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CAUCHY TYPE MEANS ON ONE–PARAMETER C_0–GROUP OF OPERATORS

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Dedicated to our parents

(*Communicated by A. Meskhi*)

Abstract. A new theory of power means is introduced on a C_0-group of continuous linear operators. A mean value theorem is proved, which builds the basis of the procedure to obtain Cauchy-type power means on a C_0-group of continuous linear operators.

1. Introduction

A significant theory of Cauchy type means has been developed [2, 3, 4, 5, 6, 7], which is both extensive and elegant. In this paper we define new means on the C_0-semigroup of bounded linear operators which also contains the inverses and hence forming a C_0-group. Later on, these means are shown to be of Cauchy-type.

This section is actually intended to give a brief exposition to few definitions and results in the theory of uniformly continuous groups(semigroups) of bounded linear operators defined on a Banach space X, which are indispensable for an understanding of the next section. Let $B(X)$ denotes the space of bounded linear operators defined on a Banach space X. A (one parameter) C_0-semigroup (or strongly continuous semigroup) of operators on a Banach space X is a family $\{Z(t)\}_{t\geqslant 0} \subset B(X)$ such that

(i) $Z(s)Z(t) = Z(s+t)$ for all $s,t \geqslant 0$.

(ii) $Z(0) = I$, the identity operator on X.

(iii) for each fixed $f \in X$, $Z(t)f \to f$ (with respect to the norm on X) as $t \to 0^+$.

If the above mentioned properties hold for $\mathbb{R}$ instead of $\mathbb{R}^+$, we call $\{Z(t)\}_{t\in\mathbb{R}}$ a *strongly continuous (one parameter) group (or C_0-group)* on X, where for $f \in X$, $Z(t)Z(-t)f = Z(0)f = f$. Therefore, $\{Z(t)\}_{t<0}$ gives the inverses of $\{Z(t)\}_{t>0}$. All the properties and characteristics of C_0-Semigroup are also possessed by C_0-group, so we shall be considering only C_0-semigroups at the moment.

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The (infinitesimal) generator of $\{Z(t)\}_{t\geqslant 0}$ is the closed linear operator $A : X \supseteq D(A) \to R(A) \subseteq X$ defined by

$$D(A) = \{f : f \in X, \lim_{t\to 0^+} A_t f \text{ exists in } X\}$$

$$Af = \lim_{t\to 0^+} A_t f \ \ (f \in D(A))$$

where, for $t > 0$,

$$A_t f = \frac{[Z(t) - I]f}{t} \ \ (f \in X).$$

Moreover $D(A)$ is a dense vector subspace of X. For $\{Z(t)\}_{t\geqslant 0} \subseteq B(X)$ a C_0-semigroup on the Banach space X, there exists constants $M > 0$ and $\omega \geqslant 0$ such that $\|Z(t)\| \leqslant Me^{\omega t}$, for all $t \geqslant 0$. See ([8], Theorem 2.14). In case $M = 1$ and $\omega = 0$, we obtain a C_0-semigroup (correspondingly group) of contractions.

The arithmetic mean on the C_0-semigroup of operators is defined as [10],

$$m(Z, f, t) = \frac{1}{t}\int_0^t Z(\tau) f d\tau.$$

Means of C_0-semigroups of operators have great importance and form the basis of *Mean Ergodic Theory*, which has been a center of interest in research for decades. (See e.g. [16, 9, 14]).

To define power-means on C_0-semigroup of operators, things need little more concentration. As the real-powers are involved, $\{Z(t)\}_{t\geqslant 0}$ should also contain the inverse operators (to define the powers like $r < 0$). One can observe that when r is any integer (positive or negative), the C_0-group property implies that $Z(t)^r = Z(rt)$. While we can generalize it for $r \in \mathbb{R}$. For example take $Z(1/2t)Z(1/2t) = Z(t)$ and thus we get $Z(t)^{1/2} = Z(1/2t)$. For $r \in \mathbb{R}$, the generator of $\{Z(rt)\}_{t\geqslant 0}$ is $(rA, D(A))$. Such semigroups are often called *rescaled semigroups*. (See e.g. [11, 13]). For $f \in X$ and $t > 0$, a C_0-semigroup(group) $\{Z(t)\}_{t\geqslant 0}$ generated by an operator A, has the form $Z(t)f = \exp[tA]f$ (see [8]). Hence $\ln[Z(t)f]$ makes sense.

In correspondence with the usual definition of power integral means, we define the power means for C_0-group of operators.

DEFINITION 1. Let X be a Banach space and $\{Z(t)\}_{t\in\mathbb{R}}$ the C_0-group of linear operators on X. For $f \in X$ and $t \in \mathbb{R}$, the power mean is defined as follows

$$M_r(Z, f, t) = \begin{cases} \left\{\frac{1}{t}\int_0^t [Z(\tau)]^r f d\tau\right\}^{1/r}, & r \neq 0 \\ \exp[\frac{1}{t}\int_0^t \ln[Z(\tau)] f d\tau], & r = 0. \end{cases} \tag{1}$$

For $t > 0$ and $r \in \mathbb{R}^+$, $Z(t)^r = Z(-t)^{-r}$. Therefore the integral domain is taken to be non-negative. Moreover for $r = 1$, $M_r(Z, f, t) = m(Z, f, t)$, the arithmetic mean, for $r = 0$ it defines the geometric mean and for $r = -1$ it defines the harmonic mean on C_0-group of operators (and hence satisfying the property of power-mean). For $r > 0$, $M_{-r}(Z, f, t)$ gives the inverse of the mean of inverse of $Z(t)^r$.

For real and continuous functions φ, χ on a closed interval $K := [k_1, k_2]$, such that φ, χ are differentiable in the interior of I and $\chi' \neq 0$, throughout the interior of I. A very well know Cauchy mean value theorem guarantees the existence of a number $\zeta \in (k_1, k_2)$, such that

$$\frac{\varphi'(\zeta)}{\chi'(\zeta)} = \frac{\varphi(k_1) - \varphi(k_2)}{\chi_(k_1) - \chi(k_2)}.$$

Now, if the function $\frac{\varphi'}{\chi'}$ is invertible, then the number ζ is unique and

$$\zeta := \Big(\frac{\varphi'}{\chi'}\Big)^{-1}\Big(\frac{\varphi(k_1) - \varphi(k_2)}{\chi_(k_1) - \chi(k_2)}\Big).$$

The number ζ is called *Cauchy's mean value* of numbers k_1, k_2. It is possible to define such a mean for several variables, in terms of divided difference. Which is given by

$$\zeta := \Big(\frac{\varphi^{n-1}}{\chi^{n-1}}\Big)^{-1}\Big(\frac{[k_1, k_2, ..., k_n]\varphi}{[k_1, k_2, ..., k_n]\chi}\Big).$$

This mean value was first defined and examined by Leach and Sholander [12]. The integral representation of Cauchy mean is given by

$$\zeta := \Big(\frac{\varphi^{n-1}}{\chi^{n-1}}\Big)^{-1}\Big(\frac{\int_{E_{n-1}} \varphi^{n-1}(k.u)du}{\int_{E_{n-1}} \chi^{n-1}(k.u)du}\Big)',$$

where $E_{n-1} := \{(u_1, u_2, ..., u_n) : u_i \geqslant 0,\ 1 \leqslant i \leqslant n,\ \sum_{i=1}^{n-1} u_i \leqslant 1\}$, is (n-1) dimensional simplex, $u = (u_1, u_2, ..., u_n)$, $u_n = 1 - \sum_{i=1}^{n-1} u_i$, $du = du_1 du_2 ... du_n$ and $k.u = \sum_{i=1}^{n} u_i k_i$.

A mean which can be expressed in the similar form as of Cauchy mean, is called *Cauchy type mean*. The purpose of our work is to introduce new means of Cauchy type defined on C_0-group of operators.

2. Main results

The present section includes a chain of results. Two mean value theorems are proved. As applications of these mean value theorems we have defined new means for C_0-group of linear operators.

LEMMA 1. *Let $\{Z(t)\}_{t \geqslant 0} \subseteq B(X)$ be a C_0-semigroup of bounded linear operators on a Banach space X. For $f \in X$ and $t > 0$,*

$$m(Z, f, t) = \frac{1}{t}\int_0^t Z(\tau) f d\tau \in X. \tag{2}$$

Proof. Let $h > 0$ and consider

$$\begin{aligned}
\frac{Z(h)-I}{h}\left\{\int_0^t Z(u)f du\right\} &= \frac{1}{h}\int_0^t \{Z(u+h)f - Z(u)f\}du \\
&= \frac{1}{h}\int_0^t Z(u+h)f du - \frac{1}{h}\int_0^t Z(u)f du \\
&= \frac{1}{h}\int_h^{t+h} Z(u)f du - \frac{1}{h}\int_0^t Z(u)f du \\
&= \frac{1}{h}\int_h^{t} Z(u)f du + \frac{1}{h}\int_t^{t+h} Z(u)f du - \frac{1}{h}\int_0^t Z(u)f du \\
&= \frac{1}{h}\int_t^{t+h} Z(u)f du - \frac{1}{h}\int_0^t Z(u)f du - \frac{1}{h}\int_t^h Z(u)f du \\
&= \frac{1}{h}\int_t^{t+h} Z(u)f du - \frac{1}{h}\int_0^t Z(u)f du
\end{aligned}$$

on letting $h \to 0^+$ and using the fundamental theorem of calculus

$$\lim_{h\to 0^+}\frac{Z(h)-I}{h}\left\{\int_0^t Z(u)f du\right\} = Z(t)f - f = [Z(t)-I]f \in D(A)$$

hence

$$\int_0^t Z(\tau)f d\tau \in D(A)$$

and since $D(A)$ is a vector subspace of X, therefore $m(Z,f,t) \in D(A)$. Also $D(A) \subset \overline{D(A)} = X$. Hence the result follows. □

COROLLARY 1. *Let $\{Z(t)\}_{t\geqslant 0} \subseteq B(X)$ be a C_0-semigroup of bounded linear operators on a Banach space X. For $f \in X$ and $t > 0$,*

$$M_r(Z,f,t) \in X,$$

where $M_r(Z,f,t)$ is defined by (1).

Proof. For $\{Z(t)\}_{t\in\mathbb{R}} \subseteq B(X)$, by group-law we have, for $\tau, r \in \mathbb{R}$,

$$[Z(\tau)]^r f = Z(r\tau)f = Z(s)f$$

where $r\tau = s$, then $Z(s) \in \{Z(t)\}_{t\in\mathbb{R}}$. By Lemma 1, we finally get that $M_r(Z,f,t) \in X$. □

REMARK 1. For a C_0-semigroup of contractions $\{Z(t)\}_{t\geqslant 0} \subseteq B(X)$, we have $\|Z(t)\| \leqslant 1$, for all $t \in \mathbb{R}$. For such C_0-groups

- The mean $m(Z,f,t)$, satisfies

$$\|m\| = Sup_{f\in X}\frac{\|m(Z,f,t)\|}{\|f\|} \leqslant 1, \text{ for } t > 0.$$

- The power mean $M_r(Z,f,t)$ is defined by (1.1). For $r>0$,

$$\|M_r\| = Sup_{f\in X}\frac{\|M_r(Z,f,t)\|}{\|f\|} \geqslant \eta, \quad f\in X,\ t>0,$$

  and for $r<0$,

$$\|M_r\| = Sup_{f\in X}\frac{\|M_r(Z,f,t)\|}{\|f\|} \leqslant \eta, \quad f\in X,\ t>0,$$

  where $\eta = \|f\|^{-(r+1)}$. Moreover for $r=0$,

$$\|M_0\| = Sup_{f\in X}\frac{\|M_0(Z,f,t)\|}{\|f\|} \leqslant 1, \quad f\in X,\ t>0.$$

- Let $\{f_n\}_{n=0}^{\infty}\subset X$, such that $f_n\to f\in X$, and $\|Z(t)\|\leqslant 1$ for $t\in\mathbb{R}$,

$$\|M_r(Z,f_n,t)-M_r(Z,f,t)\| \leqslant \|M_r\|\|f_n-f\|, \quad r\leqslant 0.$$

  Therefore, for $r\leqslant 0$, $M_r(Z,f_n,t)\to M_r(Z,f,t)$.

Next, we shall prove a mean value theorem which actually forms the basis of rest of the theory and somehow, can be regarded as the analogue of ([15],Theorem 1) to Banach spaces.

THEOREM 1. *Let X be a Banach space and $\{Z(t)\}_{t\geqslant 0}\subset B(X)$ be a C_0-semigroup of operators on X. For $\phi,\psi\in C^2(X)$ there exists some $\xi\in X$ such that*

$$\frac{\frac{1}{t}\int_0^t\phi[Z(\tau)]f\,d\tau-\phi[\frac{1}{t}\int_0^t[Z(\tau)]f\,d\tau]}{\frac{1}{t}\int_0^t\psi[Z(\tau)]f\,d\tau-\psi[\frac{1}{t}\int_0^t[Z(\tau)]f\,d\tau]} = \frac{\phi''(\xi)}{\psi''(\xi)}. \tag{3}$$

Proof. For the sake of simplicity throughout the proof, we shall denote $m(Z,f,t)$ by m_t. For $\rho\in X$, define

$$(Q\phi)(\rho) := \frac{1}{t}\int_0^t\phi[\rho[Z(\tau)f]+(1-\rho)m_t]d\tau-\phi(m_t)$$

similarly, for the operator ψ, we define $(Q\psi)(\rho)$. It is observed that,

$$(Q\phi)'(\rho) := \frac{1}{t}\int_0^t[Z(\tau)f-m_t]\phi'[\rho[Z(\tau)f]+(1-\rho)m_t]d\tau$$

and

$$(Q\phi)''(\rho) := \frac{1}{t}\int_0^t[Z(\tau)f-m_t]^2\phi''[\rho[Z(\tau)f]+(1-\rho)m_t]d\tau.$$

Here, $(.)'$ denotes the Gateaux derivative. Let us define an other operator $W(\rho)$, as follows

$$W(\rho) = (Q\psi)(1)(Q\phi)(\rho)-(Q\phi)(1)(Q\psi)(\rho).$$

It can be easily seen that

$$W(0) = W(1) = W'(0) = 0$$

where 0, 1 are the zero, identity elements of X, respectively.

After two applications of Mean Value theorem [1], we conclude that there exists an element $\eta \in X$ such that

$$W''(\eta) = 0.$$

Hence

$$\frac{1}{t}\int_0^t [Z(\tau)f - m_t]^2\{(Q\psi)(1)\phi''[\eta[Z(\tau)f] + (1-\eta)m_t] \\ -(Q\phi)(1)\psi''[\eta[Z(\tau)f] + (1-\eta)]\}m_t d\tau = 0 \tag{4}$$

A mapping $\varphi_f : [0,\infty) \to X$ defined by

$$\varphi_f(t) = Z(t)f, \quad f \in X$$

is continuous on $[0,\infty)$. See ([8], Lemma 2.4). Hence for any fixed $\eta \in X$, the expression in the braces in (4) is a continuous function of τ, so it vanishes for some value of $\tau \geqslant 0$. Corresponding to that value of $\tau \geqslant 0$, we get an element $\xi \in X$, such that

$$\xi = \eta[Z(\tau)f] + (1-\eta)m_t, \quad f \in X.$$

So that

$$(Q\psi)(1)\phi''(\xi) - (Q\phi)(1)\psi''(\xi) = 0.$$

The assertion (3) follows directly. □

COROLLARY 2. *Let X be a Banach space and $\{Z(t)\}_{t\geqslant 0} \subseteq B(X)$ be a C_0-semigroup of operators on X. For $\phi,\psi \in C^2(X)$ such that $\frac{\phi''}{\psi''}$ is invertible. Then there exists a unique $\xi \in X$ which is the mean of the Cauchy type that is*

$$\xi = \left(\frac{\phi''}{\psi''}\right)^{-1}\left(\frac{\frac{1}{t}\int_0^t \phi[Z(\tau)]fd\tau - \phi[\frac{1}{t}\int_0^t[Z(\tau)]fd\tau]}{\frac{1}{t}\int_0^t \psi[Z(\tau)]fd\tau - \psi[\frac{1}{t}\int_0^t[Z(\tau)]fd\tau]}\right). \tag{5}$$

COROLLARY 3. *Let X be a Banach space and $\{Z(t)\}_{t\geqslant 0} \subset B(X)$ be a C_0-semigroup of operators on X. For $\phi \in C^2(X)$ and some $\xi \in X$*

$$\frac{1}{t}\int_0^t \phi[Z(\tau)f]d\tau - \phi\left[\frac{1}{t}\int_0^t [Z(\tau)f]d\tau\right] = \frac{\phi''(\xi)}{2}\left\{\frac{1}{t}\int_0^t [Z(\tau)]^2 fd\tau - \left[\frac{1}{t}\int_0^t [Z(\tau)]fd\tau\right]^2\right\}. \tag{6}$$

Proof. By setting $\psi(f) = f^2$ for $f \in X$, in Theorem 1, we get the assertion (6). □

Next, let G be the group of invertible bounded linear operators from a Banach space X to itself. For $\{Z(t)\}_{t\geqslant 0} \subset B(X)$ a C_0-semigroup of operators defined on X and $H \in G$, the quasi-arithmetic mean is defined as

$$M_H^{\circ}(Z,f,t) = H^{-1}\Big\{\frac{1}{t}\int_0^t H[Z(\tau)f]d\tau\Big\}, \quad f \in X, t \geqslant 0. \tag{7}$$

By ([8], Lemma 1.85), $B(X)$ is closed under composition of operators so the above expressions exists and belongs to X. For the sake of simplicity, the set of all elements of G, whose second order derivative (in Gateaux's sense) exits, is denoted by $C^2G(X)$.

THEOREM 2. *Let X be a Banach space and let $H,F,K \in C^2G(X)$. Then*

$$\frac{H(M_H^{\circ}(Z,f,t)) - H(M_F^{\circ}(Z,f,t))}{K(M_K^{\circ}(Z,f,t)) - K(M_F^{\circ}(Z,f,t))} = \frac{H''(\eta)F'(\eta) - H'(\eta)F''(\eta)}{K''(\eta)F'(\eta) - K'(\xi)F''(\xi)}. \tag{8}$$

For some $\xi \in X$, provided that the denominator on the left hand side of (8) is non-zero.

Proof. By choosing the operators ϕ and ψ in Theorem 1, such that

$$\phi = H \circ F^{-1}, \quad \psi = K \circ F^{-1} \quad \textit{and} \quad Z(\tau)f = F[Z(\tau)f]$$

where $H,F,K \in C^2G(X)$. We find that there exists $\xi \in X$, such that

$$\frac{H(M_H^{\circ}(Z,f,t)) - H(M_F^{\circ}(Z,f,t))}{K(M_K^{\circ}(Z,f,t)) - K(M_F^{\circ}(Z,f,t))} = \frac{H''(F^{-1}(\xi))F'(F^{-1}(\xi)) - H'(F^{-1}(\xi))F''(F^{-1}(\xi))}{K''(F^{-1}(\xi))F'(F^{-1}(\xi)) - K'(F^{-1}(\xi))F''(F^{-1}(\xi))}.$$

Therefore, by setting $F^{-1}(\xi) = \eta$, we find that there exists $\eta \in X$, such that

$$\frac{H(M_H^{\circ}(Z,f,t)) - H(M_F^{\circ}(Z,f,t))}{K(M_K^{\circ}(Z,f,t)) - K(M_F^{\circ}(Z,f,t))} = \frac{H''(\eta)F'(\eta) - H'(\eta)F''(\eta)}{K''(\eta)F'(\eta) - K'(\xi)F''(\xi)},$$

which completes the proof. □

REMARK 2. For $(X, \|.\|)$ a Banach space, it follows from Theorem 2 that

$$m \leqslant \left\| \frac{H(M_H^{\circ}(Z,f,t)) - H(M_F^{\circ}(Z,f,t))}{K(M_K^{\circ}(Z,f,t)) - K(M_F^{\circ}(Z,f,t))} \right\| \leqslant M,$$

Where m and M are respectively, the minimum and maximum values of

$$\left\| \frac{H''(\eta)F'(\eta) - H'(\eta)F''(\eta)}{K''(\eta)F'(\eta) - K'(\xi)F''(\xi)} \right\|, \ \eta \in X.$$

Next, we prove an important result which lead us to define the Cauchy type means on C_0-group of operators.

COROLLARY 4. *Let* $r,s,l \in \mathbb{R}$ *and* $\{Z(t)\}_{t\in\mathbb{R}} \subset B(X)$ *be a* C_0*-semigroup of operators on a Banach space* X*. Then*

$$\frac{M_r^r(Z,f,t) - M_s^r(Z,f,t)}{M_l^l(Z,f,t) - M_s^l(Z,f,t)} = \frac{r(r-s)}{l(l-s)}\eta^{r-l}, \quad \eta \in X. \tag{9}$$

Where $M_r(Z,f,t)$ *is defined by (1).*

Proof. For $r,s,l \in \mathbb{R}$ and $f \in X$, if we set

$$H(f) = f^r,\ F(f) = f^s,\ K(f) = f^l$$

in Theorem 2, the assertion in (9) follows directly. □

REMARK 3. It follows from Corollary (4) that

$$\left|\frac{r(r-s)}{l(l-s)}\right| m \leqslant \left\|\frac{M_r^r(Z,f,t) - M_s^r(Z,f,t)}{M_l^l(Z,f,t) - M_s^l(Z,f,t)}\right\| \leqslant \left|\frac{r(r-s)}{l(l-s)}\right| M.$$

Where m and M are respectively, the minimum and maximum values of $\|\eta^{r-l}\|$, $\eta \in X$.

In the next definition we have defined means of the Cauchy type on C_0-group of linear operators.

DEFINITION 2. Let $r,s,l \in \mathbb{R}$ and $\{Z(t)\}_{t\in\mathbb{R}} \subset B(X)$ be a C_0-semigroup of operators on a Banach space X. Then

$$\mathfrak{M}_r^{l,s}(Z,f,t) = \left(\frac{l(l-s)}{r(r-s)} \frac{M_r^r(Z,f,t) - M_s^r(Z,f,t)}{M_l^l(Z,f,t) - M_s^l(Z,f,t)}\right)^{\frac{1}{r-l}}. \tag{10}$$

is a mean of the Cauchy type on C_0-group of operators. This definition is true for all $r \neq l \neq s \neq 0$ and other cases can be taken as limiting cases, as in [6].

3. Conclusion

Firstly, we have proved two mean value theorems. A systematic procedure has been used to define means on C_0-group of linear operators. These means are Cauchy type means on C_0-group of linear operators. Moreover, it can be easily proved that these means are monotonic.

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