ON AN EXTENSION OF SAKAGUCHI'S RESULT

MAMORU NUNOKAWA AND JANUSZ SOKÓŁ

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Abstract. We improve Pommerenke's result [15] by using a generalized lemma from [11]. It gives an extension of Sakaguchi's result too, [18]. Several applications of main theorems are presented. A part of them improves the previous results of this type. We consider also Sakaguchi's result under different type of the assumptions.

1. Introduction

Let $\mathscr{H}$ denote the class of analytic functions in the unit disc $\mathbb{U} = \{z : |z| < 1\}$ in the complex plane $\mathbb{C}$. Recall that a set $E \subset \mathbb{C}$ is said to be starlike with respect to a point $w_0 \in E$ if and only if the linear segment joining w_0 to every other point $w \in E$ lies entirely in E, while a set E is said to be convex if and only if it is starlike with respect to each of its points, that is if and only if the linear segment joining any two points of E lies entirely in E. A univalent function f maps $\mathbb{U}$ onto convex domain E if and only if [22]

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \quad \text{for all } z \in \mathbb{U} \tag{1.1}$$

and then f is said to be convex in $\mathbb{U}$ (or briefly convex). Let $\mathscr{A}$ denote the subclass of $\mathscr{H}$ consisting of functions normalized by $f(0) = 0$, $f'(0) = 1$. The set of all functions $f \in \mathscr{A}$ that are convex univalent in $\mathbb{U}$ we denote by $\mathscr{K}$. The set of all functions $f \in \mathscr{A}$ that are starlike univalent in $\mathbb{U}$ with respect to the origin we denote by $\mathscr{S}^*$. In [18] Sakaguchi proved that if $f \in \mathscr{A}$ and $g \in \mathscr{S}^*$, then

$$\left[\mathfrak{Re}\left\{\frac{f'(z)}{g'(z)}\right\} > 0 \quad z \in \mathbb{U}\right] \Rightarrow \left[\mathfrak{Re}\left\{\frac{f(z)}{g(z)}\right\} > 0 \quad z \in \mathbb{U}\right].$$

This result found many of the applications. It was also generalized, see [8] and [16]. In this paper we consider an extension two of different manners.

LEMMA 1.1. *Let $w(z)$ be an analytic function in $|z| < R$ of the form*

$$w(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad p \geqslant 1.$$

If the maximum of $|w(z)|$ on the circle $|z| = r < R$ is attained at $z = z_0$, then we have

$$\frac{z_0 w'(z_0)}{w(z_0)} = l \geqslant p, \tag{1.2}$$

which means $z_0 w'(z_0)/w(z_0)$ is a positive real number.

Proof. Let $w(z)$ be a function defined by

$$w(z) = z^p \varphi(z). \tag{1.3}$$

Then $\varphi(z)$ is analytic in $|z| < R$. If $|\varphi(z)|$ takes the maximum at $z = z_0$ on the circle $|z| = r$, then z moves on the circle $|z| = r$ with positive direction, $\arg\{\varphi(z)\}$ increase at $z = z_0$. Therefore, we have

$$\left(\frac{d}{d\theta} \arg\{\varphi(re^\theta)\} \right)_{z=z_0} = \mathfrak{Re}\left\{ \frac{z_0 \varphi'(z_0)}{\varphi(z_0)} \right\} \geqslant 0, \ z = re^{i\theta}$$

and so,

$$\mathfrak{Re}\left\{ \frac{z_0 w'(z_0)}{w(z_0)} \right\} = \mathfrak{Re}\left\{ p + \frac{z_0 \varphi'(z_0)}{\varphi(z_0)} \right\} = l \geqslant p.$$

On the other hand, from the hypothesis, we have

$$\left(\frac{d}{d\theta} |w(z)| \right)_{z=z_0} = 0.$$

This shows that

$$\left(\frac{d}{d\theta} (\mathfrak{Re}\{\log(w(z))\}) \right)_{z=z_0} = \mathfrak{Im}\left\{ \frac{z_0 w'(z_0)}{w(z_0)} \right\} = 0.$$

It completes the proof. □

The above proof can be found also in [1], for $p = 1$ Lemma 1.1 becomes the well known Jack's Lemma [3].

To prove the main results, we also need the following generalization of the Nunokawa's Lemma, [11], [12].

LEMMA 1.2. *Let $p(z)$ be analytic function in $|z| < 1$ of the form*

$$p(z) = 1 + \sum_{n=k}^{\infty} a_n z^n, \quad a_k \neq 0,$$

with $p(z) \neq 0$ in $|z| < 1$. If there exists a point z_0, $|z_0| < 1$, such that

$$|\arg\{p(z)\}| < \frac{\pi\alpha}{2} \quad for \quad |z| < |z_0|$$

and

$$|\arg\{p(z_0)\}| = \frac{\pi\alpha}{2}$$

for some $\alpha > 0$, then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = im\alpha,$$

where

$$m \geqslant \frac{k}{2}\left(a + \frac{1}{a}\right) \geqslant k \quad when \quad \arg\{p(z_0)\} = \frac{\pi\alpha}{2} \tag{1.4}$$

and

$$m \leqslant -\frac{k}{2}\left(a + \frac{1}{a}\right) \leqslant -k \quad when \quad \arg\{p(z_0)\} = -\frac{\pi\alpha}{2}, \tag{1.5}$$

where

$$|p(z_0)|^{1/\alpha} = a.$$

Proof. Let a function $q(z)$ be defined by

$$q(z) = \{p(z)\}^{1/\alpha}. \tag{1.6}$$

Then we easily have the following

$$q(z) = 1 + b_k z^k + \cdots, b_k \neq 0.$$

Moreover, from the hypothesis, we have

$$\mathfrak{Re}\{q(z)\} > 0 \quad \text{for} \quad |z| < |z_0| < 1$$

and

$$\mathfrak{Re}\{q(z_0)\} = 0 \neq q(z_0).$$

If we define the function ϕ by the equation

$$\phi(z) = \frac{q(z) - 1}{q(z) + 1} \quad |z| \leqslant |z_0|,$$

then we have

$$\phi(0) = \phi'(0) = \phi''(0) = \ldots = \phi^{(k-1)}(0) = 0,$$

moreover

$$|\phi(z)| < 1 \quad \text{for} \quad |z| < |z_0| \quad \text{and} \quad |\phi(z_0)| = 1.$$

Because $\phi(z)$ satisfies the assumptions of Lemma 1.1, we have

$$\frac{z_0\phi'(z_0)}{\phi(z_0)} = \frac{-2z_0q'(z_0)}{1-q^2(z_0)} = \frac{-2z_0q'(z_0)}{1+|q(z_0)|^2} = l \geqslant k,$$

where $k \leqslant l$. Therefore, we have

$$\frac{z_0q'(z_0)}{q(z_0)} = \frac{l\left(1+|q(z_0)|^2\right)}{-2q(z_0)} = \begin{cases} \frac{il}{2}\left(a+\frac{1}{a}\right) & \text{when } q(z_0)=ai, \\ -\frac{il}{2}\left(a+\frac{1}{a}\right) & \text{when } q(z_0)=-ai. \end{cases} \tag{1.7}$$

Since (1.6), we have

$$\frac{z_0q'(z_0)}{q(z_0)} = \frac{1}{\alpha}\frac{z_0p'(z_0)}{p(z_0)},$$

hence by (1.7) we obtain

$$\frac{z_0p'(z_0)}{p(z_0)} = im\alpha,$$

where

$$m: \begin{cases} m = \frac{l}{2}\left(a+\frac{1}{a}\right) & \text{when } \arg\{p(z_0)\} = \frac{\pi\alpha}{2}, \\ m = -\frac{l}{2}\left(a+\frac{1}{a}\right) & \text{when } \arg\{p(z_0)\} = -\frac{\pi\alpha}{2}. \end{cases}$$

Because $k \leqslant l$, the above m satisfies both (1.4) and (1.5). This completes the proof. $\qquad\square$

2. Main results

Ch. Pommerenke has shown in [15] the following theorem.

THEOREM A. *If $f(z)$ is analytic, $g(z)$ is convex univalent in $|z| < 1$ and*

$$\left|\arg\left\{\frac{f'(z)}{g'(z)}\right\}\right| \leqslant \frac{\alpha\pi}{2}, \ \ 0 \leqslant \alpha \leqslant 1 \ \ in \ |z| < 1, \tag{2.1}$$

then

$$\left|\arg\left\{\frac{f(z_2)-f(z_1)}{g(z_2)-g(z_1)}\right\}\right| \leqslant \frac{\alpha\pi}{2}, \ \ |z_1| < 1, \ |z_2| < 1. \tag{2.2}$$

Notice that (2.1) implies that f satisfies Ozaki's univalence condition [14], or that f strongly close-to-convex of order α with respect to g. Recall [17], that $f \in \mathscr{A}$ is said to be in the class $\mathscr{C}(\alpha)$ of strongly close-to-convex functions of order α, $0 < \alpha \leqslant 1$, if and only if there exist $g \in \mathscr{K}$, $\varphi \in \mathbb{R}$, such that

$$\left|\arg\left\{\frac{f'(z)}{e^{i\varphi}g'(z)}\right\}\right| < \frac{\pi\alpha}{2}, \ \ z \in \mathbb{U}. \tag{2.3}$$

If $\alpha = 1$, then $\mathscr{C}(\alpha)$ becomes the well known class of close-to-convex functions, Kaplan [4]. Functions defined by (2.3) with $\varphi = 0$, $\alpha = 1$ where considered earlier by Ozaki [14], see also Umezawa [23, 24]. Moreover, Lewandowski [5, 6] defined the

class of functions $f \in \mathscr{A}$ for which the complement of $f(\mathbb{U})$ with respect to the complex plane is a linearly accessible domain in the large sense. The Lewandowski's class is identical with the Kaplan's class $\mathscr{C}(1)$. It is worthy to note that $f \in \mathscr{A}$ satisfies the condition (2.2) with some convex univalent g and $\alpha = 1$ if and only if f is close-to-convex function, see [2, p. 31].

In this work we improve Theorem A in the following one.

THEOREM 2.1. *If $f(z)$ is analytic, $g(z)$ is convex univalent in $|z| < 1$, $f'(0) = g'(0)$ and*

$$\left| \arg \left\{ \frac{f'(z)}{g'(z)} \right\} \right| \leqslant \frac{\alpha \pi}{2}, \ \ in \ |z| < 1, \tag{2.4}$$

where $0 \leqslant \alpha$, then

$$\left| \arg \left\{ \frac{f(z_2) - f(z_1)}{g(z_2) - g(z_1)} \right\} \right| < \alpha \left(\frac{\pi}{2} - \log 2 \right), \ \ |z_1| < 1, \ \ |z_2| < 1, \tag{2.5}$$

where $\pi/2 - \log 2 = 0.877649147\ldots$.

Proof. In this proof we make more precise the Pommerenke's method [15] by using the idea from [13]. Let $\chi(w)$ be inverse function of $w = g(z)$, and let $H(w) = f(\chi(w))$, with $w_j = g(z_j)$, $j = 1, 2$, then we have

$$\frac{f(z_2) - f(z_1)}{g(z_2) - g(z_1)} = \frac{H(w_2) - H(w_1)}{w_2 - w_1} = \int_0^1 H'(w_1 + (w_2 - w_1)t) dt. \tag{2.6}$$

Then, applying [13], we have

$$\begin{aligned}
\left| \arg \left\{ \frac{f(z_2) - f(z_1)}{g(z_2) - g(z_1)} \right\} \right| &= \left| \arg \left\{ \int_0^1 H'(w_1 + (w_2 - w_1)t) dt \right\} \right| \\
&\leqslant \int_0^1 \left| \arg \left\{ H'(w_1 + (w_2 - w_1)t) \right\}^{\alpha} \right| dt \\
&\leqslant \int_0^1 \left| \arg \left\{ \frac{1+z}{1-z} \right\}^{\alpha} \right| dt \\
&\leqslant \alpha \int_0^1 \sin^{-1} \frac{2\rho}{1+\rho^2} dt = \alpha \left(\frac{\pi}{2} - \log 2 \right),
\end{aligned}$$

since $H'(w) = f'(\chi(w))/g'(\chi(w))$ lies in the convex sector $\{ |\arg \zeta| \leqslant (\alpha\pi)/2 \}$. $\square$

Applying Theorem 2.1 for $\alpha = \pi/(\pi - \log 4)$, we have

COROLLARY 2.2. *If $f(z)$ is analytic, $g(z)$ is convex univalent in $|z| < 1$, $f'(0) = g'(0)$ and*

$$\left| \arg \left\{ \frac{f'(z)}{g'(z)} \right\} \right| \leqslant \frac{\pi^2}{2\pi - 2\log 4}, \ \ in \ |z| < 1, \tag{2.7}$$

then

$$\left|\arg\left\{\frac{f(z_2)-f(z_1)}{g(z_2)-g(z_1)}\right\}\right| < \frac{\pi}{2}, \ |z_1| < 1, \ |z_2| < 1, \tag{2.8}$$

or

$$\Re\left\{\frac{f(z_2)-f(z_1)}{g(z_2)-g(z_1)}\right\} > 0, \ |z_1| < 1, \ |z_2| < 1. \tag{2.9}$$

If $f(0) = g(0) = 0$, then putting $z_1 = 0$ and $z_2 = z$, $|z| < 1$, instead (2.9) we have

$$\Re\left\{\frac{f(z)}{g(z)}\right\} > 0, \ |z_1| < 1, \ |z_2| < 1.$$

$$\frac{\pi^2}{2\pi-2\log 4} = \frac{\pi}{2}\frac{\pi}{\pi-\log 4}$$
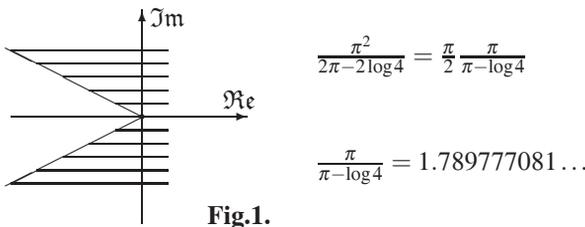
$$\frac{\pi}{\pi-\log 4} = 1.789777081\ldots$$

Fig.1.

For special choices of the convex function g in Theorem 2.1 we can get several interesting corollaries. Let us consider in succession $g(z) = z$ and $g(z) = z/(1-z)$.

COROLLARY 2.3. *If $f(z)$ is analytic in $|z| < 1$, $f'(0) = 1$ and*

$$\left|\arg\left\{f'(z)\right\}\right| \leqslant \frac{\alpha\pi}{2}, \ in \ |z| < 1,$$

where $0 \leqslant \alpha$, then

$$\left|\arg\left\{\frac{f(z_2)-f(z_1)}{z_2-z_1}\right\}\right| < \alpha\left(\frac{\pi}{2}-\log 2\right), \ |z_1| < 1, \ |z_2| < 1,$$

where $\pi/2 - \log 2 = 0.877649147\ldots$.

COROLLARY 2.4. *If $f(z)$ is analytic in $|z| < 1$, $f'(0) = 1$ and*

$$\left|\arg\left\{(1-z)^2 f'(z)\right\}\right| \leqslant \frac{\alpha\pi}{2}, \ in \ |z| < 1,$$

where $0 \leqslant \alpha$, then

$$\left|\arg\left\{\frac{(1-z_1)(1-z_2)\left(f(z_2)-f(z_1)\right)}{z_2-z_1}\right\}\right| < \alpha\left(\frac{\pi}{2}-\log 2\right), \ |z_1| < 1, \ |z_2| < 1,$$

where $\pi/2 - \log 2 = 0.877649147\ldots$.

The next corollaries we obtain by putting $z_2 = -z_1$ or $z_2 = 0$ in the previous corollaries.

COROLLARY 2.5. *If $f(z)$ is analytic in $|z| < 1$, $f'(0) = 1$ and*

$$\left| \arg\left\{ f'(z) \right\} \right| \leqslant \frac{\alpha\pi}{2}, \ \ in \ |z| < 1,$$

where $0 \leqslant \alpha$, then

$$\left| \arg\left\{ \frac{f(z) - f(-z)}{2z} \right\} \right| < \alpha\left(\frac{\pi}{2} - \log 2 \right), \ |z| < 1,$$

where $\pi/2 - \log 2 = 0.877649147\ldots$.

COROLLARY 2.6. *If $f(z)$ is analytic in $|z| < 1$, $f'(0) = 1$ and*

$$\left| \arg\left\{ (1-z)^2 f'(z) \right\} \right| \leqslant \frac{\alpha\pi}{2}, \ \ in \ |z| < 1,$$

where $0 \leqslant \alpha$, then

$$\left| \arg\left\{ \frac{(1-z)^2 \left(f(z) - f(-z) \right)}{2z} \right\} \right| < \alpha\left(\frac{\pi}{2} - \log 2 \right), \ |z| < 1,$$

where $\pi/2 - \log 2 = 0.877649147\ldots$.

COROLLARY 2.7. *If $f(z)$ is analytic in $|z| < 1$, $f(0) = 0$, $f'(0) = 1$ and*

$$\left| \arg\left\{ f'(z) \right\} \right| \leqslant \frac{\alpha\pi}{2}, \ \ in \ |z| < 1,$$

where $0 \leqslant \alpha$, then

$$\left| \arg\left\{ \frac{f(z)}{z} \right\} \right| < \alpha\left(\frac{\pi}{2} - \log 2 \right), \ |z| < 1,$$

where $\pi/2 - \log 2 = 0.877649147\ldots$.

COROLLARY 2.8. *If $f(z)$ is analytic in $|z| < 1$, $f(0) = 0$, $f'(0) = 1$ and*

$$\left| \arg\left\{ (1-z)^2 f'(z) \right\} \right| \leqslant \frac{\alpha\pi}{2}, \ \ in \ |z| < 1,$$

where $0 \leqslant \alpha$, then

$$\left| \arg\left\{ \frac{(1-z)^2 f(z)}{z} \right\} \right| < \alpha\left(\frac{\pi}{2} - \log 2 \right), \ |z| < 1,$$

where $\pi/2 - \log 2 = 0.877649147\ldots$.

If we denote $f(z) = zp(z)$, then the Corollary 2.7 becomes

$$\left|\arg\left\{p(z) + zp'(z)\right\}\right| \leqslant \frac{\alpha\pi}{2} \quad \Rightarrow \quad \left|\arg\left\{p(z)\right\}\right| < \alpha\left(\frac{\pi}{2} - \log 2\right), \quad |z| < 1. \quad (2.10)$$

If $\alpha = 3/2$, then we (2.10) becomes

$$\left|\arg\left\{p(z) + zp'(z)\right\}\right| \leqslant \frac{3\pi}{4} \quad \Rightarrow \quad \left|\arg\left\{p(z)\right\}\right| < \frac{3}{2}\left(\frac{\pi}{2} - \log 2\right), \quad |z| < 1. \quad (2.11)$$

Writing (2.11) in terms of the subordinations we get

$$p(z) + zp'(z) \prec \left(\frac{1+z}{1-z}\right)^{3/2} \quad \Rightarrow \quad p(z) \prec \left(\frac{1+z}{1-z}\right)^{\varepsilon}, \quad |z| < 1, \quad (2.12)$$

where $\varepsilon = 3/2 - \log 8/\pi = 0.838\dots$. Therefore, (2.12) improves the result from [9, p. 75], where instead of $\varepsilon = 0.838\dots$ there is 1.

Furthermore, writing (2.10) with $\alpha = 1$ we get

$$\left|\arg\left\{p(z) + zp'(z)\right\}\right| \leqslant \frac{\pi}{2} \quad \Rightarrow \quad \left|\arg\left\{p(z)\right\}\right| < \frac{\pi}{2} - \log 2, \quad |z| < 1. \quad (2.13)$$

Therefore, (2.13) improves the result from [10], see also the formula (3.1-12) in [9], where instead of $\frac{\pi}{2} - \log 2 = 0.877649\dots$ there is $\theta_0 = 0.9110\dots$.

Putting $z_2 = 0$ in Theorem 2.1, we obtain that If $f(z)$ is analytic, $g(z)$ is convex univalent in $|z| < 1$, $f'(0) = g'(0)$ and

$$\left|\arg\left\{\frac{f'(z)}{g'(z)}\right\}\right| \leqslant \frac{\alpha\pi}{2}, \quad in \ |z| < 1,$$

where $0 \leqslant \alpha$, then

$$\left|\arg\left\{\frac{f(z)}{g(z)}\right\}\right| < \alpha\left(\frac{\pi}{2} - \log 2\right), \quad |z| < 1, \quad (2.14)$$

where $\pi/2 - \log 2 = 0.877649147\dots$. For similar results see also [19] and [20]. The next theorem provides a inequality of the type (2.14) under some different assumptions.

THEOREM 2.9. *Let α be positive real number and $\beta < 1$. Let $f(z)$ and $g(z)$ be analytic in $|z| < 1$ and let it be of the form*

$$f(z) = z^p + \sum_{n=p+k}^{\infty} a_n z^n, \quad g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n$$

with $1 \leqslant p$, $1 \leqslant k$. Assume also that $0 \leqslant x < 1$, $0 \leqslant y$ and

$$\mathfrak{Re}\left\{\frac{g(z)}{zg'(z)}\right\} \geqslant \frac{x}{p} \quad in \ |z| < 1, \quad (2.15)$$

$$\left|\mathfrak{Im}\left\{\frac{g(z)}{zg'(z)}\right\}\right| \leqslant y \quad in \ |z| < 1. \quad (2.16)$$

Then

$$\left| \arg \left\{ (1-\alpha)\frac{f(z)}{g(z)} + \alpha\frac{f'(z)}{g'(z)} - \beta \right\} \right| < \frac{\pi}{2} + \tan^{-1}\frac{\alpha kx}{(\alpha ky+1)p} \quad in \ |z| < 1, \quad (2.17)$$

implies

$$\mathfrak{Re}\left\{ \frac{f(z)}{g(z)} \right\} > \beta \quad in \ |z| < 1. \quad (2.18)$$

Proof. Let us put

$$\lambda(z) = \frac{g(z)}{zg'(z)},$$

and let

$$q(z) = \frac{1}{1-\beta}\left(\frac{f(z)}{g(z)} - \beta \right) = 1 + c_k z^k + \cdots.$$

Then it follows that

$$(1-\alpha)\frac{f(z)}{g(z)} + \alpha\frac{f'(z)}{g'(z)} - \beta = (1-\beta)(q(z) + \alpha\lambda(z)zq'(z)).$$

If there exists a point z_0, $|z_0| < 1$, such that

$$|\arg\{q(z)\}| < \frac{\pi}{2} \quad \text{for } |z| < |z_0|$$

and

$$|\arg\{q(z_0)\}| = \frac{\pi}{2},$$

then from Lemma 1.2, we have

$$\frac{z_0 q'(z_0)}{q(z_0)} = im,$$

where $q(z_0) = \pm ia$, $0 < a$ and

$$m : \begin{cases} m \geqslant k\left(a+\frac{1}{a}\right)/2 & \text{when } \arg\{q(z_0)\} = \frac{\pi}{2}, \\ m \leqslant -k\left(a+\frac{1}{a}\right)/2 & \text{when } \arg\{q(z_0)\} = -\frac{\pi}{2}. \end{cases} \quad (2.19)$$

For the case $\arg\{q(z_0)\} = \pi/2$, we have

$$\begin{aligned} \arg &\left\{ (1-\alpha)\frac{f(z_0)}{g(z_0)} + \alpha\frac{f'(z_0)}{g'(z_0)} - \beta \right\} \\ &= \arg\{(1-\beta)(q(z_0) + \alpha\lambda(z_0)z_0 q'(z_0))\} \\ &= \arg\{(1-\beta)(1 + \alpha\lambda(z_0)im)\} + \frac{\pi}{2} \\ &= \arg\{I(z_0)\} + \frac{\pi}{2}. \end{aligned} \quad (2.20)$$

Then, by (2.15), it follows that

$$
\begin{aligned}
\mathfrak{Im}\{I(z_0)\} &= \mathfrak{Im}\{(1-\beta)\alpha\lambda(z_0)im\}\\
&= \alpha(1-\beta)m\mathfrak{Re}\{\lambda(z_0)\}\\
&\geqslant \alpha(1-\beta)mx/p\\
&=: Y.
\end{aligned}
$$

Moreover, it follows that

$$
\begin{aligned}
\mathfrak{Re}\{I(z_0)\} &= \mathfrak{Re}\{(1-\beta)\alpha\lambda(z_0)im+(1-\beta)\}\\
&= -\alpha(1-\beta)m\mathfrak{Im}\{\lambda(z_0)\}+(1-\beta)\\
&\leqslant \alpha(1-\beta)my+(1-\beta)\\
&=: X.
\end{aligned}
$$

Therefore, $I(z_0)$ lies in the sector

$$
\{\zeta : \mathfrak{Re}\zeta \leqslant X,\ \mathfrak{Im}\zeta \geqslant Y\},
$$

where $X, Y > 0$ so we have $\arg\{I(z_0)\} \geqslant \tan^{-1}(Y/X)$. Hence

$$
\begin{aligned}
\arg\{I(z_0)\} &\geqslant \tan^{-1}\frac{\alpha(1-\beta)mx/p}{\alpha(1-\beta)my+(1-\beta)}\\
&= \tan^{-1}\frac{\alpha mx/p}{\alpha my+1}\\
&\geqslant \tan^{-1}\frac{\alpha\frac{k}{2}\left(a+\frac{1}{a}\right)x/p}{\alpha\frac{k}{2}\left(a+\frac{1}{a}\right)y+1},
\end{aligned}
$$

because the function

$$
h(m)=\frac{\alpha mx/p}{\alpha my+1}
$$

has positive derivative for $m > 0$ and h increases in $\left[\frac{k}{2}\left(a+\frac{1}{a}\right),\infty\right)$. Moreover,

$$
\begin{aligned}
\arg\{I(z_0)\} &\geqslant \tan^{-1}\frac{\alpha\frac{k}{2}\left(a+\frac{1}{a}\right)x/p}{\alpha\frac{k}{2}\left(a+\frac{1}{a}\right)y+1}\\
&= \tan^{-1}\frac{\alpha kx/p\left(a^2+1\right)}{\alpha ky\left(a^2+1\right)+2a}\\
&\geqslant \tan^{-1}\frac{\alpha kx/p}{\alpha ky+1},
\end{aligned}
$$

because the function

$$
s(a)=\frac{\alpha k\left(a^2+1\right)x/p}{\alpha ky\left(a^2+1\right)+2a}
$$

attains its minimum at $a = 1$.

This inequality together with (2.20) contradicts hypothesis (2.17). For the case $\arg\{q(z_0)\} = -\pi\varphi/2$ applying the same method as the above one, we have

$$\arg\left\{(1-\alpha)\frac{f(z_0)}{g(z_0)} + \alpha\frac{f'(z_0)}{g'(z_0)} - \beta\right\} < -\left(\frac{\pi}{2} + \tan^{-1}\frac{\alpha kx}{(\alpha ky+1)p}\right).$$

This is also the contradiction and therefore it completes the proof. $\square$

Theorem 2.9 is closely related to Ponnusamy and Karunakaran's result [16, p. 81] of the form

$$\left[\Re\frac{\alpha g(z)}{zg'(z)} > \delta \text{ and } \Re\left\{(1-\alpha)\frac{f'(z)}{g'(z)} + \alpha\frac{f'(z)}{g'(z)}\right\} > \beta\right] \Rightarrow \Re\frac{f(z)}{g(z)} > \frac{2\beta + k\delta}{2 + k\delta},$$

where α is a complex number with $\Re\alpha > 0$.

The conditions (2.15) and (2.16) describe a strip which contains the disc $|z - (x/p+y)| \leqslant y$, thus Theorem 2.9 provides the following corollary.

COROLLARY 2.10. *Let α be positive real number and $\beta < 1$. Let $f(z)$ and $g(z)$ be analytic in $|z| < 1$ and let it be of the form*

$$f(z) = z^p + \sum_{n=p+k}^{\infty} a_n z^n, \quad g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n$$

with $1 \leqslant p$, $1 \leqslant k$. Assume also that $0 \leqslant x < 1$, $0 \leqslant y$ and

$$\left|\frac{g(z)}{zg'(z)} - (x/p+y)\right| \leqslant y \text{ in } |z| < 1.$$

Then

$$\left|\arg\left\{(1-\alpha)\frac{f(z)}{g(z)} + \alpha\frac{f'(z)}{g'(z)} - \beta\right\}\right| < \frac{\pi}{2} + \tan^{-1}\frac{\alpha kx/p}{\alpha ky+1} \text{ in } |z| < 1,$$

implies

$$\Re\left\{\frac{f(z)}{g(z)}\right\} > \beta \text{ in } |z| < 1.$$

For $\alpha = 1$ Theorem 2.9 becomes the following corollary.

COROLLARY 2.11. *Let $\beta < 1$. Let $f(z)$ and $g(z)$ be analytic in $|z| < 1$ and let it be of the form*

$$f(z) = z^p + \sum_{n=p+k}^{\infty} a_n z^n, \quad g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n$$

with $1 \leqslant p$, $1 \leqslant k$. Assume also that $0 \leqslant x < 1$, $0 \leqslant y$ and

$$\Re\left\{\frac{g(z)}{zg'(z)}\right\} \geqslant \frac{x}{p} \text{ in } |z| < 1, \tag{2.21}$$

$$\left|\Im\left\{\frac{g(z)}{zg'(z)}\right\}\right| \leqslant y \text{ in } |z| < 1. \tag{2.22}$$

Then

$$\left| \arg\left\{ \frac{f'(z)}{g'(z)} - \beta \right\} \right| < \frac{\pi}{2} + \tan^{-1} \frac{kx/p}{ky+1} \ \ in \ |z| < 1, \tag{2.23}$$

implies

$$\Re\left\{ \frac{f(z)}{g(z)} \right\} > \beta \ \ in \ |z| < 1, \tag{2.24}$$

For $x = 0$ Corollary 2.11 becomes MacGregor's result, [8], of the form

$$\left[\Re\frac{pg(z)}{zg'(z)} > 0 \ \text{and} \ \Re\frac{f'(z)}{g'(z)} > \beta \right] \Rightarrow \Re\frac{f(z)}{g(z)} > \beta,$$

which is a generalization of Libera's result [7] with $\beta = 0$.

THEOREM 2.12. *Let $f(z)$ and be analytic in $|z| < 1$ and let it be of the form*

$$h(z) = z + \sum_{n=k+1}^{\infty} c_n z^n.$$

Assume also that $\beta < 1$ and p is positive integer. If

$$\left| \arg\left\{ \frac{zh'(z)}{h^{1-p}(z)z^p} - \beta \right\} \right| < \frac{\pi}{2} + \tan^{-1}\frac{k}{p} \ \ in \ |z| < 1, \tag{2.25}$$

then

$$\Re\left\{ \frac{h(z)}{z} \right\} > \beta, \ |z| < 1. \tag{2.26}$$

Proof. Choose

$$f(z) = (h(z))^p = z^p + \sum_{n=p+k}^{\infty} a_n z^n, \quad g(z) = z^p.$$

For $\alpha = 1$ the conditions (2.21) (2.22) become $1/p \geqslant x/p$, $0 \leqslant y$ so we can allow $x = 1$, $y = 0$. Then, condition (2.23) becomes (2.25) while (2.24) becomes (2.26). $\square$

For $p = k = 1$ we get the following result.

COROLLARY 2.13. *Let $f(z)$ and be analytic in $|z| < 1$ and let it be of the form*

$$h(z) = z + \sum_{n=2}^{\infty} c_n z^n.$$

Assume also that $\beta < 1$ is positive integer. If

$$\left| \arg\left\{ h'(z) - \beta \right\} \right| < \frac{3\pi}{4} \ \ in \ |z| < 1, \tag{2.27}$$

then

$$\Re\left\{ \frac{h(z)}{z} \right\} > \beta, \ |z| < 1. \tag{2.28}$$

Recall here another result of this type [21, p. 1550] that if $h \in \mathscr{A}$, $\beta < 1$, then

$$\left[h'(\mathbb{U}) \subseteq \{ w \in \mathbb{C} : |w - 2\beta + 1| > |\mathfrak{Re}\{w\} - \beta|\} \right] \Rightarrow \left[\mathfrak{Re} \left\{ \frac{h(z)}{z} \right\} > \beta, \ |z| < 1 \right].$$

The condition $h'(\mathbb{U}) \subseteq \{ w \in \mathbb{C} : |w - 2\beta + 1| > |\mathfrak{Re}\{w\} - \beta|\}$ means that $h'(z)$, $z \in \mathbb{U}$, lies in on the right of a parabola but it hasn't a simple relation with the sector described in (2.27).

THEOREM 2.14. *Let $f(z)$ and $g(z)$ be analytic in $|z| < 1$ and let it be of the form*

$$f(z) = z^p + \sum_{n=p+k}^{\infty} a_n z^n, \quad g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n.$$

Assume also that $0 < \alpha$, $\beta < 1$, $0 \leqslant x < 1$, $0 \leqslant y$ and

$$\mathfrak{Re} \left\{ \frac{g(z)}{z g'(z)} \right\} \geqslant \frac{x}{p} \ \text{in} \ |z| < 1, \tag{2.29}$$

$$\left| \mathfrak{Im} \left\{ \frac{g(z)}{z g'(z)} \right\} \right| \leqslant y \ \text{in} \ |z| < 1. \tag{2.30}$$

If

$$\left| \arg \left\{ (1 - \alpha) \frac{f(z)}{g(z)} + \alpha \frac{f'(z)}{g'(z)} - \beta \right\} \right| \leqslant \frac{\pi}{2} \varphi + \tan^{-1} \frac{\alpha \varphi k x}{(\alpha \varphi k y + 1) p} \ \text{in} \ |z| < 1, \tag{2.31}$$

then we have

$$\left| \arg \left\{ \frac{f(z)}{g(z)} - \beta \right\} \right| < \frac{\pi}{2} \varphi \ \text{in} \ |z| < 1. \tag{2.32}$$

Proof. Putting

$$q(z) = \frac{1}{1 - \beta} \left(\frac{f(z)}{g(z)} - \beta \right) = 1 + c_k z^k + \cdots, \quad \lambda(z) = \frac{g(z)}{z g'(z)},$$

and applying the same method as in the proof of Theorem 2.9, we have

$$(1 - \alpha) \frac{f(z)}{g(z)} + \alpha \frac{f'(z)}{g'(z)} - \beta = (1 - \beta)(q(z) + \alpha \lambda(z) z q'(z)).$$

Then, if there exists a point z_0, $|z_0| < 1$, such that

$$|\arg\{q(z)\}| < \frac{\pi \varphi}{2} \ \text{for} \ |z| < |z_0|$$

and

$$|\arg\{q(z_0)\}| = \frac{\pi \varphi}{2},$$

then from Lemma 1.2, we have

$$\frac{z_0 q'(z_0)}{q(z_0)} = i\varphi m \ \text{ where } \ m \geqslant k.$$

For the case $\arg\{q(z_0)\} = \pi\varphi/2$, it follows that

$$\arg\left\{(1-\alpha)\frac{f(z_0)}{g(z_0)} + \alpha\frac{f'(z_0)}{g'(z_0)} - \beta\right\}$$
$$= \arg\left\{(1-\beta)(q(z_0) + \alpha\lambda(z_0)z_0 q'(z_0))\right\}$$
$$\geqslant \arg\left\{(q(z_0) + \alpha\lambda(z_0)z_0 q'(z_0))\right\}$$
$$= \arg\left\{q(z_0)\right\} + \arg\left\{(1 + \alpha\lambda(z_0)\frac{z_0 q'(z_0)}{q(z_0)})\right\}$$
$$= \frac{\pi}{2}\varphi + \arg\left\{(1 + \alpha\lambda(z_0)i\varphi m\right\}$$
$$> \frac{\pi}{2}\varphi + \tan^{-1}\frac{\alpha\varphi kx}{(\alpha\varphi ky+1)p}.$$

This contradicts hypothesis (2.31) and for the case $\arg\{q(z_0)\} = -\pi\varphi/2$ applying the same method as the above one, we have

$$\arg\left\{(1-\alpha)\frac{f(z_0)}{g(z_0)} + \alpha\frac{f'(z_0)}{g'(z_0)} - \beta\right\} < -\left(\frac{\pi}{2}\varphi + \tan^{-1}\frac{\alpha\varphi kx}{(\alpha\varphi ky+1)p}\right),$$

which contradicts hypothesis (2.31) too. This completes the proof. □

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Mamoru Nunokawa
University of Gunma
Hoshikuki-cho 798-8, Chuou-Ward, Chiba, 260-0808, Japan
e-mail: `mamoru_nuno@doctor.nifty.jp`

Janusz Sokół
Department of Mathematics
Rzeszów University of Technology
Al. Powstańców Warszawy 12, 35-959 Rzeszów, Poland
e-mail: `jsokol@prz.edu.pl`