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TWO–WEIGHT INEQUALITIES FOR HARDY OPERATOR AND COMMUTATORS

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Abstract. For the maximal operator N related to the Hardy operator P and its adjoint Q, we give the characterizations for weights (u,v) such that N is bounded from $L^p(v)$ to $L^{p,\infty}(u)$ and from $L^p(v)$ to $L^p(u)$ respectively. We also obtain some A_p type conditions which are sufficient for the two-weight inequalities for the Hardy operator P, the adjoint operator Q and the commutators of these operators with CMO functions.

1. Introduction

Let P and Q be the Hardy operator and its adjoint on $(0,\infty)$,

$$Pf(x) = \frac{1}{x}\int_0^x f(y)dy, \quad Qf(x) = \int_x^\infty \frac{f(y)}{y}dy.$$

Hardy [8, 9] established the Hardy integral inequalities

$$\int_0^\infty |Pf(y)|^p dy \leqslant p'^p \int_0^\infty |f(y)|^p dy, \quad p>1,$$

and

$$\int_0^\infty |Qf(y)|^p dy \leqslant p^p \int_0^\infty |f(y)|^p dy, \quad p>1,$$

where $p' = p/(p-1)$.

The two inequalities above go by the name of Hardy's integral inequalities. For the earlier development of this kind of inequality and many applications in analysis, see [10, 15].

The Calderón operator S is defined as $S = P+Q$ and plays a significant role in the theory of real interpolation, see [1]. Duoandikoetxea, Martin-Reyes and Ombrosi in [7] introduced the maximal operator N related to the Calderón operator and obtained a

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new characterization for the weighted inequalities on S. Given a measurable function f on $(0,\infty)$, the maximal operator N is defined as

$$Nf(x)=\sup_{t>x}\frac{1}{t}\int_0^t|f(y)|dy.$$

Notice that Nf is a decreasing function, and that $|Pf|\leqslant Nf\leqslant S(|f|)$ for any f.

Let $1\leqslant p<\infty$, we say b is a one-side dyadic CMO^p function, if

$$\sup_{j\in\mathbb{Z}}\Big(\frac{1}{2^j}\int_0^{2^j}|b(y)-b_{(0,2^j]}|^p dy\Big)^{1/p}=\|b\|_{\mathrm{CMO}^q}<\infty,$$

where $b_{(0,2^j]}=\frac{1}{2^j}\int_0^{2^j}f(x)dx$, we then say that $b\in\mathrm{CMO}^p$.

It is easy to see $\mathrm{BMO}(0,\infty)\subsetneqq\mathrm{CMO}^p$, where $1\leqslant p<\infty$. $\mathrm{CMO}^q\subsetneqq\mathrm{CMO}^p$ for $1\leqslant p<q<\infty$.

Let b be a locally integrable function on $(0,\infty)$, we define the commutators of the Calderón operator S with b as $S_b=P_b+Q_b$, where

$$P_bf(x)=\frac{1}{x}\int_0^x(b(x)-b(y))f(y)dy,\ \ Q_bf(x)=\int_x^\infty\frac{(b(x)-b(y))f(y)}{y}dy.$$

Long and Wang in [12] established the Hardy's integral inequalities for commutators generated by P and Q with one-sided dyadic CMO functions.

For operator T such as N, P, Q, S and S_b, it is natural to consider the problem of characterizing the pairs (u,v) of nonnegative measurable functions such that

$$\Big(\int_0^\infty|Tf(y)|^q u(y)dy\Big)^{1/q}\leqslant C\Big(\int_0^\infty|f(y)|^p v(y)dy\Big)^{1/p}\tag{1.1}$$

holds with a positive constant C independent of f, where $0<p,q<\infty$.

For $p>1$, Muckenhoupt in [14] proved that P is bounded from $L^p(v)$ to $L^p(u)$ if and only if there exists $C>0$ such that for all $t>0$ it holds that

$$\Big(\int_t^\infty\frac{u(y)}{y^p}dy\Big)^{1/p}\Big(\int_0^t v^{1-p'}(y)dy\Big)^{1/p'}\leqslant C.$$

Bradley [2] and Maz'ya [13] obtained the similar results for the case $1<p\leqslant q<\infty$. For operator Q, they also obtained the similar results.

For $1<p<\infty$, we say a pair of weights (u,v) satisfies the two-weight $A_{p,0}$ condition, denoted $(u,v)\in A_{p,0}$, if

$$[u,v]_p=\sup_{t>0}\Big(\frac{1}{t}\int_0^t u(y)dy\Big)\Big(\frac{1}{t}\int_0^t v(y)^{-p'/p}dy\Big)^{p/p'}<\infty.$$

For $p=1$, we write $(u,v)\in A_{1,0}$, for the class of nonnegative functions such that $Nu(y)\leqslant Cv(y)$, $a.e.$ and $[u,v]_1$ denotes the constant for which the inequality holds.

When $u = v$, the classes of $A_{p,0}$ weights were introduced by Duoandikoetxea, Martin-Reyes and Ombrosi in [7]. They proved that the Calderón operator S are bounded on $L^p(w)$ if and only if $w \in A_{p,0}$ when $p > 1$. For N, the result is same.

In this paper, we obtain the characterizations for weights (u,v) such that N is bounded from $L^p(v)$ to $L^{p,\infty}(u)$ and from $L^p(v)$ to $L^p(u)$ respectively. We also give some A_p type conditions which are sufficient for the two-weight strong (p,p) inequalities for the operators P, P_b, Q and Q_b. Our conditions differ from the conditions in Muckenhoupt in [14], Bradley [2] and Maz'ya [13].

THEOREM 1.1. *For $1 \leqslant p < \infty$, N is bounded from $L^p(v)$ to $L^{p,\infty}(u)$ if and only if $(u,v) \in A_{p,0}$. More precisely,*

$$\sup_{\lambda>0} \lambda u(\{x : Nf(x) > \lambda\})^{1/p} \leqslant [u,v]_p^{1/p} \|f\|_{L^p(v)}.$$

THEOREM 1.2. *For $1 < p < \infty, 0 < q < \infty$, N is bounded from $L^p(v)$ to $L^q(u)$ if and only if for any $t > 0$, (u,v) satisfies*

$$\Big(\int_0^t [N(v^{1-p'}\chi_{(0,b)})(y)]^q u(y)dy\Big)^{1/q} \leqslant C\Big(\int_0^t v(y)^{1-p'}dy\Big)^{1/p} < \infty. \tag{1.2}$$

But for $1 < p < \infty$, N is not bounded from $L^p(v)$ to $L^p(u)$ if $(u,v) \in A_{p,0}$, the proof is same as the case for the Hardy-Littlewood maximal function on $\mathbb{R}^n$, see [6]. Notice that $|Pf| \leqslant Nf \leqslant S(|f|)$, by Theorem 1.1, we have that $(u,v) \in A_{p,0}$ is necessary but not sufficient for S is bounded from $L^p(v)$ to $L^p(u)$.

THEOREM 1.3. *Let $1 < p < \infty$.*

(1) If (u,v) is a pair of weights for which there exists $r > 1$ such that, for every $t > 0$,

$$\Big(\frac{1}{t}\int_0^t u(y)dy\Big)\Big(\frac{1}{t}\int_0^t v(y)^{-rp'/p}dy\Big)^{p/rp'} \leqslant C < \infty. \tag{1.3}$$

Then

$$\int_0^\infty |Pf(x)|^p u(x)dx \leqslant C\int_0^\infty |f(x)|^p v(x)dx. \tag{1.4}$$

(2) If (u,v) is a pair of weights for which there exists $r > 1$ such that, for every $t > 0$,

$$\Big(\frac{1}{t}\int_0^t u(y)^r dy\Big)^{1/r}\Big(\frac{1}{t}\int_0^t v(y)^{-p'/p}dy\Big)^{p/p'} \leqslant C < \infty. \tag{1.5}$$

Then

$$\int_0^\infty |Qf(x)|^p u(x)dx \leqslant C\int_0^\infty |f(x)|^p v(x)dx. \tag{1.6}$$

THEOREM 1.4. *Let $1 < p < \infty$, $b \in \mathrm{CMO}^{r'\max\{p,p'\}}$ and (u,v) be a pair of weights for which there exists $r > 1$ such that for every $t > 0$,*

$$\Big(\frac{1}{t}\int_0^t u(y)^r dy\Big)^{1/r}\Big(\frac{1}{t}\int_0^t v(y)^{-rp'/p}dy\Big)^{p/rp'} \leqslant C < \infty, \tag{1.7}$$

then

$$\int_0^\infty |P_b f(x)|^p u(x)dx \leqslant C\int_0^\infty |f(x)|^p v(x)dx, \tag{1.8}$$

and

$$\int_0^\infty |Q_b f(x)|^p u(x)dx \leqslant C\int_0^\infty |f(x)|^p v(x)dx. \tag{1.9}$$

2. The Proofs of Theorem 1.1 and Theorem 1.2

In the proofs of Theorem 1.1 and Theorem 1.2, we need the maximal operator N_g associated to a fixed positive measurable function g. We defined N_g as

$$N_g f(x) = \sup_{t>x} \frac{\int_0^t |f(y)|g(y)dy}{\int_0^t g(y)dy}.$$

THEOREM 2.1. [7] *Let g be a nonnegative measurable function such that $0 < g(0,b) = \int_0^b g(y)dy < \infty$ for all $b > 0$.*

(i) N_g is of weak type $(1,1)$ with respect to the measure $g(t)dt$. Actually,

$$\int_{\{x:N_g f(x)>\lambda\}} g(y)dy \leqslant \frac{1}{\lambda}\int_{\{x:N_g f(x)>\lambda\}} |f(y)|g(y)dy$$

for all $\lambda > 0$ and all measurable functions f.

(ii) N_g is of strong type (p,p), $1 < p < \infty$, with respect to the measure $g(t)dt$. More precisely,

$$\int_0^\infty |N_g f(y)|^p g(y)dy \leqslant (p')^p \int_0^\infty |f(y)|^p g(y)dy.$$

Proof of Theorem 1.1. For $1 \leqslant p < \infty$, the proof for the necessity of $A_{p,0}$ weights is standard, we omitted here. For sufficiency, we observe that Nf is decreasing and continuous. Therefore, if $\{x : Nf(x) > \lambda\}$ is not empty, then it is a bounded interval $(0,d)$, thus

$$d\lambda = \int_0^d |f(y)|dy.$$

Then

$$\begin{aligned}
&\lambda u(\{x : Nf(x) > \lambda\})^{1/p} = \lambda\Big(\int_0^d u(y)dy\Big)^{1/p} \\
&\leqslant \frac{1}{d}\Big(\int_0^d |f(y)|^p v(y)dy\Big)^{1/p}\Big(\int_0^d u(y)dy\Big)^{1/p}\Big(\int_0^d v(y)^{-p/p'}dy\Big)^{1/p'} \\
&\leqslant [u,v]_p^{1/p}\Big(\int_0^d |f(y)|^p v(y)dy\Big)^{1/p}.
\end{aligned}$$

This ends the proof. □

Proof of Theorem 1.2. Denote $\sigma = v^{1-p'}$. The necessity of (1.2) follows by a standard argument if we substitute $f = \sigma\chi_{(0,b)}$ into $\|Nf\|_{L^q(u)} \leqslant C\|f\|_{L^p(v)}$.

To show that (1.2) is sufficient, fix a bounded nonnegative function f with compact support. Since Nf is decreasing and continuous, for each $k \in \mathbb{Z}$, if $\{x \in (0,\infty) : Nf(x) > 2^k\}$ is not empty, then there exists d_k such that $\{x \in (0,\infty) : Nf(x) > 2^k\} = (0,d_k)$. Thus $0 < d_{k+1} \leqslant d_k$, $\Omega_k = \{x \in (0,\infty) : 2^k < Nf(x) \leqslant 2^{k+1}\} = [d_{k+1},d_k)$ and

$$2^k d_k = \int_0^{d_k} f(y)dy.$$

Fix a large integer $K > 0$, which will go to infinity later, and let $\Lambda_K = \{k \in \mathbb{Z} : |k| \leqslant K\}$. We have

$$\begin{aligned}
\mathscr{I}_K &= \int_{\cup_{k=-K}^{K}\Omega_k} (Nf(y))^q u(y)dy \leqslant \sum_{k=-K}^{K} 2^{(k+1)q}\int_{d_{k+1}}^{d_k} u(y)dy \\
&= 2^q \sum_{k=-K}^{K} \int_{d_{k+1}}^{d_k} u(y)dy\Big(\frac{1}{d_k}\int_0^{d_k} f(y)dy\Big)^q \\
&= 2^q \sum_{k=-K}^{K} \int_{d_{k+1}}^{d_k} u(y)dy\Big(\frac{1}{d_k}\int_0^{d_k} \sigma(y)dy\Big)^q\Big(\frac{\int_0^{d_k}(f\sigma^{-1})(y)\sigma(y)dy}{\int_0^{d_k}\sigma(y)dy}\Big)^q \\
&= 2^q \int_{\mathbb{Z}} T_K(f\sigma^{-1})^q d\nu,
\end{aligned}$$

where ν is the measure on $\mathbb{Z}$ given by

$$\nu(k) = \int_{d_{k+1}}^{d_k} u(y)dy\Big(\frac{1}{d_k}\int_0^{d_k} \sigma(y)dy\Big)^q,$$

and, for every measurable function h, the operator T_K is defined by

$$T_K h(k) = \frac{\int_0^{d_k} h(y)\sigma(y)dy}{\int_0^{d_k}\sigma(y)dy}\chi_{\Lambda_K}(k).$$

If we prove that T_K is uniformly bounded from $L^p((0,\infty),\sigma)$ to $L^q(\mathbb{Z},\nu)$ independently of K, we shall obtain

$$\begin{aligned}
\mathscr{I}_K &\leqslant C\int_{\mathbb{Z}} T_K(f\sigma^{-1})^q d\nu \leqslant C\Big(\int_0^\infty [(f\sigma^{-1})(y)]^p\sigma(y)dy\Big)^{q/p} \\
&= C\Big(\int_0^\infty f(y)^p v(y)dy\Big)^{q/p}.
\end{aligned}$$

The uniformity in K of this estimate and the monotone convergence theorem will lead to the desired inequality.

Now we prove that T_K is a bounded operator from $L^p((0,\infty),\sigma)$ to $L^q(\mathbb{Z},\nu)$. It is clear that $T_K : L^\infty((0,\infty),\sigma) \to L^\infty(\mathbb{Z},\nu)$ with constant less than or equal 1. The Marcinkiewicz interpolation theorem says that it is enough to prove the uniform boundedness of the operators T_K from $L^1((0,\infty),\sigma)$ to $L^{q/p,\infty}(\mathbb{Z},\nu)$. For this, fix $h \geqslant 0$ a bounded function with compact support and put $F_\lambda = \{k \in \mathbb{Z} : T_K h(k) > \lambda\} = \{|k| \leqslant K : T_K h(k) > \lambda\}$, and $k_0 = \min\{k : k \in F_\lambda\}$. Using (1.2), we have

$$\begin{aligned}
\nu(F_\lambda) &= \sum_{k\in F_\lambda} \int_{d_{k+1}}^{d_k} \Big(\frac{1}{d_k}\int_0^{d_k} \sigma(y)dy\Big)^q u(x)dx \\
&\leqslant \sum_{k\in F_\lambda} \int_{d_{k+1}}^{d_k} (N(\sigma\chi_{(0,d_k)})(x)^q u(x)dx \\
&\leqslant \sum_{k\in F_\lambda} \int_{d_{k+1}}^{d_k} (N(\sigma\chi_{(0,d_{k_0})})(x)^q u(x)dx \\
&\leqslant \int_0^{d_{k_0}} (N(\sigma\chi_{(0,d_{k_0})})(x)^q u(x)dx \\
&\leqslant C\Big(\int_0^{d_{k_0}} h(y)\sigma(y)dy\Big)^{q/p} \\
&\leqslant C\Big(\frac{1}{\lambda}\int_0^{d_{k_0}} h(y)\sigma(y)\sigma(y)dy\Big)^{q/p} \\
&\leqslant C\Big(\frac{1}{\lambda}\int_0^{\infty} h(y)\sigma(y)dy\Big)^{q/p},
\end{aligned}$$

where the constant C does not depend on K. This ends the proof. □

3. The Proofs of Theorem 1.3 and Theorem 1.4

LEMMA 3.1. [12] *Let* $b \in \mathrm{CMO}^1$, $j,k \in \mathbb{Z}$, *then*

$$|b(t) - b_{(0,2^{j+1}]}| \leqslant |b(t) - b_{(0,2^{k+1}]}| + 2|j-k|\|b\|_{\mathrm{CMO}^1}.$$

Proof of Theorem 1.3. We first prove (1.4). By Hölder inequality and condition (1.3), we have

$$\begin{aligned}
&\int_0^\infty |Pf(x)|^p u(x)dx \\
&= \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} \Big|\frac{1}{x}\int_0^x f(y)dy\Big|^p u(x)dx \\
&\leqslant \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} \Big|\frac{1}{2^j}\sum_{k=-\infty}^{j}\int_{2^k}^{2^{k+1}} |f(y)|dy\Big|^p u(x)dx \\
&\leqslant \sum_{j=-\infty}^{\infty} \Big(\frac{1}{2^j}\sum_{k=-\infty}^{j}\Big(\int_{2^k}^{2^{k+1}} |f(y)|^p v(y)dy\Big)^{1/p}\Big(\int_{2^k}^{2^{k+1}} v(y)^{-p'/p}dy\Big)^{1/p'}\Big)^p \int_{2^j}^{2^{j+1}} u(x)dx \\
&\leqslant \sum_{j=-\infty}^{\infty} \frac{1}{2^{jp}}\int_{2^j}^{2^{j+1}} u(x)dx \\
&\quad\times\Big(\sum_{k=-\infty}^{j}\Big(\int_{2^k}^{2^{k+1}} |f(y)|^p v(y)dy\Big)^{1/p}\Big(\int_{2^k}^{2^{k+1}} v(y)^{-rp'/p}dy\Big)^{1/rp'}\Big(\int_{2^k}^{2^{k+1}} dy\Big)^{1/r'p'}\Big)^p
\end{aligned}$$

$$\leqslant C\sum_{j=-\infty}^{\infty}\Big(\sum_{k=-\infty}^{j}2^{\frac{(k-j)}{r'p'}}\Big(\int_{2^k}^{2^{k+1}}|f(y)|^p v(y)dy\Big)^{1/p}\Big)^p$$
$$\leqslant C\sum_{j=-\infty}^{\infty}\Big(\sum_{k=-\infty}^{j}2^{\frac{(k-j)}{2r'}}\Big)^{p/p'}\sum_{k=-\infty}^{j}2^{\frac{(k-j)p}{2r'p'}}\int_{2^k}^{2^{k+1}}|f(y)|^p v(y)dy$$
$$\leqslant C\int_0^{\infty}|f(y)|^p v(y)dy.$$

Now we prove (1.6). By Hölder inequality and condition (1.5), we have

$$\begin{aligned}
&\int_0^{\infty}|Qf(x)|^p u(x)dx\\
&=\sum_{j=-\infty}^{\infty}\int_{2^j}^{2^{j+1}}\Big|\int_x^{\infty}\frac{f(y)}{y}dy\Big|^p u(x)dx\\
&\leqslant\sum_{j=-\infty}^{\infty}\int_{2^j}^{2^{j+1}}\Big|\sum_{k=j}^{\infty}\frac{1}{2^k}\int_{2^k}^{2^{k+1}}|f(y)|dy\Big|^p u(x)dx\\
&\leqslant\sum_{j=-\infty}^{\infty}\int_{2^j}^{2^{j+1}}u(x)dx\\
&\quad\times\Big(\sum_{k=j}^{\infty}\frac{1}{2^k}\Big(\int_{2^k}^{2^{k+1}}|f(y)|^p v(y)dy\Big)^{1/p}\Big(\int_{2^k}^{2^{k+1}}v(y)^{-p'/p}dy\Big)^{1/p'}\Big)^p\\
&\leqslant\sum_{j=-\infty}^{\infty}\Big(\int_{2^j}^{2^{j+1}}u(x)^r dx\Big)^{1/r}2^{\frac{j}{r'}}\\
&\quad\times\Big(\sum_{k=j}^{\infty}\frac{1}{2^k}\Big(\int_{2^k}^{2^{k+1}}|f(y)|^p v(y)dy\Big)^{1/p}\Big(\int_{2^k}^{2^{k+1}}v(y)^{-p'/p}dy\Big)^{1/p'}\Big)^p\\
&\leqslant C\sum_{j=-\infty}^{\infty}\Big(\sum_{k=j}^{\infty}2^{\frac{j-k}{r'p}}\Big(\int_{2^k}^{2^{k+1}}|f(y)|^p v(y)dy\Big)^{1/p}\Big)^p\\
&\leqslant C\sum_{j=-\infty}^{\infty}\Big(\sum_{k=j}^{\infty}2^{\frac{(j-k)p'}{2r'p}}\Big)^{p/p'}\sum_{k=j}^{\infty}2^{\frac{(j-k)}{2r'}}\int_{2^k}^{2^{k+1}}|f(y)|^p v(y)dy\\
&\leqslant C\int_0^{\infty}|f(y)|^p v(y)dy.
\end{aligned}$$

This ends the proof. □

Proof of Theorem 1.4. We first prove (1.8).

$$\begin{aligned}
&\int_0^{\infty}|P_b f(x)|^p u(x)dx\\
&=\sum_{j=-\infty}^{\infty}\int_{2^j}^{2^{j+1}}\Big|\frac{1}{x}\int_0^x(b(x)-b(y))f(y)dy\Big|^p u(x)dx
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \sum_{j=-\infty}^{\infty}\int_{2^j}^{2^{j+1}}\Big|\frac{1}{2^j}\sum_{k=-\infty}^{j}\int_{2^k}^{2^{k+1}}|(b(x)-b(y))f(y)|dy\Big|^p u(x)dx\\
&\leqslant 2^{p/p'}\sum_{j=-\infty}^{\infty}\int_{2^j}^{2^{j+1}}\Big|\frac{1}{2^j}\sum_{k=-\infty}^{j}\int_{2^k}^{2^{k+1}}|(b(x)-b_{(0,2^{j+1}]})f(y)|dy\Big|^p u(x)dx\\
&\quad +2^{p/p'}\sum_{j=-\infty}^{\infty}\int_{2^j}^{2^{j+1}}\Big|\frac{1}{2^j}\sum_{k=-\infty}^{j}\int_{2^k}^{2^{k+1}}|(b(y)-b_{(0,2^{j+1}]})f(y)|dy\Big|^p u(x)dx\\
&= \mathrm{I}+\mathrm{II}.
\end{aligned}
$$

For I, by Hölder inequality and condition (1.7), we have

$$
\begin{aligned}
\mathrm{I} &= 2^{p/p'}\sum_{j=-\infty}^{\infty}\frac{1}{2^{jp}}\int_{2^j}^{2^{j+1}}|b(x)-b_{(0,2^{j+1}]}|^p u(x)dx\Big(\sum_{k=-\infty}^{j}\int_{2^k}^{2^{k+1}}|f(y)|dy\Big)^p\\
&\leqslant 2^{p/p'}\sum_{j=-\infty}^{\infty}\frac{1}{2^{jp}}\Big(\int_0^{2^{j+1}}|b(x)-b_{(0,2^{j+1}]}|^{pr'}dx\Big)^{1/r'}\Big(\int_0^{2^{j+1}}u(x)^r dx\Big)^{1/r}\\
&\quad\times\Big(\sum_{k=-\infty}^{j}\Big(\int_{2^k}^{2^{k+1}}|f(y)|^p v(y)dy\Big)^{1/p}\Big(\int_{2^k}^{2^{k+1}}v(y)^{-rp'/p}dy\Big)^{1/rp'}\Big(\int_{2^k}^{2^{k+1}}dy\Big)^{1/r'p'}\Big)^p\\
&\leqslant C\|b\|^p_{\mathrm{CMO}^{pr'}}\sum_{j=-\infty}^{\infty}\frac{2^{(j+1)}}{2^{jp}}\Big(\frac{1}{2^{j+1}}\int_0^{2^{j+1}}u(x)^r dx\Big)^{1/r}\\
&\quad\times\Big(\sum_{k=-\infty}^{j}2^{\frac{(j+1)}{rp'}+\frac{(k+1)}{r'p'}}\Big(\int_{2^k}^{2^{k+1}}|f(y)|^p v(y)dy\Big)^{1/p}\Big(\frac{1}{2^{j+1}}\int_0^{2^{j+1}}v(y)^{-p'r/p}dy\Big)^{1/rp'}\Big)^p\\
&\leqslant C\|b\|^p_{\mathrm{CMO}^{pr'}}\sum_{j=-\infty}^{\infty}\Big(\sum_{k=-\infty}^{j}2^{\frac{(k-j)}{r'p'}}\Big(\int_{2^k}^{2^{k+1}}|f(y)|^p v(y)dy\Big)^{1/p}\Big)^p\\
&\leqslant C\|b\|^p_{\mathrm{CMO}^{pr'}}\sum_{j=-\infty}^{\infty}\Big(\sum_{k=-\infty}^{j}2^{\frac{(k-j)}{2r'}}\Big)^{p/p'}\sum_{k=-\infty}^{j}2^{\frac{(k-j)p}{2r'p'}}\int_{2^k}^{2^{k+1}}|f(y)|^p v(y)dy\\
&\leqslant C\|b\|^p_{\mathrm{CMO}^{pr'}}\int_0^{\infty}|f(y)|^p v(y)dy.
\end{aligned}
$$

For II, by Lemma 3.1, we have

$$
\begin{aligned}
\mathrm{II} &= 2^{p/p'}\sum_{j=-\infty}^{\infty}\frac{1}{2^{jp}}\int_{2^j}^{2^{j+1}}\Big|\sum_{k=-\infty}^{j}\int_{2^k}^{2^{k+1}}|(b(y)-b_{(0,2^{k+1}]})f(y)|dy\Big|^p u(x)dx\\
&\quad +2^{p/p'}\sum_{j=-\infty}^{\infty}\frac{1}{2^{jp}}\int_{2^j}^{2^{j+1}}\Big|\sum_{k=-\infty}^{j}\int_{2^k}^{2^{k+1}}2(j-k)C\|b\|_{\mathrm{CMO}^1}|f(y)|dy\Big|^p u(x)dx\\
&= \mathrm{II}_1+\mathrm{II}_2.
\end{aligned}
$$

For II_1, by Hölder inequality and condition (1.7), we have

$$\begin{aligned}\mathrm{II}_1 &= 2^{p/p'}\sum_{j=-\infty}^{\infty}\frac{1}{2^{jp}}\int_{2^j}^{2^{j+1}}u(x)dx\Big(\sum_{k=-\infty}^{j}\Big(\int_{2^k}^{2^{k+1}}|b(y)-b_{(0,2^{k+1}]}|^{r'p'}dy\Big)^{1/r'p'}\\
&\quad\times\Big(\int_{2^k}^{2^{k+1}}|f(y)|^p v(y)dy\Big)^{1/p}\Big(\int_{2^k}^{2^{k+1}}v(y)^{-rp'/p}dy\Big)^{1/rp'}\Big)^p\\
&\leqslant C\|b\|^p_{\mathrm{CMO}^{p'r'}}\sum_{j=-\infty}^{\infty}\frac{2^{(j+1)}}{2^{jp}}\Big(\sum_{k=-\infty}^{j}2^{\frac{(k+1)}{r'p'}+\frac{(j+1)}{rp'}}\Big(\int_{2^k}^{2^{k+1}}|f(y)|^p v(y)dy\Big)^{1/p}\Big)^p\\
&\leqslant C\|b\|^p_{\mathrm{CMO}^{p'r'}}\int_0^\infty|f(y)|^p v(y)dy.\end{aligned}$$

For II_2, we have

$$\begin{aligned}\mathrm{II}_2 &= C\|b\|^p_{\mathrm{CMO}^1}\sum_{j=-\infty}^{\infty}\frac{1}{2^{jp}}\int_{2^j}^{2^{j+1}}u(x)dx\Big(\sum_{k=-\infty}^{j}(j-k)\Big(\int_{2^k}^{2^{k+1}}|f(y)|^p v(y)dy\Big)^{1/p}\\
&\quad\times\Big(\int_{2^k}^{2^{k+1}}v(y)^{-rp'/p}dy\Big)^{1/rp'}\Big(\int_{2^k}^{2^{k+1}}dy\Big)^{1/r'p'}\Big)^p\\
&\leqslant C\|b\|^p_{\mathrm{CMO}^{p'r'}}\sum_{j=-\infty}^{\infty}\Big(\sum_{k=-\infty}^{j}(j-k)2^{\frac{(k-j)}{r'p'}}\Big(\int_{2^k}^{2^{k+1}}|f(y)|^p v(y)dy\Big)^{1/p}\Big)^p\\
&\leqslant C\|b\|^p_{\mathrm{CMO}^{p'r'}}\int_0^\infty|f(y)|^p v(y)dy.\end{aligned}$$

Now we prove (1.9). We have

$$\begin{aligned}&\int_0^\infty|Q_b f(x)|^p u(x)dx\\
&=\sum_{j=-\infty}^{\infty}\int_{2^j}^{2^{j+1}}\Big|\int_x^\infty\frac{(b(x)-b(y))f(y)}{y}dy\Big|^p u(x)dx\\
&\leqslant\sum_{j=-\infty}^{\infty}\int_{2^j}^{2^{j+1}}\Big|\sum_{k=j}^{\infty}\frac{1}{2^k}\int_{2^k}^{2^{k+1}}|(b(x)-b(y))f(y)|dy\Big|^p u(x)dx\\
&\leqslant 2^{p/p'}\sum_{j=-\infty}^{\infty}\int_{2^j}^{2^{j+1}}\Big|\sum_{k=j}^{\infty}\frac{1}{2^k}\int_{2^k}^{2^{k+1}}|(b(x)-b_{(0,2^{j+1}]})f(y)|dy\Big|^p u(x)dx\\
&\quad+2^{p/p'}\sum_{j=-\infty}^{\infty}\int_{2^j}^{2^{j+1}}\Big|\sum_{k=j}^{\infty}\frac{1}{2^k}\int_{2^k}^{2^{k+1}}|(b(y)-b_{(0,2^{j+1}]})f(y)|dy\Big|^p u(x)dx\\
&=\mathrm{J}+\mathrm{JJ}.\end{aligned}$$

For J, by Hölder inequality and condition (1.7), we have

$$\begin{aligned}
\mathrm{J} &= 2^{p/p'}\sum_{j=-\infty}^{\infty}\int_{2^j}^{2^{j+1}}|b(x)-b_{(0,2^{j+1}]}|^p u(x)dx\Big(\sum_{k=j}^{\infty}\frac{1}{2^k}\int_{2^k}^{2^{k+1}}|f(y)|dy\Big)^p \\
&\leqslant 2^{p/p'}\sum_{j=-\infty}^{\infty}\Big(\int_0^{2^{j+1}}|b(x)-b_{(0,2^{j+1}]}|^{pr'}dx\Big)^{1/r'}\Big(\int_0^{2^{j+1}}u(x)^r dx\Big)^{1/r} \\
&\quad\times\Big(\sum_{k=j}^{\infty}\frac{1}{2^k}\big(\int_{2^k}^{2^{k+1}}|f(y)|^p v(y)dy\big)^{1/p}\big(\int_{2^k}^{2^{k+1}}v(y)^{-p'/p}dy\big)^{1/p'}\Big)^p \\
&\leqslant C\|b\|^p_{\mathrm{CMO}^{pr'}}\sum_{j=-\infty}^{\infty}2^{j/r'}\Big(\frac{1}{2^{k+1}}\int_0^{2^{k+1}}u(x)^r dx\Big)^{1/r} \\
&\quad\times\Big(\sum_{k=j}^{\infty}2^{-k+\frac{k}{rp}+\frac{k}{p'}}\big(\int_{2^k}^{2^{k+1}}|f(y)|^p v(y)dy\big)^{1/p}\big(\frac{1}{2^{k+1}}\int_0^{2^{k+1}}v(y)^{-p'/p}dy\big)^{1/p'}\Big)^p \\
&\leqslant C\|b\|^p_{\mathrm{CMO}^{pr'}}\sum_{j=-\infty}^{\infty}\Big(\sum_{k=j}^{\infty}2^{\frac{j-k}{r'p}}\big(\int_{2^k}^{2^{k+1}}|f(y)|^p v(y)dy\big)^{1/p}\Big)^p \\
&\leqslant C\|b\|^p_{\mathrm{CMO}^{pr'}}\sum_{j=-\infty}^{\infty}\Big(\sum_{k=j}^{\infty}2^{\frac{(j-k)p'}{2r'p}}\Big)^{p/p'}\sum_{k=j}^{\infty}2^{\frac{(j-k)}{2r'}}\int_{2^k}^{2^{k+1}}|f(y)|^p v(y)dy \\
&\leqslant C\|b\|^p_{\mathrm{CMO}^{pr'}}\int_0^{\infty}|f(y)|^p v(y)dy.
\end{aligned}$$

For JJ, by Lemma 3.1, we have

$$\begin{aligned}
\mathrm{JJ} &= 2^{p/p'}\sum_{j=-\infty}^{\infty}\int_{2^j}^{2^{j+1}}\Big|\sum_{k=j}^{\infty}\frac{1}{2^k}\int_{2^k}^{2^{k+1}}|(b(y)-b_{(0,2^{k+1}]})f(y)|dy\Big|^p u(x)dx \\
&\quad+2^{p/p'}\sum_{j=-\infty}^{\infty}\int_{2^j}^{2^{j+1}}\Big|\sum_{k=j}^{\infty}\frac{1}{2^k}\int_{2^k}^{2^{k+1}}2(k-j)C\|b\|_{\mathrm{CMO}^1}|f(y)|dy\Big|^p u(x)dx \\
&= \mathrm{JJ}_1+\mathrm{JJ}_2.
\end{aligned}$$

For JJ_1, by Hölder inequality and condition (1.7), we have

$$\begin{aligned}
\mathrm{JJ}_1 &= 2^{p/p'}\sum_{j=-\infty}^{\infty}\int_{2^j}^{2^{j+1}}u(x)dx\Big|\sum_{k=j}^{\infty}\frac{1}{2^k}\int_{2^k}^{2^{k+1}}|(b(y)-b_{(0,2^{k+1}]})f(y)|dy\Big|^p \\
&\leqslant C\sum_{j=-\infty}^{\infty}2^{j/r'}\Big(\int_0^{2^{j+1}}u(x)^r dx\Big)^{1/r}\Big(\sum_{k=j}^{\infty}\frac{1}{2^k}\Big(\int_{2^k}^{2^{k+1}}|b(y)-b_{(0,2^{k+1}]}|^{r'p'}dy\Big)^{1/r'p'} \\
&\quad\times\Big(\int_{2^k}^{2^{k+1}}|f(y)|^p v(y)dy\Big)^{1/p}\Big(\int_{2^k}^{2^{k+1}}v(y)^{-rp'/p}dy\Big)^{1/rp'}\Big)^p \\
&\leqslant C\|b\|^p_{\mathrm{CMO}^{p'r'}}\sum_{j=-\infty}^{\infty}2^{j/r'}\Big(\sum_{k=j}^{\infty}2^{\frac{k}{rp}-k+\frac{(k+1)}{r'p'}+\frac{(k+1)}{rp'}}\Big(\int_{2^k}^{2^{k+1}}|f(y)|^p v(y)dy\Big)^{1/p}\Big)^p
\end{aligned}$$

$$\leqslant C\|b\|^p_{\mathrm{CMO}^{p'r'}}\sum_{j=-\infty}^{\infty}\Big(\sum_{k=j}^{\infty}2^{\frac{j-k}{r'p}}\Big(\int_{2^k}^{2^{k+1}}|f(y)|^p v(y)dy\Big)^{1/p}\Big)^p$$

$$\leqslant C\|b\|^p_{\mathrm{CMO}^{p'r'}}\int_0^{\infty}|f(y)|^p v(y)dy.$$

For JJ_2, we have

$$\mathrm{JJ}_2 = C\|b\|^p_{\mathrm{CMO}^1}\sum_{j=-\infty}^{\infty}\int_{2^j}^{2^{j+1}}u(x)dx\Big(\sum_{k=j}^{\infty}\frac{k-j}{2^k}\Big(\int_{2^k}^{2^{k+1}}|f(y)|^p v(y)dy\Big)^{1/p}$$

$$\times\Big(\int_{2^k}^{2^{k+1}}v(y)^{-p'/p}dy\Big)^{1/p'}\Big)^p$$

$$\leqslant C\|b\|^p_{\mathrm{CMO}^{p'r'}}\sum_{j=-\infty}^{\infty}\Big(\sum_{k=j}^{\infty}(k-j)2^{\frac{j-k}{r'p}}\Big(\int_{2^k}^{2^{k+1}}|f(y)|^p v(y)dy\Big)^{1/p}\Big)^p$$

$$\leqslant C\|b\|^p_{\mathrm{CMO}^{p'r'}}\int_0^{\infty}|f(y)|^p v(y)dy.$$

This ends the proof. □

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