\Lambda_{p,i}K)}.
\end{aligned}$$

Take $Q = \Lambda_{p,i}\phi K$ in the above formula, and note that $\widetilde{W}_{-p,i}(\phi^{-t}\Lambda_{p,i}K, \phi^{-t}\Lambda_{p,i}K) = \widetilde{W}_i(\phi^{-t}\Lambda_{p,i}K)$, we have

$$\widetilde{W}_{-p,i}(\phi^{-t}\Lambda_{p,i}K, \phi^{-t}\Lambda_{p,i}K) = \widetilde{W}_{-p,i}(\phi^{-t}\Lambda_{p,i}K, \Lambda_{p,i}\phi K).$$

By the above equation and Lemma 3.5, we immediately yields (45).

(ii) Suppose $K \in \mathscr{F}_{i,o}^n$ and $\lambda > 0$, then $h_K(\lambda x) = \lambda h_K(x)$ for any $x \in \mathbb{R}^n$. From this and (8) and (24), and note that $S_i(\lambda K, \cdot) = \lambda^{n-i-1}S_i(K, \cdot)$, we have

$$\begin{aligned}
f_{p,i}(\lambda K, \cdot) &= h^{1-p}(\lambda K, \cdot) f_i(\lambda K, \cdot) = \lambda^{1-p} h^{1-p}(K, \cdot)\frac{dS_i(\lambda K, \cdot)}{dS} \\
&= \lambda^{n-p-i} h^{1-p}(K, \cdot)\frac{dS_i(K, \cdot)}{dS} = \lambda^{n-p-i} h^{1-p}(K, \cdot) f_i(K, \cdot) \\
&= \lambda^{n-p-i} f_{p,i}(K, \cdot).
\end{aligned} \tag{55}$$

This eq. (55), together with the definition (27) of ith L_p-curvature image, eq. (52) with $L=K$ and $\alpha=\beta=\lambda>0$, we have that for $K\in\mathscr{F}^n_{i,o}$,

$$\frac{\rho(\Lambda_{p,i}\lambda K,\cdot)^{n+p-i}}{\widetilde{W}_i(\Lambda_{p,i}\lambda K)}=\frac{f_{p,i}(\lambda K,\cdot)}{\omega_n}=\frac{\lambda^{n-p-i}f_{p,i}(K,\cdot)}{\omega_n}$$
$$=\frac{\lambda^{n-p-i}\rho(\Lambda_{p,i}K,\cdot)^{n+p-i}}{\widetilde{W}_i(\Lambda_{p,i}K)}=\frac{\rho(\lambda^{\frac{n-p-i}{p}}\Lambda_{p,i}K,\cdot)^{n+p-i}}{\widetilde{W}_i(\lambda^{\frac{n-p-i}{p}}\Lambda_{p,i}K)},$$

i.e.,

$$\rho(\Lambda_{p,i}\lambda K,\cdot)=\left(\frac{\widetilde{W}_i(\Lambda_{p,i}\lambda K)}{\widetilde{W}_i(\lambda^{\frac{n-p-i}{p}}\Lambda_{p,i}K)}\right)^{\frac{1}{n+p-i}}\rho(\lambda^{\frac{n-p-i}{p}}\Lambda_{p,i}K,\cdot). \tag{56}$$

This (56), combined with the formula (21) of the dual quermassintegrals, we have

$$\widetilde{W}_i(\Lambda_{p,i}\lambda K)=\widetilde{W}_i(\lambda^{\frac{n-p-i}{p}}\Lambda_{p,i}K).$$

Therefore, from (56) we get

$$\rho(\Lambda_{p,i}\lambda K,\cdot)=\rho(\lambda^{\frac{n-p-i}{p}}\Lambda_{p,i}K,\cdot).$$

This immediately yields (46).

(iii) Because of $f_{p,i}(B,\cdot)=1$, this combined with (36), we give

$$\Omega^{(i)}_{p,j}(B)=\int_{S^{n-1}}f_{p,i}(B,u)^{\frac{n-i-j}{n+p-i}}dS(u)=\int_{S^{n-1}}dS(u)=n\omega_n.$$

Also according to (43), we can get

$$\widetilde{W}_i(\Lambda_{p,i}B)^{\frac{p}{n+p-i}}=\frac{1}{n}\omega_n^{\frac{-(n-i)}{n+p-i}}\Omega^{(i)}_{p,i}(B)=\omega_n^{\frac{p}{n+p-i}},$$

therefore, $\widetilde{W}_i(\Lambda_{p,i}B)=\omega_n$. Further, by definition (27) of the ith L_p-curvature image, we have

$$\rho(\Lambda_{p,i}B,\cdot)^{n+p-i}=\frac{\widetilde{W}_i(\Lambda_{p,i}B)}{\omega_n}f_{p,i}(B,\cdot)=1,$$

i.e., $\rho(\Lambda_{p,i}B,\cdot)=1$, this yields (47). $\square$

PROPOSITION 3.4. *If $p\geqslant 1$, $L\in\mathscr{F}^n_{i,o}$, then*

$$\Omega^{(i)}_p(L)^{n+p-i}\leqslant n^{n+p-i}W_{p,i}(L,K^*)^{n-i}\widetilde{W}_i(K)^p, \tag{57}$$

for all $K\in\mathscr{K}^n_o$, with equality if and only if K and $\Lambda_{p,i}L$ are dilates.

Proof. For $L \in \mathscr{F}_{i,o}^n$ and each $K \in \mathscr{K}_o^n$, from (9), (11), (20), (21) and Hölder's inequality, we have

$$\begin{aligned}
&\Omega_p^{(i)}(L)^{n+p-i}\\
&= \left[\int_{S^{n-1}} f_{p,i}(L,u)^{\frac{n-i}{n+p-i}} dS(u)\right]^{n+p-i}\\
&= \left[\int_{S^{n-1}} \left(\rho(K,u)^{-p} f_{p,i}(L,u)\right)^{\frac{n-i}{n+p-i}} \left(\rho(K,u)^{n-i}\right)^{\frac{p}{n+p-i}} dS(u)\right]^{n+p-i}\\
&\leqslant n^{n+p-i}\left(\frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{-p} f_{p,i}(L,u) dS(u)\right)^{n-i} \left(\frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n-i} dS(u)\right)^{p}\\
&= n^{n+p-i}\left(\frac{1}{n}\int_{S^{n-1}} h(K^*,u)^{p} f_{p,i}(L,u) dS(u)\right)^{n-i} \left(\frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n-i} dS(u)\right)^{p}\\
&= n^{n+p-i} W_{p,i}(L,K^*)^{n-i} \widetilde{W}_i(K)^p.
\end{aligned}$$

From this, we immediately get (57). According to the condition of equality holds in Hölder's inequality, we know that equality holds in (57) if and only if

$$\frac{\rho(K,u)^{-p} f_{p,i}(L,u)}{\rho(K,u)^{n-i}} = c$$

for any $u \in S^{n-1}$, where c is a constant. Combined with the definition (27) of ith L_p-curvature image, for any $u \in S^{n-1}$, we have

$$\frac{\rho(\Lambda_{p,i}L,u)^{n+p-i}}{\rho(K,u)^{n+p-i}} = \frac{c\widetilde{W}_i(\Lambda_{p,i}L)}{\omega_n},$$

this shows that K and $\Lambda_{p,i}L$ are dilates. Therefore, the equality holds in the inequality (57) if and only if K and $\Lambda_{p,i}L$ are dilates. The proof is complete. □

According to Proposition 3.4, we can give the extension definition of the $(i,0)$ type L_p-affine surface area of $K \in \mathscr{K}_o^n$ as follows:

DEFINITION 3.3. If $K \in \mathscr{K}_o^n$, $p \geqslant 1$, then the $(i,0)$ type L_p-affine surface area, $\Omega_p^{(i)}(K)$, of K is defined by

$$n^{-\frac{p}{n-i}} \Omega_p^{(i)}(K)^{\frac{n+p-i}{n-i}} = \inf\left\{ nW_{p,i}(K,Q^*)\widetilde{W}_i(Q)^{\frac{p}{n-i}} : Q \in \mathscr{S}_o^n \right\}. \tag{58}$$

Note that for $i=0$, this definition is just definition of L_p-affine surface area by Lutwak proposed in [6].

Since for any $K \in \mathscr{K}_o^n$, the ith L_p-surface area measure, $S_{p,i}(K,\cdot)$ is well-defined, we can gave a natural extension of eq. (34) of $(i,-p)$ type the L_p-mixed affine surface area $\Omega_{p,-p}$ from $\mathscr{F}_{i,o}^n \times \mathscr{F}_{i,o}^n$ to $\mathscr{K}_o^n \times \mathscr{F}_{i,o}^n$. Specifically, for $K \in \mathscr{K}_o^n$ and $L \in \mathscr{F}_{i,o}^n$, let

$$\Omega_{p,-p}^{(i)}(K,L) = \int_{S^{n-1}} f_{p,i}(L,u)^{\frac{-p}{n+p-i}} dS_{p,i}(K,u). \tag{59}$$

It is defined that for $K \in \mathscr{F}_{i,o}^n$, $dS_{p,i}(K,\cdot) = f_{p,i}(K,\cdot)dS(\cdot)$. Thus (59) boils down to (34) for $K \in \mathscr{F}_{i,o}^n$. Note that the case $i=0$ was studied by Lv and Leng in [10].

PROPOSITION 3.5. *Suppose $n-i \neq p \geqslant 1$, $i=0,1,\ldots,n-1$, and $K \in \mathscr{K}_o^n$, then*

$$\Omega_p^{(i)}(K)^{\frac{n+p-i}{n-i}} = \inf\left\{\Omega_{p,-p}^{(i)}(K,L)\Omega_p^{(i)}(L)^{\frac{p}{n-i}} : \forall L \in \mathscr{F}_{i,o}^n\right\}. \tag{60}$$

Proof. For $K \in \mathscr{K}_o^n$, $L \in \mathscr{F}_{i,o}^n$, using (18), (11), (27) and (59), we have

$$\begin{aligned} W_{p,i}(K,\Lambda_{p,i}^* L) &= \frac{1}{n}\int_{S^{n-1}} h(\Lambda_{p,i}^* L,u)^p dS_{p,i}(K,u) \\ &= \frac{1}{n}\int_{S^{n-1}} \rho(\Lambda_{p,i}L,u)^{-p} dS_{p,i}(K,u) \\ &= \frac{1}{n}\left(\frac{\widetilde{W}_i(\Lambda_{p,i}L)}{\omega_n}\right)^{-\frac{p}{n+p-i}} \int_{S^{n-1}} f_{p,i}(L,u)^{-\frac{p}{n+p-i}} dS_{p,i}(K,u) \\ &= \frac{1}{n}\left(\frac{\widetilde{W}_i(\Lambda_{p,i}L)}{\omega_n}\right)^{-\frac{p}{n+p-i}} \Omega_{p,-p}^{(i)}(K,L), \end{aligned}$$

i.e., for $K \in \mathscr{K}_o^n$, $L \in \mathscr{F}_{i,o}^n$,

$$\omega_n^{\frac{p}{n+p-i}}\Omega_{p,-p}^{(i)}(K,L) = n\widetilde{W}_i(\Lambda_{p,i}L)^{\frac{p}{n+p-i}} W_{p,i}(K,\Lambda_{p,i}^* L). \tag{61}$$

Together with (61) and (43), it shows that

$$n^{-\frac{p}{n-i}}\Omega_{p,-p}^{(i)}(K,L)\Omega_p^{(i)}(L)^{\frac{p}{n-i}} = nW_{p,i}(K,\Lambda_{p,i}^* L)\widetilde{W}_i(\Lambda_{p,i}L)^{\frac{p}{n-i}}. \tag{62}$$

On the other hand, from the definition (58) we know that for $K \in \mathscr{K}_o^n$ and any $Q \in \mathscr{S}_o^n$,

$$n^{-\frac{p}{n-i}}\Omega_p^{(i)}(K)^{\frac{n+p-i}{n-i}} \leqslant nW_{p,i}(K,Q^*)\widetilde{W}_i(Q)^{\frac{p}{n-i}}.$$

Take $Q = \Lambda_{p,i}L$ for $L \in \mathscr{F}_{i,o}^n$ in the above inequality, then we have

$$n^{-\frac{p}{n-i}}\Omega_p^{(i)}(K)^{\frac{n+p-i}{n-i}} \leqslant n\widetilde{W}_i(\Lambda_{p,i}L)^{\frac{p}{n-i}} W_{p,i}(K,\Lambda_{p,i}^* L). \tag{63}$$

Combined with (62) and (63), we immediately get for $K \in \mathscr{K}_o^n$,

$$\Omega_p^{(i)}(K)^{\frac{n+p-i}{n-i}} \leqslant \Omega_{p,-p}^{(i)}(K,L)\Omega_p^{(i)}(L)^{\frac{p}{n-i}},\ \ \forall L \in \mathscr{F}_{i,o}^n. \tag{64}$$

From (64), we immediately get the results of Proposition 3.5. The proof is complete. □

Proposition 3.5 is the generalization of result by Lv and Leng proved in [10].

From Proposition 3.5, the following corollary is obvious.

COROLLARY 3.1. *If $n-i \neq p \geqslant 1$, $i=0,1,\ldots,n-1$ and $K \in \mathscr{K}_o^n$, $L \in \mathscr{F}_{i,o}^n$, then*

$$\Omega_{p,-p}^{(i)}(K,L) \geqslant \Omega_p^{(i)}(K)^{\frac{n+p-i}{n-i}}\Omega_p^{(i)}(L)^{\frac{-p}{n-i}}. \tag{65}$$

Note that if $p \geqslant 1$ and $K,L \in \mathscr{F}_{i,o}^n$, then from the Hölder inequality, we obtain that equality holds in (65) for $n-i \neq p = 1$ and $0 \leqslant i < n-1$ if and only if K and L are homothetic, for $n-i \neq p > 1$ and $0 \leqslant i < n$ if and only if K and L are dilates.

4. The generalized L_p-Winternitz monotonicity problem

LEMMAS 4.1. *If $K,L\in\mathscr{K}_o^n$, $p\geqslant 1$ and $i=0,1,\cdots,n-1$, then*

$$W_{p,i}(L,\Pi_{p,i}K)=W_{p,i}(K,\Pi_{p,i}L). \tag{66}$$

Proof. From (6), (18) and Fubini theorem, it is easy to prove Lemmas 4.1. □

THEOREM 4.1. *Let $L\in\mathscr{F}_{i,c}^n$, $p\geqslant 1$ and $i=0,1,\cdots,n-1$. If $L\in\mathscr{W}_{p,i}^n$, then for all $K,M\in\mathscr{K}_o^n$,*

$$\frac{\Omega_{p,-p}^{(i)}(K,L)}{\Omega_{p,-p}^{(i)}(M,L)}\leqslant\max_{u\in S^{n-1}}\frac{h(\Pi_{p,i}K,u)^p}{h(\Pi_{p,i}M,u)^p}. \tag{67}$$

Proof. Because $L\in\mathscr{W}_{p,i}^n$, then there exists $Z\in\Pi_{p,i}^n$ such that $f_{p,i}(L,u)^{\frac{-p}{n+p-i}}=h(Z,u)^p$. Since every ith L_p-projection body is origin-symmetric convex body in $\mathbb{R}^n$, let $Z=\Pi_{p,i}Q$ for some $Q\in\mathscr{K}_o^n$. From (59), (18) and Lemma 4.1, we have

$$\begin{aligned}\frac{\Omega_{p-p}^{(i)}(K,L)}{\Omega_{p-p}^{(i)}(M,L)}&=\frac{\int_{S^{n-1}}f_{p,i}(L,u)^{\frac{-p}{n+p-i}}dS_{p,i}(K,u)}{\int_{S^{n-1}}f_{p,i}(L,u)^{\frac{-p}{n+p-i}}dS_{p,i}(M,u)}\\&=\frac{\int_{S^{n-1}}h(Z,u)^pdS_{p,i}(K,u)}{\int_{S^{n-1}}h(Z,u)^pdS_{p,i}(M,u)}\\&=\frac{W_{p,i}(K,Z)}{W_{p,i}(M,Z)}=\frac{W_{p,i}(K,\Pi_{p,i}Q)}{W_{p,i}(M,\Pi_{p,i}Q)}=\frac{W_{p,i}(Q,\Pi_{p,i}K)}{W_{p,i}(Q,\Pi_{p,i}M)}\\&=\frac{\int_{S^{n-1}}h(\Pi_{p,i}K,u)^pdS_{p,i}(Q,u)}{\int_{S^{n-1}}h(\Pi_{p,i}M,u)^pdS_{p,i}(Q,u)}\\&\leqslant\max_{S^{n-1}}\frac{h(\Pi_{p,i}K,u)^p}{h(\Pi_{p,i}M,u)^p}.\end{aligned}$$

The proof is complete. □

The following result is an immediate consequence of Theorem 4.1.

COROLLARY 4.1. *Let $p\geqslant 1$, $L\in\mathscr{F}_{i,c}^n$, $i=0,1,\cdots,n-1$. If $L\in\mathscr{W}_{p,i}^n$, then*

$$\Omega_{p,-p}^{(i)}(K,L)\leqslant\Omega_{p,-p}^{(i)}(M,L)$$

holds for all $K,M\in\mathscr{K}_o^n$ satisfying $\Pi_{p,i}K\subseteq\Pi_{p,i}M$.

COROLLARY 4.2. *If $n-i\neq p\geqslant 1$, $i=0,1,\cdots,n-1$, $L\in\mathscr{W}_{p,i}^n$ and $K\in\mathscr{K}_o^n$. Then*

$$\left(\frac{\Omega_p^{(i)}(K)}{\Omega_p^{(i)}(L)}\right)^{\frac{n+p-i}{n-i}}\leqslant\max_{u\in S^{n-1}}\frac{h(\Pi_{p,i}K,u)^p}{h(\Pi_{p,i}L,u)^p}. \tag{68}$$

Proof. Take $M = L$ in Theorem 4.1. From the inequality (65) and eq. (40), we have

$$\max_{S^{n-1}} \frac{h(\Pi_{p,i}K,u)^p}{h(\Pi_{p,i}L,u)^p} \geqslant \frac{\Omega^{(i)}_{p,-p}(K,L)}{\Omega^{(i)}_{p,-p}(L,L)} \geqslant \frac{\Omega^{(i)}_p(K)^{\frac{n+p-i}{n-i}}\Omega^{(i)}_p(L)^{\frac{-p}{n-i}}}{\Omega^{(i)}_p(L)} = \left(\frac{\Omega^{(i)}_p(K)}{\Omega^{(i)}_p(L)}\right)^{\frac{n+p-i}{n-i}}.$$

This proves the corollary. □

Proof of Theorem 1.1. From Corollary 4.2, we immediately get Theorem 1.1. □

Theorem 1.1 for the case $i = 0$ were established by Lv and Leng in [10].

Proof of Theorem 1.2. For $K \in \mathscr{K}^n_o$, $L \in \mathscr{W}^n_{p,i}$, and for all $Q \in \mathscr{K}^n_o$ such that

$$W_{p,i}(K,Q) \leqslant W_{p,i}(L,Q).$$

Because $L \in \beth^n_{p,i}$ and every ith L_p-projection body is origin-symmetric convex body in $\mathbb{R}^n$. Taking $Q = \Lambda^*_{p,i}L$, then

$$W_{p,i}(K,\Lambda^*_{p,i}L) \leqslant W_{p,i}(L,\Lambda^*_{p,i}L),$$

using the formula (18) of the L_p-mixed quermassintegral, we have

$$\int_{S^{n-1}} h(\Lambda^*_{p,i}L,u)^p dS_{p,i}(K,u) \leqslant \int_{S^{n-1}} h(\Lambda^*_{p,i}L,u)^p dS_{p,i}(L,u),$$

from eq. (11), this is equivalent to

$$\int_{S^{n-1}} \rho(\Lambda_{p,i}L,u)^{-p} dS_{p,i}(K,u) \leqslant \int_{S^{n-1}} \rho(\Lambda_{p,i}L,u)^{-p} dS_{p,i}(L,u). \tag{69}$$

From (69) and (27), we get

$$\int_{S^{n-1}} f_{p,i}(L,u)^{\frac{-p}{n+p-i}} dS_{p,i}(K,u) \leqslant \int_{S^{n-1}} f_{p,i}(L,u)^{\frac{-p}{n+p-i}} dS_{p,i}(L,u). \tag{70}$$

Together with (59), (40) and (70), we have

$$\Omega^{(i)}_{p,-p}(K,L) \leqslant \Omega^{(i)}_p(L).$$

Using the inequality (65) in the above inequality, then

$$\Omega^{(i)}_p(L) \geqslant \Omega^{(i)}_{p,-p}(K,L) \geqslant \Omega^{(i)}_p(K)^{\frac{n+p-i}{n-i}}\Omega^{(i)}_p(L)^{\frac{-p}{n-i}},$$

this implies

$$\Omega^{(i)}_p(K) \leqslant \Omega^{(i)}_p(L).$$

The proof is complete. □

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Tongyi Ma
School of Mathematics and Statistics, Hexi University
Zhangye, Gansu 734000
People's Republic of China
e-mail: matongyi@126.com, matongyi_123@163.com,
gsmatongyi@hotmail.com