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REVERSES OF ANDO AND DAVIS–CHOI INEQUALITY

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Abstract. Reverse Ando's inequalities for positive linear maps are deduced. A difference reverse inequality is proven for solidarities one of which is of a special type that includes connections. A quotient reverse inequality (or an operator reverse Hölder inequality) for connections and for the special type of solidarities that includes connections is given. As an application an operator reverse of Hölder's inequality for the general weighted power mean is given in a difference and a quotient form. An analogous inequality in a difference form for the relative operator entropy is directly proven. An another type of reverse inequalities is given for Ando's and Davis-Choi's inequality. The estimations in this case are expressed using a kind of variation of involved family of operators.

1. Introduction and preliminaries

The theory for connections and means of pairs of positive operators has been developed by Kubo and Ando in [18]. A binary operation $(A,B) \mapsto A\sigma B$ on the set of positive invertible operators is a connection if it satisfies the following axiomatic properties:

C1. $A \leqslant C$, $B \leqslant D$ implies $A\sigma B \leqslant C\sigma D$,

C2. $T^*(A\sigma B)T \leqslant (T^*AT)\sigma(T^*BT)$ for every operator T,

C3. $A_k \downarrow A$ and $B_k \downarrow B$ imply $A_k\sigma B_k \downarrow A\sigma B$.

A mean is a connection with normalization condition
C4. $I\sigma I = I$.
The correspondence $\sigma \mapsto f_\sigma$ defined by

$$f_\sigma(t)I = I\sigma(tI) \quad (t>0), \tag{1.1}$$

establishes an isomorphism between the class of connections and the class of non-negative operator monotone functions on $(0,\infty)$. Moreover, for positive invertible operators A,B

$$A\sigma B = A^{1/2} f_\sigma\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}. \tag{1.2}$$

If the conditions C1 and C3 are replaced by conditions

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S1. $B \leqslant C$ implies $A\sigma B \leqslant A\sigma C$,

S3. $B_n \downarrow B$ implies $A\sigma B_n \downarrow A\sigma B$,

S4. $A_n \to A$ strongly implies $A_n\sigma I \to A\sigma I$,

then an extension of Kubo-Ando theory is obtained as developed in [8]. A binary operator σ satisfying S1., C2., S3. and S4. is called a solidarity. The same correspondence as given in (1.1) defines an isomorphism between the class of solidarities and the class of operator monotone functions on $(0,\infty)$ and (1.2) also holds. The function f in this correspondence is called the representing function for σ. If f_σ is nonnegative, then σ is a connection.

Let s be the solidarity for an operator monotone function f. For positive invertible operators A and B, it follows from (1.2) that $A\ s\ B$ exists as a bounded operator. For a noninvertible case, the solidarity $s = s_f$ for f is defined as the following limit if it exists:

$$A\ s\ B = s - \lim_{\varepsilon\downarrow 0} A\ s\ (B+\varepsilon). \tag{1.3}$$

By the definition (1.3), it follows that $A\ s\ B$ exists if a set $\{A\ s\ (B+\varepsilon) | \varepsilon > 0\}$ is bounded below.

The relative operator entropy introduced in [9] as

$$S(A|B) = A^{1/2}\log\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}$$

and Tsallis relative operator entropy $T_\alpha(A|B)$ introduced in [12] as

$$T_\alpha(A|B) = \frac{1}{\alpha}\left(A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\alpha A^{1/2} - A\right) = \frac{1}{\alpha}(A\#_\alpha B - A),\ \ \alpha\in(0,1],$$

defined for positive invertible operators A,B, are the basic examples of solidarities, with obvious representing functions $f(t) = \log t$ and $f_\alpha(t) = \frac{1}{\alpha}(t^\alpha - 1)$, respectively. $A\#_\alpha B$ denotes the standard weighted geometric mean.

The weighted (operator) power mean defined by

$$A m_{p,\alpha} B = A^{1/2}\left[(1-\alpha)I + \alpha\left(A^{-1/2}BA^{-1/2}\right)^p\right]^{1/p} A^{1/2}, 0\leqslant\alpha\leqslant 1, -1\leqslant p\leqslant 1,$$

with representing function $f_{p,\alpha}(t) = [1-\alpha+\alpha t^p]^{1/p}$, is an important example of means which contains many operator means as special cases. For example

$$A\#_\alpha B = \lim_{p\to 0} A m_{p,\alpha} B.$$

The following properties are proved in [8].

THEOREM 1.1. *If s is a solidarity, then*

1. $(A+B)\ s\ (C+D) \geqslant A\ s\ C + B\ s\ D$ *(subaditivity)*

2. $(\lambda A_1+(1-\lambda)A_2)\ s\ (\lambda B_1+(1-\lambda)B_2) \geqslant \lambda\,(A_1\ s\ B_1)+(1-\lambda)\,(A_2\ s\ B_2)$ *for* $0\leqslant\lambda\leqslant 1$ *(joint concavity).*

A simple consequence is:

COROLLARY 1.2. *Let* A_i, B_i *be positive operators for* $i=1,\ldots,n$. *Then*

$$\sum_{i=1}^{n} A_i\ s\ B_i \leqslant \left(\sum_{i=1}^{n} A_i\right)\ s\ \left(\sum_{i=1}^{n} B_i\right) \tag{1.4}$$

for any solidarity s.

The following inequality is a generalization of Ando's inequality $\Phi(A \sharp B) \leqslant \Phi(A) \sharp \Phi(B)$ (see [2, 3]):

THEOREM 1.3. *If* Φ *is a positive linear map, then for any connection* σ *and for each* A, B *positive,*

$$\Phi(A\sigma B) \leqslant \Phi(A)\sigma\Phi(B). \tag{1.5}$$

The analogous result for the Tsallis relative operator entropy is proven in [13] (see also [12] and for a simple proof see [14]) and for the relative operator entropy see [15].

REMARK 1.4. It follows from Theorem 1.3 that the inequality

$$\sum_{j=1}^{n} \Phi_j(A_j\sigma B_j) \leqslant \Big(\sum_{j=1}^{n} \Phi_j(A_j)\Big)\sigma\Big(\sum_{j=1}^{n} \Phi_j(B_j)\Big) \tag{1.6}$$

also holds, where σ is a connection, A_j, B_j are positive invertible operators, Φ_j are positive linear maps for $j=1,\ldots,n$.

In this way the Hölder inequality for positive invertible operators A_i, B_i and $i=1,\ldots,n$

$$\sum_{i=1}^{n} A_i \#_{p,\alpha} B_i \leqslant \left(\sum_{i=1}^{n} A_i\right) \#_{p,\alpha} \left(\sum_{i=1}^{n} B_i\right) \tag{1.7}$$

holds, where $0\leqslant\alpha\leqslant 1$, $-1\leqslant p\leqslant 1$.

Although it is common to call inequality (1.7) the Hölder inequality, it is worthwhile to mention that in the real case inequality (1.7) reduces to

$$\sum_{i=1}^{n} \left[(1-\alpha)a_i^p+\alpha b_i^p\right]^{\frac{1}{p}} \leqslant \left[(1-\alpha)\left(\sum_{i=1}^{n} a_i\right)^p + \alpha\left(\sum_{i=1}^{n} b_i\right)^p\right]^{\frac{1}{p}},$$

which holds for $p<1$ and the reversed inequality holds for $p>1$. This is discrete Minkowski's inequality. A more general form of which is

$$\sum_{i=1}^{n} p_i \left(\sum_{j=1}^{m} q_j a_{i,j}^p\right)^{\frac{1}{p}} \leqslant \left[\sum_{j=1}^{m} q_j \left(\sum_{i=1}^{n} p_i a_{i,j}\right)^p\right]^{\frac{1}{p}}, \tag{1.8}$$

where $a_{i,j} > 0$, $p_i, q_j \geqslant 0$, $i = 1, \dots, n$, $j = 1, \dots, m$, $p < 1$. For $p > 1$ the reversed inequality holds in (1.8).

Note that inequality (1.8) is, due to homogeneous property, equivalent to inequality

$$\sum_{i=1}^{n}\left(\sum_{j=1}^{m} a_{i,j}^{p}\right)^{\frac{1}{p}} \leqslant \left[\sum_{j=1}^{m}\left(\sum_{i=1}^{n} a_{i,j}\right)^{p}\right]^{\frac{1}{p}}.$$

In the operator case the only known result of this type is proven in [1] for the harmonic mean:

$$\sum_{i=1}^{n}\left(\sum_{j=1}^{m} A_{i,j}^{-1}\right)^{-1} \leqslant \left[\sum_{j=1}^{m}\left(\sum_{i=1}^{n} A_{i,j}\right)^{-1}\right]^{-1},$$

where $A_{i,j}$, $i = 1, \dots, n$, $j = 1, \dots, m$ are positive invertible operators.

Our main goal in this paper is to give reverse inequalities to (1.4) of a difference and quotient type for the special type of solidarities that includes connections. In the case of the weighted power mean the explicit estimations of reverse inequalities of (1.7) in a difference and quotient form are obtained. The difference case for the relative operator entropy is also given without using the geometric mean. In Section 3 reverse inequalities of difference type are given for Ando's and Davis-Choi's inequality using a kind of variation of the involved family of operators and the divided difference of the second order.

The methods used in this paper and related results can be found in the monograph [15]. For some recent results in this area see the monograph [11].

The next result is proven in [4] (see also [20] and [15]).

THEOREM 1.5. *Let A_i, B_i be positive invertible operators such that $mA_i \leqslant B_i \leqslant MA_i$ for $i = 1, \dots, n$ and some scalars $0 < m < M$ and $\alpha \in [0,1]$. Then*

$$\left(\sum_{i=1}^{n} A_i\right) \#_{\alpha} \left(\sum_{i=1}^{n} B_i\right) \leqslant \frac{1}{K(m,M,\alpha)} \sum_{i=1}^{n} A_i \#_{\alpha} B_i, \tag{1.9}$$

where

$$K(m,M,\alpha) = \frac{Mm^{\alpha} - mM^{\alpha}}{(1-\alpha)(M-m)}\left(\frac{1-\alpha}{\alpha}\frac{M^{\alpha} - m^{\alpha}}{Mm^{\alpha} - mM^{\alpha}}\right)^{\alpha} \quad (\alpha \in \mathbb{R})$$

is the generalized Kantorovich constant.

A difference reverse type result is given in [10] (see also [20] and [15]).

THEOREM 1.6. *Let A_i, B_i be positive invertible operators such that $mA_i \leqslant B_i \leqslant MA_i$ for some scalars $0 < m \leqslant M$ and $i = 1, \dots, n$. Then for each $\alpha \in [0,1]$*

$$\left(\sum_{i=1}^{n} A_i\right) \#_{\alpha} \left(\sum_{i=1}^{n} B_i\right) - \sum_{i=1}^{n} A_i \#_{\alpha} B_i \leqslant C(m,M,\alpha) \sum_{i=1}^{n} A_i, \tag{1.10}$$

where

$$C(m,M,\alpha)=(1-\alpha)\left(\frac{M^{\alpha}-m^{\alpha}}{\alpha(M-m)}\right)^{\frac{\alpha}{\alpha-1}}-\frac{Mm^{\alpha}-mM^{\alpha}}{M-m}\quad(\alpha\in\mathbb{R}).$$

In the same paper, using Theorem 1.6 and $S(A|B)=\lim_{\alpha\to 0}\frac{A\#_{\alpha}B-A}{A}$, the following corollary is proven.

COROLLARY 1.7. *Let A_i, B_i be positive invertible operators such that $mA_i\leqslant B_i\leqslant MA_i$ for some scalars $0<m\leqslant M$ and $i=1,\dots,n$. Then*

$$S\left(\sum_{i=1}^{n}A_i|\sum_{i=1}^{n}B_i\right)-\sum_{i=1}^{n}S\left(A_i|B_i\right)\leqslant\log S(h)\sum_{i=1}^{n}A_i, \tag{1.11}$$

where

$$S(h)=\frac{(h-1)h^{\frac{1}{h-1}}}{e\log h}\ (h\neq 1)\quad\text{and}\quad S(1)=1$$

is the Specht ratio and $h=\frac{M}{m}$.

We also give reverse inequalities to the famous Davis-Choi inequality, which states that for an operator convex function $f:I\to\mathbb{R}$, $I\subseteq\mathbb{R}$ an interval, Φ a normalized positive linear map, it holds

$$f(\Phi(A))\leqslant\Phi\left(f(A)\right), \tag{1.12}$$

where A is a selfadjoint operator with $\mathrm{Sp}(A)\subseteq I$ (see [6], [5]).

2. Reverse inequalities for solidarities and connections

In this section we prove reverse type inequalities for inequality (1.5). The following theorem is the Lah-Ribarič inequality for the special type of solidarities that includes connections.

THEOREM 2.1. *Let A,B be positive invertible operators such that $mA\leqslant B\leqslant MA$ for some scalars $0<m\leqslant M$, Φ a positive linear map. Let s be the solidarity for an operator monotone function f. Then*

$$\Phi(A\ s\ B)\geqslant\frac{M\Phi(A)-\Phi(B)}{M-m}f(m)+\frac{\Phi(B)-m\Phi(A)}{M-m}f(M). \tag{2.1}$$

Proof. Since f is concave on $(0,\infty)$, by the Lah-Ribarič inequality (see [21]) we have

$$f(x)\geqslant\frac{M-x}{M-m}f(m)+\frac{x-m}{M-m}f(M),\ x\in[m,M]. \tag{2.2}$$

It easily follows from $mA \leqslant B \leqslant MA$ that $mI \leqslant A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leqslant MI$. Using the functional calculus and (2.2), we obtain

$$f\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right) \geqslant \frac{M - A^{-\frac{1}{2}}BA^{-\frac{1}{2}}}{M-m} f(m) + \frac{A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - m}{M-m} f(M). \tag{2.3}$$

Multiplying (2.3) twice by $A^{\frac{1}{2}}$, acting by Φ, it follows

$$\Phi\left(A^{\frac{1}{2}} f\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right) A^{\frac{1}{2}}\right) \geqslant \frac{M\Phi(A) - \Phi(B)}{M-m} f(m) + \frac{\Phi(B) - m\Phi(A)}{M-m} f(M). \quad \square$$

A difference counterpart of an operator Hölder's inequality (1.5) is given in the following remark. The proof is analogous as in the case of connections (see [15]).

REMARK 2.2. Let A, B be positive invertible operators such that $mA \leqslant B \leqslant MA$ for some scalars $0 < m \leqslant M$, Φ a positive linear map, and $\lambda \in \mathbb{R}$. Suppose that solidarities s_1 and s_2 have representing functions f_1 and f_2 respectively. Then

$$\lambda\left(\Phi(A)\right)\ s_1\ \left(\Phi(B)\right) - \Phi\left(A\ s_2\ B\right) \leqslant \max_{m \leqslant t \leqslant M} \left[\lambda f_1(t) - \left(\frac{M-t}{M-m} f_2(m) + \frac{t-m}{M-m} f_2(M)\right)\right] \Phi(A). \tag{2.4}$$

The operator reverse Hölder's inequality of the quotient type can be deduced from the difference type in the usual way (see [15]).

REMARK 2.3. Let A, B be positive invertible operators such that $mA \leqslant B \leqslant MA$ for some $0 < m \leqslant M$, Φ a positive linear map. Let σ be the connection for f, and s the solidarity for g. Then

$$\Phi(A\ s\ B) \geqslant \min_{m \leqslant t \leqslant M} \frac{\frac{M-t}{M-m} g(m) + \frac{t-m}{M-m} g(M)}{f(t)} \Phi(A)\ \sigma\ \Phi(B). \tag{2.5}$$

As an application we give the following theorem which is a generalization of Theorem 1.5. Let $f(t) = [1 - \alpha + \alpha t^p]^{1/p}$ be the representing function for $\sharp_{p,\alpha}$. For convenience, we define:

$$\mu_f = \frac{(1-\alpha+\alpha M^p)^{\frac{1}{p}} - (1-\alpha+\alpha m^p)^{\frac{1}{p}}}{M-m}$$
$$\nu_f = \frac{M(1-\alpha+\alpha m^p)^{\frac{1}{p}} - m(1-\alpha+\alpha M^p)^{\frac{1}{p}}}{M-m}. \tag{2.6}$$

THEOREM 2.4. *Let $-1 \leqslant p \leqslant 1$, $0 \leqslant \alpha \leqslant 1$, A, B be positive invertible operators such that $mA \leqslant B \leqslant MA$ for some scalars $0 < m \leqslant M$, Φ a positive linear map. Then*

$$\Phi(A)\#_{p,\alpha}\Phi(B) \leqslant K(M,m,p,\alpha)\Phi(A\#_{p,\alpha}B), \tag{2.7}$$

where

$$K(m,M,p,\alpha)=\frac{1}{\nu_f}\left[(1-\alpha)^{\frac{1}{1-p}}+\alpha^{\frac{1}{1-p}}\left(\frac{\nu_f}{\mu_f}\right)^{\frac{p}{1-p}}\right]^{\frac{1-p}{p}}$$

and μ_f *and* ν_f *are defined as* (2.6).

Proof. We give a sketch of long but routine calculations. Set in (2.4) $f_1(t)=f_2(t)=(1-\alpha+\alpha t^p)^{1/p}$. Define

$$F(t)=\frac{(M-m)(1-\alpha+\alpha t^p)^{1/p}}{(M-t)(1-\alpha+\alpha m^p)^{1/p}+(t-m)(1-\alpha+\alpha M^p)^{1/p}}.$$

By straightforward calculations $F'(t)=0$ is equivalent to equation

$$\begin{aligned}&\alpha t^{p-1}\left[(M-t)(1-\alpha+\alpha m^p)^{1/p}+(t-m)(1-\alpha+\alpha M^p)^{1/p}\right]\\&=(1-\alpha+\alpha t^p)\left[(1-\alpha+\alpha M^p)^{1/p}-(1-\alpha+\alpha m^p)^{1/p}\right],\end{aligned}$$

which by obvious cancelling is equivalent to equation

$$\begin{aligned}&\alpha\left(M(1-\alpha+\alpha m^p)^{1/p}-m(1-\alpha+\alpha M^p)^{1/p}\right)\\&=(1-\alpha)t^{1-p}\left((1-\alpha+\alpha M^p)^{1/p}-(1-\alpha+\alpha m^p)^{1/p}\right),\end{aligned}$$

which finally gives

$$t=\left(\frac{\alpha}{1-\alpha}\right)^{\frac{1}{1-p}}\left(\frac{M(1-\alpha+\alpha m^p)^{1/p}-m(1-\alpha+\alpha M^p)^{1/p}}{(1-\alpha+\alpha M^p)^{1/p}-(1-\alpha+\alpha m^p)^{1/p}}\right)^{\frac{1}{1-p}}.$$

Plugging this value in F and rearranging, the constant $K(m,M,p,\alpha)$ can be easily obtained. □

The following corollary follows by setting $p=-1$ in the previous theorem. This is a weighted generalization of the known reverse inequality for the harmonic mean (see [15]).

COROLLARY 2.5. *Let* $0\leqslant\alpha\leqslant 1$, A,B *be positive invertible operators such that* $mA\leqslant B\leqslant MA$ *for some scalars* $0<m\leqslant M$, Φ *a positive linear map. Then*

$$\Phi(A)!_\alpha\Phi(B)\leqslant K(M,m,-1,\alpha)\Phi(A!_\alpha B), \tag{2.8}$$

where

$$K(m,M,-1,\alpha)=\frac{((1-\alpha)m+\alpha)((1-\alpha)M+\alpha)}{\left((1-\alpha)\sqrt{mM}+\alpha\right)^2}.$$

In particular,

$$\Phi(A)\,!\,\Phi(B)\leqslant\frac{(m+1)(M+1)}{(\sqrt{mM}+1)^2}\Phi(A\,!\,B).$$

Generalization of Theorem 1.6 is given in the following theorem.

THEOREM 2.6. *Let* $-1 \leqslant p \leqslant 1$, $0 \leqslant \alpha \leqslant 1$, A, B *be positive invertible operators such that* $mA \leqslant B \leqslant MA$ *for some scalars* $0 < m \leqslant M$, Φ *a positive linear map. Then*

$$\Phi(A) \#_{p,\alpha} \Phi(B) - \Phi(A \#_{p,\alpha} B) \leqslant C(m, M, p, \alpha) \Phi(A), \tag{2.9}$$

where

$$C(m, M, p, \alpha) = \left(\frac{1-\alpha}{\alpha}\right)^{\frac{1}{p}} \frac{\mu_f}{\left(\alpha^{\frac{1}{p-1}} \mu_f^{\frac{p}{1-p}} - 1\right)^{\frac{1-p}{p}}} - \nu_f,$$

and μ_f *and* ν_f *are defined as* (2.6).

Proof. We use Remark 2.2 for $\lambda = 1$, $f_1(t) = f_2(t) = f(t) = [1 - \alpha + \alpha t^p]^{1/p}$. Set

$$F(t) = [1 - \alpha + \alpha t^p]^{1/p} - \mu_f t - \nu_f.$$

It is easy to see that equation $F'(t) = 0$ gives

$$t = \left(\frac{1-\alpha}{\alpha}\right)^{\frac{1}{p}} \frac{1}{\left[\alpha^{\frac{1}{p-1}} \mu_f^{\frac{p}{1-p}} - 1\right]^{\frac{1}{p}}}.$$

Plugging this value in F and rearranging the constant $C(m, M, p, \alpha)$ easily follows. □

Theorem 1.5 can be deduced from the previous theorem using $\lim_{p \to 0}$ or from Remark 2.2 by setting $\lambda = 1$, $g_\alpha(t) = f_1(t) = f_2(t) = t^\alpha$, $0 \leqslant \alpha \leqslant 1$. An immediate consequence of Remark 2.2 is the following difference reverse result for Tsallis relative operator entropy. This result is obtained in [14].

THEOREM 2.7. *Let* A, B *be positive invertible operators such that* $mA \leqslant B \leqslant MA$ *for some scalars* $0 < m \leqslant M$, Φ *a positive linear map. Then for every* $\alpha \in (0, 1]$

$$T_\alpha(\Phi(A) | \Phi(B)) - \Phi(T_\alpha(A|B)) \leqslant \frac{1}{\alpha} C(m, M, \alpha) \Phi(A), \tag{2.10}$$

where $C(m, M, \alpha)$ *is defined in Theorem* 1.5.

Proof. Denote $F(t) = f_\alpha(t) - \left(\frac{M-t}{M-m} f_\alpha(m) + \frac{t-m}{M-m} f_\alpha(M)\right)$, where $f_\alpha(t) = \frac{1}{\alpha}(t^\alpha - 1)$, and $G(t) = g_\alpha(t) - \left(\frac{M-t}{M-m} g_\alpha(m) + \frac{t-m}{M-m} g_\alpha(M)\right)$, where $g_\alpha(t) = t^\alpha$. Obviously $F(t) = \frac{1}{\alpha} G(t)$ and (2.10) follows using Theorem 2.2. □

We give the direct proof of a generalization of Corollary 1.7 using Remark 2.2.

COROLLARY 2.8. *Let A,B be positive invertible operators such that $mA \leqslant B \leqslant MA$ for some scalars $0 < m \leqslant M$, Φ a positive linear map. Then*

$$S(\Phi(A)|\Phi(B)) - \Phi(S(A|B)) \leqslant \log S(h)\Phi(A), \tag{2.11}$$

where $S(h)$ is defined in Corollary 1.7.

Proof. Set in Remark 2.2 $f_1(t) = f_2(t) = \log t$ and $\lambda = 1$. Define

$$F(t) = \log t - \frac{M-t}{M-m}\log m - \frac{t-m}{M-m}\log M.$$

Trivially $F'(t) = 0$ is equivalent to

$$t = \frac{M-m}{\log M - \log m}.$$

It is straightforward to check that

$$F\left(\frac{M-m}{\log M - \log m}\right) = \log S(h),$$

where $h = M/m$. $\square$

3. Difference reverses of Ando and Davis-Choi inequalities of another type

In this section a difference generated by Ando's and Davis-Choi's inequality is estimated from above with a kind of variation of operators included in these inequalities. The method is based on [7].

The following theorem is given in [16] in integral form. For the sake of completeness we will prove the discrete version.

THEOREM 3.1. *Let Φ_j be positive linear maps and let A_j be self-adjoint operators with $Sp(A_j) \subseteq [m,M]$ for some scalars $m < M$ and $j = 1,\dots,n$. If f is a concave function on an open interval $J \supset [m,M]$ and $p_1,\dots,p_n$ are positive real numbers such that $\sum_{j=1}^n p_j = 1$ and $\sum_{j=1}^n p_j\Phi_j(I) = I$, then*

$$\begin{aligned} & f\Big(\sum_{j=1}^n p_j\Phi_j(A_j)\Big) - \sum_{j=1}^n p_j\Phi_j(f(A_j)) \\ & \leqslant - \inf_{t\in\langle m,M\rangle} \Psi_f(t;m,M)\Big(M - \sum_{j=1}^n p_j\Phi_j(A_j)\Big)\Big(\sum_{j=1}^n p_j\Phi_j(A_j) - m\Big) \\ & \leqslant \frac{f'_+(m) - f'_-(M)}{M-m}\Big(M - \sum_{j=1}^n p_j\Phi_j(A_j)\Big)\Big(\sum_{j=1}^n p_j\Phi_j(A_j) - m\Big) \qquad (3.1) \\ & \leqslant \frac{1}{4}(M-m)(f'_+(m) - f'_-(M)), \end{aligned}$$

where $\Psi_f(\cdot;m,M)\colon (m,M)\to\mathbb{R}$ *is defined by*

$$\Psi_f(t;m,M)=\frac{1}{M-m}\Big(\frac{f(M)-f(t)}{M-t}-\frac{f(t)-f(m)}{t-m}\Big)=[m,t,M;f], \tag{3.2}$$

and $[m,t,M;f]$ *denotes the divided difference of the second order.*

Proof. Since f is concave, we have from the Lah-Ribarič inequality:

$$f(t)\geqslant\frac{M-t}{M-m}f(m)+\frac{t-m}{M-m}f(M) \tag{3.3}$$

for every $t\in[m,M]$, so we can replace t with A_j in (3.3) and then apply Φ_j:

$$\Phi_j(f(A_j))\geqslant\frac{M-\Phi_j(A_j)}{M-m}f(m)+\frac{\Phi_j(A_j)-m}{M-m}f(M). \tag{3.4}$$

Multiplying the inequality (3.4) with p_j, and then summing it, we get

$$\sum_{j=1}^{n}p_j\Phi_j(f(A_j))\geqslant\frac{M-\sum_{j=1}^{n}p_j\Phi_j(A_j)}{M-m}f(m)+\frac{\sum_{j=1}^{n}p_j\Phi_j(A_j)-m}{M-m}f(M). \tag{3.5}$$

Using functional calculus and (3.5) we obtain

$$\begin{aligned}
&f\Big(\sum_{j=1}^{n}p_j\Phi_j(A_j)\Big)-\sum_{j=1}^{n}p_j\Phi_j(f(A_j))\\
&\leqslant f\Big(\sum_{j=1}^{n}p_j\Phi_j(A_j)\Big)-\frac{M-\sum_{j=1}^{n}p_j\Phi_j(A_j)}{M-m}f(m)-\frac{\sum_{j=1}^{n}p_j\Phi_j(A_j)-m}{M-m}f(M)\\
&=-\frac{1}{M-m}\Big(M-\sum_{j=1}^{n}p_j\Phi_j(A_j)\Big)\Big(\sum_{j=1}^{n}p_j\Phi_j(A_j)-m\Big)\\
&\Bigg(\frac{f(M)-f\Big(\sum_{j=1}^{n}p_j\Phi_j(A_j)\Big)}{M-\sum_{j=1}^{n}p_j\Phi_j(A_j)}-\frac{f\Big(\sum_{j=1}^{n}p_j\Phi_j(A_j)\Big)-f(m)}{\sum_{j=1}^{n}p_j\Phi_j(A_j)-m}\Bigg)\\
&=\Big(M-\sum_{j=1}^{n}p_j\Phi_j(A_j)\Big)\Big(\sum_{j=1}^{n}p_j\Phi_j(A_j)-m\Big)\Big(-\Psi_f\Big(\sum_{j=1}^{n}p_j\Phi_j(A_j);m,M\Big)\Big)\\
&\leqslant-\inf_{t\in\langle m,M\rangle}(\Psi_f(t;m,M))\Big(M-\sum_{j=1}^{n}p_j\Phi_j(A_j)\Big)\Big(\sum_{j=1}^{n}p_j\Phi_j(A_j)-m\Big),
\end{aligned}$$

since $m\leqslant\sum_{j=1}^{n}p_j\Phi_j(A_j)\leqslant M$. The last two inequalities in (3.1) follow directly from:

$$-\inf_{t\in(m,M)}\Psi_f(t;m,M)\leqslant\frac{f'_+(m)-f'_-(M)}{M-m}$$

and $(M-t)(t-m)\leqslant\frac{1}{4}(M-m)^2$. □

REMARK 3.2. We need to observe that if in Theorem 3.1 we take $p_1 = 1$, we obtain a converse of the Davis-Choi inequality:

$$\begin{aligned} f(\Phi(A)) - \Phi(f(A)) &\leqslant - \inf_{t\in(m,M)} \Psi_f(t;m,M)(M-\Phi(A))(\Phi(A)-m) \\ &\leqslant (M-\Phi(A))(\Phi(A)-m)\frac{f'_+(m)-f'_-(M)}{M-m} \\ &\leqslant \frac{1}{4}(M-m)(f'_+(m)-f'_-(M)). \end{aligned} \tag{3.6}$$

THEOREM 3.3. *Let A_j, B_j be positive invertible operators such that $mA_j \leqslant B_j \leqslant MA_j$ for some scalars $0 < m \leqslant M$ and $j = 1,\dots,n$. Let s be the solidarity for an operator monotone function f. If Φ_j are positive linear maps, then*

$$\begin{aligned} &\left(\sum_{j=1}^n \Phi_j(A_j)\right) s \left(\sum_{j=1}^n \Phi_j(B_j)\right) - \frac{M\sum_{j=1}^n \Phi_j(A_j) - \sum_{j=1}^n \Phi_j(B_j)}{M-m} f(m) \\ &\quad - \frac{\sum_{j=1}^n \Phi_j(B_j) - m\sum_{j=1}^n \Phi_j(A_j)}{M-m} f(M) \\ &\leqslant - \inf_{t\in(m,M)} \Psi_f(t;m,M) \\ &\Big(M\sum_{j=1}^n \Phi_j(A_j) - \sum_{j=1}^n \Phi_j(B_j)\Big)\Big(\sum_{j=1}^n \Phi_j(A_j)\Big)^{-1}\Big(\sum_{j=1}^n \Phi_j(B_j) - m\sum_{j=1}^n \Phi_j(A_j)\Big) \\ &\leqslant \frac{f'_+(m)-f'_-(M)}{M-m} \\ &\Big(M\sum_{j=1}^n \Phi_j(A_j) - \sum_{j=1}^n \Phi_j(B_j)\Big)\Big(\sum_{j=1}^n \Phi_j(A_j)\Big)^{-1}\Big(\sum_{j=1}^n \Phi_j(B_j) - m\sum_{j=1}^n \Phi_j(A_j)\Big) \\ &\leqslant \frac{1}{4}(M-m)(f'_+(m)-f'_-(M))\Big(\sum_{j=1}^n \Phi_j(A_j)\Big), \end{aligned} \tag{3.7}$$

where $\Psi_f(\cdot;m,M)$ is defined by (3.2).

Proof. Let J be an open interval of real numbers such that $[m,M] \subset J$ and let us suppose that $\phi : J \to \mathbb{R}$ is a concave function. Then from the Lah-Ribarič inequality we easily get

$$\begin{aligned} &\phi(t) - \frac{M-t}{M-m}\phi(m) - \frac{t-m}{M-m}\phi(M) \\ &= -(M-t)(t-m)\frac{1}{M-m}\Big(\frac{\phi(M)-\phi(t)}{M-t} - \frac{\phi(t)-\phi(m)}{t-m}\Big) \\ &= -(M-t)(t-m)\Psi_\Phi(t;m,M) \leqslant (M-t)(t-m)\sup_{t\in(m,M)}(-\Psi_\Phi(t;m,M)) \\ &\leqslant (M-t)(t-m)\frac{\phi'_+(m)-\phi'_-(M)}{M-m} \leqslant \frac{1}{4}(M-m)(\phi'_+(m)-\phi'_-(M)) \end{aligned} \tag{3.8}$$

for every $t \in (m,M)$.

It follows from $mA_j \leqslant B_j \leqslant MA_j$ that

$$m \leqslant \Big(\sum_{j=1}^{n} \Phi_j(A_j)\Big)^{-\frac{1}{2}}\Big(\sum_{j=1}^{n} \Phi_j(B_j)\Big)\Big(\sum_{j=1}^{n} \Phi_j(A_j)\Big)^{-\frac{1}{2}} \leqslant M,$$

so using functional calculus and (3.8) we get the following inequalities:

$$\begin{aligned}
&f\Big(\Big(\sum_{j=1}^{n} \Phi_j(A_j)\Big)^{-\frac{1}{2}}\Big(\sum_{j=1}^{n} \Phi_j(B_j)\Big)\Big(\sum_{j=1}^{n} \Phi_j(A_j)\Big)^{-\frac{1}{2}}\Big)\\
&\quad-\frac{M-\Big(\sum_{j=1}^{n} \Phi_j(A_j)\Big)^{-\frac{1}{2}}\Big(\sum_{j=1}^{n} \Phi_j(B_j)\Big)\Big(\sum_{j=1}^{n} \Phi_j(A_j)\Big)^{-\frac{1}{2}}}{M-m}f(m)\\
&\quad-\frac{\Big(\sum_{j=1}^{n} \Phi_j(A_j)\Big)^{-\frac{1}{2}}\Big(\sum_{j=1}^{n} \Phi_j(B_j)\Big)\Big(\sum_{j=1}^{n} \Phi_j(A_j)\Big)^{-\frac{1}{2}}-m}{M-m}f(M)\\
&\leqslant -\inf_{t\in(m,M)} \Psi_f(t;m,M)\Big(M-\Big(\sum_{j=1}^{n} \Phi_j(A_j)\Big)^{-\frac{1}{2}}\Big(\sum_{j=1}^{n} \Phi_j(B_j)\Big)\Big(\sum_{j=1}^{n} \Phi_j(A_j)\Big)^{-\frac{1}{2}}\Big)\\
&\Big(\Big(\sum_{j=1}^{n} \Phi_j(A_j)\Big)^{-\frac{1}{2}}\Big(\sum_{j=1}^{n} \Phi_j(B_j)\Big)\Big(\sum_{j=1}^{n} \Phi_j(A_j)\Big)^{-\frac{1}{2}}-m\Big)\\
&\leqslant \frac{f'_+(m)-f'_-(M)}{M-m}\Big(M-\Big(\sum_{j=1}^{n} \Phi_j(A_j)\Big)^{-\frac{1}{2}}\Big(\sum_{j=1}^{n} \Phi_j(B_j)\Big)\Big(\sum_{j=1}^{n} \Phi_j(A_j)\Big)^{-\frac{1}{2}}\Big)\\
&\Big(\Big(\sum_{j=1}^{n} \Phi_j(A_j)\Big)^{-\frac{1}{2}}\Big(\sum_{j=1}^{n} \Phi_j(B_j)\Big)\Big(\sum_{j=1}^{n} \Phi_j(A_j)\Big)^{-\frac{1}{2}}-m\Big) \qquad (3.9)\\
&\leqslant \frac{1}{4}(M-m)(f'_+(m)-f'_-(M)).
\end{aligned}$$

Now, if we multiply the inequalities (3.9) twice by $\Big(\sum_{j=1}^{n} \Phi_j(A_j)\Big)^{\frac{1}{2}}$, inequalities (3.7) follow. $\square$

As an immediate consequence of Theorems 2.1 and 3.3 we have the following corollary which is a difference reverse of Ando's inequality (1.6).

COROLLARY 3.4. *Let A_j, B_j be positive invertible operators such that $mA_j \leqslant B_j \leqslant MA_j$ for some scalars $0 < m \leqslant M$ and $j = 1,\dots,n$. Let s be the solidarity for an operator monotone function f. If Φ_j are positive linear maps, then*

$$\begin{aligned}
&\Big(\sum_{j=1}^{n} \Phi_j(A_j)\Big)\sigma\Big(\sum_{j=1}^{n} \Phi_j(B_j)\Big)-\sum_{j=1}^{n} \Phi_j(A_j\sigma B_j)\\
&\quad\leqslant -\inf_{t\in(m,M)} \Psi_f(t;m,M)\\
&\quad\Big(M\sum_{j=1}^{n} \Phi_j(A_j)-\sum_{j=1}^{n} \Phi_j(B_j)\Big)\Big(\sum_{j=1}^{n} \Phi_j(A_j)\Big)^{-1}\Big(\sum_{j=1}^{n} \Phi_j(B_j)-m\sum_{j=1}^{n} \Phi_j(A_j)\Big)
\end{aligned}$$

$$\leqslant \frac{f'_+(m)-f'_-(M)}{M-m}\times \tag{3.10}$$
$$\Big(M\sum_{j=1}^{n}\Phi_j(A_j)-\sum_{j=1}^{n}\Phi_j(B_j)\Big)\Big(\sum_{j=1}^{n}\Phi_j(A_j)\Big)^{-1}\Big(\sum_{j=1}^{n}\Phi_j(B_j)-m\sum_{j=1}^{n}\Phi_j(A_j)\Big)$$
$$\leqslant \frac{1}{4}(M-m)(f'_+(m)-f'_-(M))\Big(\sum_{j=1}^{n}\Phi_j(A_j)\Big),$$

where $\Psi_f(\cdot;m,M)$ *is defined by (3.2).*

COROLLARY 3.5. *Let* A,B *be positive invertible operators such that* $mA\leqslant B\leqslant MA$ *for some scalars* $0<m\leqslant M$. *Let* s *be the solidarity for an operator monotone function* f. *If* Φ *is a positive linear map, then*

$$\begin{aligned}
&\Phi(A)\ s\ \Phi(B)-\Phi(A\ s\ B)\\
&\quad\leqslant -\inf_{t\in(m,M)}\Psi_f(t;m,M)(M\Phi(A)-\Phi(B))\Phi(A)^{-1}(\Phi(B)-m\Phi(A))\\
&\quad\leqslant \frac{f'_+(m)-f'_-(M)}{M-m}(M\Phi(A)-\Phi(B))\Phi(A)^{-1}(\Phi(B)-m\Phi(A))\\
&\quad\leqslant \frac{1}{4}(M-m)(f'_+(m)-f'_-(M))\Phi(A),
\end{aligned}\tag{3.11}$$

where $\Psi_f(\cdot;m,M)$ *is defined by (3.2).*

Proof. We give an alternative proof using the reverse of Choi-Davis inequality (3.6).

Let us define the map

$$\Psi(X)=\Phi(A)^{-\frac{1}{2}}\Phi(A^{\frac{1}{2}}XA^{\frac{1}{2}})\Phi(A)^{-\frac{1}{2}}.$$

Then Ψ is a normalized positive linear map. So we have by (3.6)

$$\begin{aligned}
&f(\Psi(X))-\Psi(f(X))\\
&\quad\leqslant -\inf_{t\in(m,M)}\Psi_f(t;m,M)(M-\Psi(X))(\Psi(X)-m)\\
&\quad\leqslant \frac{f'_+(m)-f'_-(M)}{M-m}(M-\Psi(X))(\Psi(X)-m)\\
&\quad\leqslant \frac{1}{4}(M-m)(f'_+(m)-f'_-(M)),
\end{aligned}\tag{3.12}$$

for a selfadjoint operator X with $\mathrm{Sp}(X)\in[m,M]$. From $mA\leqslant B\leqslant MA$ we easily get

$m \leqslant A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leqslant M$, so we can put $X = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ in (3.12) and we have

$$\begin{aligned}
&f(\Phi(A)^{-\frac{1}{2}}\Phi(B)\Phi(A)^{-\frac{1}{2}}) - \Phi(A)^{-\frac{1}{2}}\Phi(A^{\frac{1}{2}}f(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}})\Phi(A)^{-\frac{1}{2}} \\
&\leqslant -\inf_{t\in(m,M)} \Psi_f(t;m,M)(M - \Phi(A)^{-\frac{1}{2}}\Phi(B)\Phi(A)^{-\frac{1}{2}})(\Phi(A)^{-\frac{1}{2}}\Phi(B)\Phi(A)^{-\frac{1}{2}} - m) \\
&\leqslant \frac{f'_+(m) - f'_-(M)}{M-m}\left(M - \Phi(A)^{-\frac{1}{2}}\Phi(B)\Phi(A)^{-\frac{1}{2}}\right)\left(\Phi(A)^{-\frac{1}{2}}\Phi(B)\Phi(A)^{-\frac{1}{2}} - m\right) \\
&\leqslant \frac{1}{4}(M-m)(f'_+(m) - f'_-(M)). \qquad (3.13)
\end{aligned}$$

The inequalities (3.11) are now obtained by multiplying the inequalities (3.13) twice with $\Phi(A)^{\frac{1}{2}}$. □

As applications of Corollary 3.4 we give reverses of this type for basic examples of operator means and relative operator entropy.

COROLLARY 3.6. *Let A_i, B_i be positive invertible operators such that $mA_i \leqslant B_i \leqslant MA_i$ for some scalars $0 < m \leqslant M$ and $i = 1, \dots, n$.*

1. *If $\alpha \in [0,1]$ and $p \in [-1,1]$, then*

$$\begin{aligned}
&\left(\sum_{i=1}^n A_i\right) \sharp_{p,\alpha} \left(\sum_{i=1}^n B_i\right) - \sum_{i=1}^n A_i \sharp_{p,\alpha} B_i \\
&\leqslant -\inf_{t\in(m,M)} \Psi_f(t;m,M)\left(M\sum_{i=1}^n A_i - \sum_{i=1}^n B_i\right)\left(\sum_{i=1}^n A_i\right)^{-1}\left(\sum_{i=1}^n B_i - m\sum_{i=1}^n A_i\right) \\
&\leqslant \alpha\frac{(\alpha+(1-\alpha)m^{-p})^{\frac{1-p}{p}} - (\alpha+(1-\alpha)M^{-p})^{\frac{1-p}{p}}}{M-m} \\
&\left(M\sum_{i=1}^n A_i - \sum_{i=1}^n B_i\right)\left(\sum_{i=1}^n A_i\right)^{-1}\left(\sum_{i=1}^n B_i - m\sum_{i=1}^n A_i\right).
\end{aligned}$$

2. *If $\alpha \in [0,1]$, then*

$$\begin{aligned}
&\left(\sum_{i=1}^n A_i\right) \sharp_{\alpha} \left(\sum_{i=1}^n B_i\right) - \sum_{i=1}^n A_i \sharp_{\alpha} B_i \\
&\leqslant -\inf_{t\in(m,M)} \Psi_\alpha(t;m,M)\left(M\sum_{i=1}^n A_i - \sum_{i=1}^n B_i\right)\left(\sum_{i=1}^n A_i\right)^{-1}\left(\sum_{i=1}^n B_i - m\sum_{i=1}^n A_i\right) \\
&\leqslant \frac{\alpha(m^{\alpha-1} - M^{\alpha-1})}{M-m}\left(M\sum_{i=1}^n A_i - \sum_{i=1}^n B_i\right)\left(\sum_{i=1}^n A_i\right)^{-1}\left(\sum_{i=1}^n B_i - m\sum_{i=1}^n A_i\right).
\end{aligned}$$

3. If $\alpha \in [0,1]$*, then*

$$\begin{aligned}
&\Big(\sum_{i=1}^{n} A_i\Big) !_\alpha \Big(\sum_{i=1}^{n} B_i\Big) - \sum_{i=1}^{n} A_i !_\alpha B_i \\
&\leqslant -\inf_{t\in(m,M)} \Psi_f(t;m,M)\Big(M\sum_{i=1}^{n} A_i - \sum_{i=1}^{n} B_i\Big)\Big(\sum_{i=1}^{n} A_i\Big)^{-1}\Big(\sum_{i=1}^{n} B_i - m\sum_{i=1}^{n} A_i\Big) \\
&\leqslant \alpha\frac{(\alpha+(1-\alpha)m)^{-2}-(\alpha+(1-\alpha)M)^{-2}}{M-m}\times \\
&\Big(M\sum_{i=1}^{n} A_i - \sum_{i=1}^{n} B_i\Big)\Big(\sum_{i=1}^{n} A_i\Big)^{-1}\Big(\sum_{i=1}^{n} B_i - m\sum_{i=1}^{n} A_i\Big).
\end{aligned}$$

4. It holds

$$\begin{aligned}
&S\Big(\sum_{i=1}^{n} A_i \Big| \sum_{i=1}^{n} B_i\Big) - \sum_{i=1}^{n} S(A_i|B_i) \\
&\leqslant -\inf_{t\in\langle m,M\rangle} \Psi_f(t;m,M)\Big(M\sum_{i=1}^{n} A_i - \sum_{i=1}^{n} B_i\Big)\Big(\sum_{i=1}^{n} A_i\Big)^{-1}\Big(\sum_{i=1}^{n} B_i - m\sum_{i=1}^{n} A_i\Big) \\
&\leqslant \frac{1}{Mm}\Big(M\sum_{i=1}^{n} A_i - \sum_{i=1}^{n} B_i\Big)\Big(\sum_{i=1}^{n} A_i\Big)^{-1}\Big(\sum_{i=1}^{n} B_i - m\sum_{i=1}^{n} A_i\Big).
\end{aligned}$$

In each case $\Psi_f(\cdot;m,M)\colon (m,M) \to \mathbb{R}$ *is defined by (3.2), where* f *is the appropriate generating function.*

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