TWO–WEIGHT INEQUALITIES FOR HARDY OPERATOR AND COMMUTATORS

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Abstract. For the maximal operator $N$ related to the Hardy operator $P$ and its adjoint $Q$, we give the characterizations for weights $(u, v)$ such that $N$ is bounded from $L^p(v)$ to $L^{p, \infty}(u)$ and from $L^p(v)$ to $L^p(u)$ respectively. We also obtain some $A_p$ type conditions which are sufficient for the two-weight inequalities for the Hardy operator $P$, the adjoint operator $Q$ and the commutators of these operators with CMO functions.

1. Introduction

Let $P$ and $Q$ be the Hardy operator and its adjoint on $(0, \infty)$,

$$Pf(x) = \frac{1}{x} \int_0^x f(y) dy, \quad Qf(x) = \int_x^\infty \frac{f(y)}{y} dy.$$  

Hardy [8, 9] established the Hardy integral inequalities

$$\int_0^\infty |Pf(y)|^p dy \leq p' p \int_0^\infty |f(y)|^p dy, \quad p > 1,$$

and

$$\int_0^\infty |Qf(y)|^p dy \leq p' p \int_0^\infty |f(y)|^p dy, \quad p > 1,$$

where $p' = p/(p - 1)$.

The two inequalities above go by the name of Hardy’s integral inequalities. For the earlier development of this kind of inequality and many applications in analysis, see [10, 15].

The Calderón operator $S$ is defined as $S = P + Q$ and plays a significant role in the theory of real interpolation, see [1]. Duoandikoetxea, Martín-Reyes and Ombrosi in [7] introduced the maximal operator $N$ related to the Calderón operator and obtained a


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new characterization for the weighted inequalities on $S$. Given a measurable function $f$ on $(0, \infty)$, the maximal operator $N$ is defined as

$$\text{NF}(x) = \sup_{t > x} \frac{1}{t} \int_0^t |f(y)|dy.$$ 

Notice that $\text{NF}$ is a decreasing function, and that $|Pf| \leq NF \leq S(|f|)$ for any $f$.

Let $1 \leq p < \infty$, we say $b$ is a one-side dyadic CMO$^p$ function, if

$$\sup_{j \in \mathbb{Z}} \left( \frac{1}{2^j} \int_0^{2^j} |b(y) - b_{[0,2^j]}|^p dy \right)^{1/p} = \|b\|_{\text{CMO}^p} < \infty,$$

where $b_{[0,2^j]} = \frac{1}{2^j} \int_0^{2^j} f(x)dx$, we then say that $b \in \text{CMO}^p$.

It is easy to see $\text{BMO}(0,\infty) \subseteq \text{CMO}^p$, where $1 \leq p < \infty$. $\text{CMO}^q \not\subseteq \text{CMO}^p$ for $1 \leq p < q < \infty$.

Let $b$ be a locally integrable function on $(0, \infty)$, we define the commutators of the Calderón operator $S$ with $b$ as $S_b = P_b + Q_b$, where

$$P_b f(x) = \frac{1}{x} \int_0^x (b(x) - b(y)) f(y)dy, \quad Q_b f(x) = \int_x^\infty \frac{(b(x) - b(y)) f(y)}{y}dy.$$

Long and Wang in [12] established the Hardy’s integral inequalities for commutators generated by $P$ and $Q$ with one-sided dyadic CMO functions.

For operator $T$ such as $N$, $P$, $Q$, $S$ and $S_b$, it is natural to consider the problem of characterizing the pairs $(u,v)$ of nonnegative measurable functions such that

$$(\int_0^\infty |T f(y)|^q u(y)dy)^{1/q} \leq C \left( \int_0^\infty |f(y)|^p v(y)dy \right)^{1/p}$$

holds with a positive constant $C$ independent of $f$, where $0 < p, q < \infty$.

For $p > 1$, Muckenhoupt in [14] proved that $P$ is bounded from $L^p(v)$ to $L^p(u)$ if and only if there exists $C > 0$ such that for all $t > 0$ it holds that

$$(\int_t^\infty u(y) \frac{dy}{yp})^{1/p} (\int_0^t v^{-p'}(y)dy)^{1/p'} \leq C.$$

Bradley [2] and Maz’ya [13] obtained the similar results for the case $1 < p \leq q < \infty$. For operator $Q$, they also obtained the similar results.

For $1 < p < \infty$, we say a pair of weights $(u,v)$ satisfies the two-weight $A_{p,0}$ condition, denoted $(u,v) \in A_{p,0}$, if

$$[u,v]_p = \sup_{t > 0} \left( \frac{1}{t} \int_0^t u(y)dy \right) \left( \frac{1}{t} \int_0^t v(y)^{-p'/p'} dy \right)^{p/p'} < \infty.$$

For $p = 1$, we write $(u,v) \in A_{1,0}$, for the class of nonnegative functions such that $Nu(y) \leq Cv(y)$, $a.e.$ and $[u,v]_1$ denotes the constant for which the inequality holds.
When \( u = v \), the classes of \( A_{p,0} \) weights were introduced by Duoandikoetxea, Martin-Reyes and Ombrosi in [7]. They proved that the Calderón operator \( S \) are bounded on \( L^p(w) \) if and only if \( w \in A_{p,0} \) when \( p > 1 \). For \( N \), the result is same.

In this paper, we obtain the characterizations for weights \((u, v)\) such that \( N \) is bounded from \( L^p(v) \) to \( L^{p,\infty}(u) \) and from \( L^p(v) \) to \( L^p(u) \) respectively. We also give some \( A_p \) type conditions which are sufficient for the two-weight strong \((p, p)\) inequalities for the operators \( P, P_b, Q \) and \( Q_b \). Our conditions differ from the conditions in Muckenhoupt in [14], Bradley [2] and Maz’ya [13].

**Theorem 1.1.** For \( 1 \leq p < \infty \), \( N \) is bounded from \( L^p(v) \) to \( L^{p,\infty}(u) \) if and only if \((u, v) \in A_{p,0} \). More precisely,

\[
\sup_{\lambda > 0} \lambda u(\{x : N f(x) > \lambda \})^{1/p} \leq \|u\|_p \|f\|_{L^p(v)}.
\]

**Theorem 1.2.** For \( 1 < p < \infty, 0 < q < \infty \), \( N \) is bounded from \( L^p(v) \) to \( L^q(u) \) if and only if for any \( t > 0 \), \((u, v)\) satisfies

\[
\left( \int_0^t [N^q(\chi_{(0,b)}(y))] v(y) dy \right)^{1/q} \leq C \left( \int_0^t v(y)^{1-p'} dy \right)^{1/p} < \infty.
\]

But for \( 1 < p < \infty \), \( N \) is not bounded from \( L^p(v) \) to \( L^p(u) \) if \((u, v) \in A_{p,0} \), the proof is same as the case for the Hardy-Littlewood maximal function on \( \mathbb{R}^n \), see [6]. Notice that \(|Pf| \leq Nf \leq |f|\), by Theorem 1.1, we have that \((u, v) \in A_{p,0} \) is necessary but not sufficient for \( S \) is bounded from \( L^p(v) \) to \( L^p(u) \).

**Theorem 1.3.** Let \( 1 < p < \infty \).

1. If \((u, v)\) is a pair of weights for which there exists \( r > 1 \) such that, for every \( t > 0 \),

\[
\left( \frac{1}{t} \int_0^t u(y) dy \right) \left( \frac{1}{t} \int_0^t v(y)^{-r/p'} dy \right)^{p'/r'} \leq C < \infty.
\]

Then

\[
\int_0^\infty |Pf(x)|^p u(x) dx \leq C \int_0^\infty |f(x)|^p v(x) dx.
\]

2. If \((u, v)\) is a pair of weights for which there exists \( r > 1 \) such that, for every \( t > 0 \),

\[
\left( \frac{1}{t} \int_0^t u(y)^r dy \right)^{1/r} \left( \frac{1}{t} \int_0^t v(y)^{-p'/r'} dy \right)^{p'/r'} \leq C < \infty.
\]

Then

\[
\int_0^\infty |Qf(x)|^p u(x) dx \leq C \int_0^\infty |f(x)|^p v(x) dx.
\]

**Theorem 1.4.** Let \( 1 < p < \infty \), \( b \in \text{CMO}^{r, \max\{p, p'\}} \) and \((u, v)\) be a pair of weights for which there exists \( r > 1 \) such that for every \( t > 0 \),

\[
\left( \frac{1}{t} \int_0^t u(y)^r dy \right)^{1/r} \left( \frac{1}{t} \int_0^t v(y)^{-r/p'} dy \right)^{p'/r'} \leq C < \infty.
\]
then
\[ \int_0^\infty |P_b f(x)|^p u(x) dx \leq C \int_0^\infty |f(x)|^p v(x) dx, \quad (1.8) \]

and
\[ \int_0^\infty |Q_b f(x)|^p u(x) dx \leq C \int_0^\infty |f(x)|^p v(x) dx. \quad (1.9) \]

2. The Proofs of Theorem 1.1 and Theorem 1.2

In the proofs of Theorem 1.1 and Theorem 1.2, we need the maximal operator \( N_g \) associated to a fixed positive measurable function \( g \). We defined \( N_g \) as
\[
N_g f(x) = \sup_{t > x} \frac{\int_0^t |f(y)| g(y) dy}{\int_0^t g(y) dy}.
\]

**Theorem 2.1.** [7] \textit{Let \( g \) be a nonnegative measurable function such that \( 0 < g(0, b) = \int_0^b g(y) dy < \infty \) for all \( b > 0 \).}

(i) \( N_g \) is of weak type \((1, 1)\) with respect to the measure \( g(t) dt \). Actually,
\[
\int \{x : N_g f(x) > \lambda\} g(y) dy \leq \frac{1}{\lambda} \int \{x : N_g f(x) > \lambda\} |f(y)| g(y) dy
\]
for all \( \lambda > 0 \) and all measurable functions \( f \).

(ii) \( N_g \) is of strong type \((p, p)\), \( 1 < p < \infty \), with respect to the measure \( g(t) dt \). More precisely,
\[
\int_0^\infty |N_g f(y)|^p g(y) dy \leq (p')^p \int_0^\infty |f(y)|^p g(y) dy.
\]

**Proof of Theorem 1.1.** For \( 1 \leq p < \infty \), the proof for the necessity of \( A_{p,0} \) weights is standard, we omitted here. For sufficiency, we observe that \( Nf \) is decreasing and continuous. Therefore, if \( \{x : Nf(x) > \lambda\} \) is not empty, then it is a bounded interval \((0, d)\), thus
\[
d\lambda = \int_0^d |f(y)| dy.
\]

Then
\[
\lambda u(\{x : Nf(x) > \lambda\})^{1/p} = \lambda \left( \int_0^d u(y) dy \right)^{1/p}
\leq \frac{1}{d} \left( \int_0^d |f(y)|^p v(y) dy \right)^{1/p} \left( \int_0^d u(y) dy \right)^{1/p} \left( \int_0^d v(y)^{-p'/p'} dy \right)^{1/p'}
\leq [u, v]_{p'}^{1/p} \left( \int_0^d |f(y)|^p v(y) dy \right)^{1/p}.
\]

This ends the proof. \( \square \)
Proof of Theorem 1.2. Denote \( \sigma = v^{1-p'} \). The necessity of (1.2) follows by a standard argument if we substitute \( f = \sigma \chi_{(0, b)} \) into \( \|Nf\|_{L^q(\nu)} \leq C\|f\|_{L^p(v)} \).

To show that (1.2) is sufficient, fix a bounded nonnegative function \( f \) with compact support. Since \( Nf \) is decreasing and continuous, for each \( k \in \mathbb{Z} \), if \( \{ x \in (0, \infty) : Nf(x) > 2^k \} \) is not empty, then there exists \( d_k \) such that \( \{ x \in (0, \infty) : Nf(x) > 2^k \} = (0, d_k) \). Thus \( 0 < d_{k+1} \leq d_k \), \( \Omega_k = \{ x \in (0, \infty) : 2^k < Nf(x) \leq 2^{k+1} \} = [d_{k+1}, d_k) \) and

\[
2^k d_k = \int_0^{d_k} f(y)dy.
\]

Fix a large integer \( K > 0 \), which will go to infinity later, and let \( \Lambda_K = \{ k \in \mathbb{Z} : |k| \leq K \} \). We have

\[
\mathcal{J}_K = \int_{-K}^{K} (Nf(y))^q u(y)dy \leq \sum_{k=-K}^{K} 2^{(k+1)q} \int_{d_k}^{d_{k+1}} u(y)dy
\]

\[
= 2^q \sum_{k=-K}^{K} \int_{d_k}^{d_{k+1}} u(y)dy \left( \frac{1}{d_k} \int_0^{d_k} f(y)dy \right)^q
\]

\[
= 2^q \sum_{k=-K}^{K} \int_{d_k}^{d_{k+1}} u(y)dy \left( \frac{1}{d_k} \int_0^{d_k} \sigma(y)dy \right)^q \left( \frac{\int_0^{d_k} (f \sigma^{-1})(y) \sigma(y)dy}{\int_0^{d_k} \sigma(y)dy} \right)^q
\]

\[
= 2^q \int_{\mathbb{Z}} T_K (f \sigma^{-1})^q dv,
\]

where \( v \) is the measure on \( \mathbb{Z} \) given by

\[
v(k) = \int_{d_{k+1}}^{d_k} u(y)dy \left( \frac{1}{d_k} \int_0^{d_k} \sigma(y)dy \right)^q,
\]

and, for every measurable function \( h \), the operator \( T_K \) is defined by

\[
T_K h(k) = \frac{\int_0^{d_k} h(y) \sigma(y)dy}{\int_0^{d_k} \sigma(y)dy} \chi_{\Lambda_K}(k).
\]

If we prove that \( T_K \) is uniformly bounded from \( L^p((0, \infty), \sigma) \) to \( L^q(\mathbb{Z}, v) \) independently of \( K \), we shall obtain

\[
\mathcal{J}_K \leq C \int_{\mathbb{Z}} T_K (f \sigma^{-1})^q dv \leq C \left( \int_0^\infty [(f \sigma^{-1})(y)]^p \sigma(y)dy \right)^{q/p}
\]

\[
= C \left( \int_0^\infty f(y)^p v(y)dy \right)^{q/p}.
\]

The uniformity in \( K \) of this estimate and the monotone convergence theorem will lead to the desired inequality.

Now we prove that \( T_K \) is a bounded operator from \( L^p((0, \infty), \sigma) \) to \( L^q(\mathbb{Z}, v) \). It is clear that \( T_K : L^\infty((0, \infty), \sigma) \rightarrow L^\infty(\mathbb{Z}, v) \) with constant less than or equal 1. The Marcinkiewicz interpolation theorem says that it is enough to prove the uniform boundedness of the operators \( T_K \) from \( L^1((0, \infty), \sigma) \) to \( L^q/p, \infty(\mathbb{Z}, v) \). For this, fix \( h \geq 0 \) a
bounded function with compact support and put \( F_\lambda = \{ k \in \mathbb{Z} : T_K h(k) > \lambda \} = \{ |k| \leq K : T_K h(k) > \lambda \} \), and \( k_0 = \min \{ k : k \in F_\lambda \} \). Using (1.2), we have

\[
\nu(F_\lambda) = \sum_{k \in F_\lambda} \int_{dk} \left( \frac{1}{dk} \int_0^{dk} \sigma(y) dy \right)^q u(x) dx
\]

\[
\leq \sum_{k \in F_\lambda} \int_{dk} \left( N(\sigma \chi_{(0,dk)}) (x)^q u(x) dx \right)
\]

\[
\leq \int_{0}^{dk} \left( N(\sigma \chi_{(0,dk)}) (x)^q u(x) dx \right)
\]

\[
\leq C \left( \int_0^{dk} h(y) \sigma(y) dy \right)^{q/p}
\]

\[
\leq C \left( \frac{1}{k} \int_0^{dk} h(y) \sigma(y) \sigma(y) dy \right)^{q/p}
\]

\[
\leq C \left( \frac{1}{k} \int_0^{dk} h(y) \sigma(y) dy \right)^{q/p},
\]

where the constant \( C \) does not depend on \( K \). This ends the proof. \( \square \)

3. The Proofs of Theorem 1.3 and Theorem 1.4

**Lemma 3.1.** [12] Let \( b \in \text{CMO}^1 \), \( j, k \in \mathbb{Z} \), then

\[
|b(t) - b_{(0,2^{j+1})}| \leq |b(t) - b_{(0,2^{k+1})}| + 2|j - k||b|_{\text{CMO}^1}.
\]

**Proof of Theorem 1.3.** We first prove (1.4). By Hölder inequality and condition (1.3), we have

\[
\int_0^\infty |Pf(x)|^p u(x) dx
\]

\[
= \sum_{j = -\infty}^{\infty} \int_{2^j}^{2^{j+1}} \left| \frac{1}{x} \int_0^x f(y) dy \right|^p u(x) dx
\]

\[
\leq \sum_{j = -\infty}^{\infty} \int_{2^j}^{2^{j+1}} \left| \frac{1}{2^j} \sum_{k = -\infty}^{j} \int_{2^k}^{2^{k+1}} |f(y)| dy \right|^p u(x) dx
\]

\[
\leq \sum_{j = -\infty}^{\infty} \left( \frac{1}{2^j} \sum_{k = -\infty}^{j} \left( \int_{2^k}^{2^{k+1}} |f(y)|^p v(y) dy \right)^{1/p} \left( \int_{2^k}^{2^{k+1}} v(y)^{1/p'} dy \right)^{1/p'} \right)^p \int_{2^j}^{2^{j+1}} u(x) dx
\]

\[
\leq \sum_{j = -\infty}^{\infty} \frac{1}{2^j} \int_{2^j}^{2^{j+1}} u(x) dx
\]

\[
\times \left( \sum_{k = -\infty}^{j} \left( \int_{2^k}^{2^{k+1}} |f(y)|^p v(y) dy \right)^{1/p} \left( \int_{2^k}^{2^{k+1}} v(y)^{-rp'/p} dy \right)^{1/rp'} \left( \int_{2^k}^{2^{k+1}} dy \right)^{1/rp'} \right)^p
\]
Now we prove (1.6). By Hölder inequality and condition (1.5), we have

\[
\int_i^\infty |Qf(x)|^p u(x)dx = \sum_{j=-\infty}^\infty \int_{2j}^{2j+1} \left( \int_{2j}^{2j+1} \frac{f(y)}{y} dy \right)^p u(x)dx 
\leq \sum_{j=-\infty}^\infty \int_{2j}^{2j+1} \left( \sum_{k=j}^{2j+1} \frac{1}{2^k} \int_{2k}^{2k+1} |f(y)|^p v(y)dy \right)^p u(x)dx 
\leq \sum_{j=-\infty}^\infty \int_{2j}^{2j+1} f(x)^p u(x)dx 
\leq C \sum_{j=-\infty}^\infty \left( \sum_{k=j}^{2j+1} 2^{j-k} \left( \int_{2k}^{2k+1} |f(y)|^p v(y)dy \right)^{1/p'} \right)^{1/p'} 
\leq C \sum_{j=-\infty}^\infty \left( \sum_{k=j}^{2j+1} 2^{j-k} \left( \int_{2k}^{2k+1} |f(y)|^p v(y)dy \right)^{1/p'} \right)^{1/p'} 
\leq C \int_0^\infty |f(y)|^p v(y)dy.
\]

This ends the proof. □

**Proof of Theorem 1.4.** We first prove (1.8).

\[
\int_0^\infty |P_b f(x)|^p u(x)dx = \sum_{j=-\infty}^\infty \int_{2j}^{2j+1} \left| \frac{1}{x} \int_0^x (b(x) - b(y)) f(y) dy \right|^p u(x)dx
\]
\[ \begin{align*} &\leq \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} \left| \frac{1}{2^j} \sum_{k=-\infty}^{j} \int_{2^k}^{2^{k+1}} (b(x) - b(y)) f(y) \, dy \right|^p u(x) \, dx \\ &\leq 2^{p/p'} \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} \left| \frac{1}{2^j} \sum_{k=-\infty}^{j} \int_{2^k}^{2^{k+1}} (b(x) - b(0,2^{j+1})) f(y) \, dy \right|^p u(x) \, dx \\ &\quad + 2^{p/p'} \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} \left| \frac{1}{2^j} \sum_{k=-\infty}^{j} \int_{2^k}^{2^{k+1}} (b(y) - b(0,2^{j+1})) f(y) \, dy \right|^p u(x) \, dx \\ &= I + II. \end{align*} \]

For I, by Hölder inequality and condition (1.7), we have

\[ \begin{align*} I &= 2^{p/p'} \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} \left| b(x) - b(0,2^{j+1}) \right|^p u(x) \, dx \left( \sum_{k=-\infty}^{j} \int_{2^k}^{2^{k+1}} |f(y)|^p \, dy \right)^{1/p'} \\ &\leq 2^{p/p'} \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} \left| b(x) - b(0,2^{j+1}) \right|^p u(x) \, dx \left( \int_{2^k}^{2^{k+1}} |f(y)|^p \, dy \right)^{1/p'} \left( \int_{2^k}^{2^{k+1}} u(x) \, dx \right)^{1/p} \\ &\leq C\|b\|_{CMO^{p'}}^{p} \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} \left| f(y) \right|^p v(y) \, dy \left( \int_{2^k}^{2^{k+1}} u(x) \, dx \right)^{1/p} \\ &\quad \times \left( \sum_{k=-\infty}^{j} \int_{2^k}^{2^{k+1}} \left| f(y) \right|^p v(y) \, dy \right)^{1/p'} \left( \int_{2^k}^{2^{k+1}} v(y)^{-r'/p' \, dy} \right)^{1/p'} \\ &\leq C\|b\|_{CMO^{p'}}^{p} \sum_{j=-\infty}^{\infty} \left( \sum_{k=-\infty}^{j} \int_{2^k}^{2^{k+1}} \left| f(y) \right|^p v(y) \, dy \right)^{1/p} \\ &\leq C\|b\|_{CMO^{p'}}^{p} \sum_{j=-\infty}^{\infty} \left( \sum_{k=-\infty}^{j} 2^{(k-j)/p'} \int_{2^k}^{2^{k+1}} \left| f(y) \right|^p v(y) \, dy \right)^{1/p} \\ &\leq C\|b\|_{CMO^{p'}}^{p} \int_{0}^{\infty} \left| f(y) \right|^p v(y) \, dy. \end{align*} \]

For II, by Lemma 3.1, we have

\[ \begin{align*} II &= 2^{p/p'} \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} \left| \sum_{k=-\infty}^{j} \int_{2^k}^{2^{k+1}} (b(y) - b(0,2^{j+1})) \, dy \right|^p u(x) \, dx \\ &\quad + 2^{p/p'} \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} \left| \sum_{k=-\infty}^{j} \int_{2^k}^{2^{k+1}} 2(j-k)C\|b\|_{CMO^{1}} \, dy \right|^p u(x) \, dx \\ &= II_1 + II_2. \end{align*} \]
For $\text{II}_1$, by Hölder inequality and condition (1.7), we have

$$\text{II}_1 = 2^{p/p'} \sum_{j=-\infty}^{\infty} \frac{1}{2jp} \int_{2^j}^{2^{j+1}} u(x)dx \left( \sum_{k=-\infty}^{j} \left( \int_{2^k}^{2^{k+1}} |b(y) - b_{(0,2^k+1)}| r^p dy \right)^{1/r'} r' \right)^{1/p'}$$

$$\leq C \|b\|_{\text{CMO}^{p,r}}^p \sum_{j=-\infty}^{\infty} \frac{2^{(j+1)}}{2jp} \left( \sum_{k=-\infty}^{j} 2^{(k+1)(k-j)/r} \left( \int_{2^k}^{2^{k+1}} |f(y)|^p v(y)dy \right)^{1/p} \right)^p$$

For $\text{II}_2$, we have

$$\text{II}_2 = C \|b\|_{\text{CMO}^1}^p \sum_{j=-\infty}^{\infty} \frac{1}{2jp} \int_{2^j}^{2^{j+1}} u(x)dx \left( \sum_{k=-\infty}^{j} (j-k) \left( \int_{2^k}^{2^{k+1}} v(y)^{-r/p'} dy \right)^{1/r'} \right)^{1/p}$$

$$\leq C \|b\|_{\text{CMO}^{p,r}}^p \sum_{j=-\infty}^{\infty} \left( \sum_{k=-\infty}^{j} (j-k) 2^{(j-k)/p} \left( \int_{2^k}^{2^{k+1}} |f(y)|^p v(y)dy \right)^{1/p} \right)^p$$

Now we prove (1.9). We have

$$\int_{\mathbb{R}} |Q_b f(x)|^p u(x)dx$$

$$= \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} \left| \int_{x}^{\infty} \frac{(b(x) - b(y))f(y)}{y} dy \right|^p u(x)dx$$

$$\leq \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} \left| \sum_{k=j}^{\infty} \frac{1}{2^k} \int_{2^k}^{2^{k+1}} |(b(x) - b(y))f(y)| dy \right|^p u(x)dx$$

$$\leq 2^{p/p'} \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} \left| \sum_{k=j}^{\infty} \frac{1}{2^k} \int_{2^k}^{2^{k+1}} |(b(x) - b_{(0,2^k+1)})f(y)| dy \right|^p u(x)dx$$

$$+ 2^{p/p'} \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} \left| \sum_{k=j}^{\infty} \frac{1}{2^k} \int_{2^k}^{2^{k+1}} |(b(y) - b_{(0,2^k+1)})f(y)| dy \right|^p u(x)dx$$

$$= J + J'$$
For $J$, by Hölder inequality and condition (1.7), we have

\[
J = 2^{p/p'} \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} |b(x) - b_{(0,2^{j+1})}|^p u(x) dx \left( \sum_{k=j}^{\infty} \frac{1}{2^k} \int_{2^k}^{2^{k+1}} |f(y)|dy \right)^p
\]

\[
\leq 2^{p/p'} \sum_{j=-\infty}^{\infty} \left( \int_{0}^{2^{j+1}} |b(x) - b_{(0,2^{j+1})}|^{p'} dx \right)^{1/p'} \left( \int_{0}^{2^{j+1}} u(x)^r dx \right)^{1/r} \times \left( \sum_{k=j}^{\infty} \frac{1}{2^k} \int_{2^k}^{2^{k+1}} |f(y)|^{p} v(y)dy \right)^{1/p} \left( \int_{2^k}^{2^{k+1}} v(y)^{-p'/r} dy \right)^{1/p'}
\]

\[
\leq C \|b\|_{\text{CMO}^{p'}} \sum_{j=-\infty}^{\infty} \int_{0}^{2^{j+1}} |f(y)|^{p} v(y)dy
\]

For $JJ$, by Lemma 3.1, we have

\[
JJ = 2^{p/p'} \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} \left| \sum_{k=j}^{\infty} \frac{1}{2^k} \int_{2^k}^{2^{k+1}} |(b(y) - b_{(0,2^{k+1})})f(y)|dy \right|^p u(x) dx
\]

\[
+2^{p/p'} \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} \left| \sum_{k=j}^{\infty} \frac{1}{2^k} \int_{2^k}^{2^{k+1}} 2(k-j)C \|b\|_{\text{CMO}^{p'}} |f(y)|dy \right|^p u(x) dx
\]

\[
= JJ_1 + JJ_2.
\]

For $JJ_1$, by Hölder inequality and condition (1.7), we have

\[
JJ_1 = 2^{p/p'} \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} u(x) dx \left| \sum_{k=j}^{\infty} \frac{1}{2^k} \int_{2^k}^{2^{k+1}} |(b(y) - b_{(0,2^{k+1})})f(y)|dy \right|^p
\]

\[
\leq C \sum_{j=-\infty}^{\infty} 2^{j/r'} \left( \int_{0}^{2^{j+1}} u(x)^r dx \right)^{1/r} \left( \sum_{k=j}^{\infty} \frac{1}{2^k} \int_{2^k}^{2^{k+1}} |b(y) - b_{(0,2^{k+1})}|^{r'} dy \right)^{1/r'} \times \left( \int_{2^k}^{2^{k+1}} |f(y)|^{p} v(y)dy \right)^{1/p} \left( \int_{2^k}^{2^{k+1}} v(y)^{-r'/p} dy \right)^{1/p'}
\]

\[
\leq C \|b\|_{\text{CMO}^{p'}} \sum_{j=-\infty}^{\infty} 2^{j/r'} \sum_{k=j}^{\infty} \frac{2^{k+1}}{r'} \left( \int_{2^k}^{2^{k+1}} |f(y)|^{p} v(y)dy \right)^{1/p} \left( \int_{2^k}^{2^{k+1}} v(y)^{-r'/p} dy \right)^{1/p'}
\]
\[ \leq C\|b\|_{CMO^{p'}}^p \sum_{j=-\infty}^{\infty} \left( \sum_{k=j}^{\infty} 2^{j-1} \left( \int_{2^k}^{2^{k+1}} |f(y)|^p v(y) dy \right)^{1/p} \right)^p \]
\[ \leq C\|b\|_{CMO^{p'}}^p \int_0^\infty |f(y)|^p v(y) dy. \]

For JJ2, we have
\[ JJ2 = C\|b\|_{CMO^1}^p \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} u(x) dx \left( \sum_{k=j}^{\infty} 2^{j-1} \left( \int_{2^k}^{2^{k+1}} |f(y)|^p v(y) dy \right)^{1/p} \right)^p \]
\[ \times \left( \int_{2^k}^{2^{k+1}} v(y)^{-p'/p} dy \right)^{1/p'} \]
\[ \leq C\|b\|_{CMO^{p'}}^p \sum_{j=-\infty}^{\infty} \left( \sum_{k=j}^{\infty} (k-j) 2^{j-1} \left( \int_{2^k}^{2^{k+1}} |f(y)|^p v(y) dy \right)^{1/p} \right)^p \]
\[ \leq C\|b\|_{CMO^{p'}}^p \int_0^\infty |f(y)|^p v(y) dy. \]

This ends the proof. \(\square\)

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\textbf{REFERENCES}


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