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INEQUALITIES FOR THE GENERALIZED TRIGONOMETRIC, HYPERBOLIC AND JACOBIAN ELLIPTIC FUNCTIONS

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Abstract. This paper deals with the inequalities for the generalized trigonometric, hyperbolic and the Jacobian elliptic functions. These families of higher transcendental functions are of great importance in the studies of some problems that arose in the theory of differential equations. Among the main results established in this paper the Wilker- and Huygens- type inequalities for the functions under discussion are obtained.

1. Introduction

In the past several years many researchers dealt with the one-parameter generalization of trigonometric functions often called the p-trigonometric functions. The latter have been introduced in 1879 by E. Lundberg and studied systematically by P. Lindqvist in [17]. For more details regarding this family of functions see [3, 16, 32, 33, 34] and the references therein. Importance of this class of functions was justified by the fact that they play a crucial role in some problems that arise in theory of differential equations. A problem which stimulated an interest in a two-parameter generalization of the p-trigonometric functions was discussed by P. Drábek and R. Manásevich in [11]. In this paper the authors gave a solution to the following problem with the Dirichlet boundary conditions. For $T,\lambda > 0$ and $p,q > 1$ the problem in question is formulated as follows

$$(\Phi_p(u'(t)))' + \lambda\Phi_q(u(t)) = 0, \qquad t \in (0,T),$$
$$u(0) = u(T) = 0,$$

where $\Phi_p(s) = |s|^{m-2}s$ if $s \neq 0$ and $\Phi_p(0) = 0$. They found that a complete solution to this problem involves the (p,q)-trigonometric function, namely the (p,q)-sine function which is denoted in the sequel by $\sin_{p,q}$. Its definition is recalled in Section 2.

Another problem, which also has its origin in theory of differential equations, is the following bifurcation problem (see [41]):

$$(\Phi_p(u'(t)))' + \lambda\Phi_q(u(t))(1-|u(t)|^q) = 0, \qquad t \in (0,T), \qquad p,q > 1,$$
$$u(0) = u(T) = 0,$$

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where Φ_p is defined above. It is known that its solution is expressed in terms of the generalized Jacobian elliptic sine function whose definition is given in the next section of this paper.

One of the main mathematical tools used in this paper is the (p,q)-version of the Schwab-Borchardt mean. Its classical version, denoted by $SB(x,y) \equiv SB$, where $x > 0$ and $y > 0$, is defined as follows

$$SB(x,y) = \begin{cases} \dfrac{\sqrt{y^2-x^2}}{\cos^{-1}(x/y)}, & x < y \\ \dfrac{\sqrt{x^2-y^2}}{\cosh^{-1}(x/y)}, & y < x \\ x, & x = y \end{cases} \tag{1.1}$$

(see [5, Thm. 8.4], [9, (2.3)]). It follows from (1.1) that $SB(x,y)$ is not symmetric in its arguments and is a homogeneous function of degree 1 in x and y. This mean has been studied extensively in [5, 9, 35, 36, 21].

This paper, which can be regarded as continuation of our earlier research whose outcome is contained in [30, 31, 32, 33, 34], is organized as follows. Definitions of three families of the generalized trigonometric, hyperbolic and Jacobian elliptic functions, often called in the sequel the (p,q)-functions, are given in Section 2. In this paper we will also utilize the R-hypergeometric functions of two variables. Their definition and some basic properties are given in Section 3. The two parameter generalization of the SB mean, denoted here by $SB_{p,q}$ is introduced in Section 4. Therein some basic properties of the new mean are discussed. Three inequalities for the mean $SB_{p,q}$ are established in the next section. The main results of this paper are presented in Section 6. Therein, among other things, the Wilker- and Huygens- type inequalities involving the (p,q)-trigonometric, hyperbolic and Jacobian elliptic functions are obtained. Other functional inequalities for functions discussed in this paper are also derived in this section.

2. The (p,q)-trigonometric, hyperbolic and Jacobian elliptic functions

For the reader's convenience we recall first definition of the celebrated Gauss hypergeometric function $F(\alpha,\beta;\gamma;z)$:

$$F(\alpha,\beta;\gamma;z) = \sum_{n=0}^{\infty} \frac{(\alpha,n)(\beta,n)}{(\gamma,n)} \frac{z^n}{n!}, \qquad |z| < 1,$$

where $(\alpha,n) = \alpha(\alpha+1)\dots(\alpha+n-1)$ $(n \neq 0)$ is the shifted factorial or Appell symbol, with $(\alpha,0) = 1$ if $\alpha \neq 0$, and $\gamma \neq 0,-1,-2,\dots$.

In what follows we will always assume that the parameters p and q are strictly greater than 1. We will adopt notation and definitions used in [3]. Let

$$\pi_{p,q} = 2\int_0^1 (1-t^q)^{-1/p} dt = \frac{2}{q} B\Big(1-\frac{1}{p}, \frac{1}{q}\Big).$$

Further, let

$$m_{p,q} = 2^{-1/p} F\left(1, \frac{1}{p}; 1 + \frac{1}{q}; \frac{1}{2}\right).$$

Also, let $I = (0,1)$. The generalized trigonometric and hyperbolic functions needed in this paper are the following homeomorphisms

$$\sin_{p,q} : (0, \pi_{p,q}/2) \to I, \qquad \cos_{p,q} : (0, \pi_{p,q}/2) \to I,$$

and

$$\sinh_{p,q} : (0, m_{p,q}) \to I.$$

The inverse functions of $\sin_{p,q}$ and $\sinh_{p,q}$ can be represented as follows [16]:

$$\sin_{p,q}^{-1} x = \int_0^x (1 - t^q)^{-1/p} dt = xF\left(\frac{1}{p}, \frac{1}{q}; 1 + \frac{1}{q}; x^p\right) \tag{2.1}$$

$$\sinh_{p,q}^{-1} x = \int_0^x (1 + t^q)^{-1/p} dt = xF\left(\frac{1}{p}, \frac{1}{q}; 1 + \frac{1}{q}; -x^p\right). \tag{2.2}$$

For later use let us record some known definitions and formulas. Let us begin with the definition of (p,q)-version of the function cosine. We follow [12] to define

$$\cos_{p,q} x = \frac{d}{dx} \sin_{p,q} x, \qquad x \in \mathbb{R}. \tag{2.3}$$

This in conjunction with (2.1) yields

$$|\cos_{p,q} x|^p + |\sin_{p,q} x|^q = 1, \qquad x \in \mathbb{R}. \tag{2.4}$$

Also,

$$\cos_{p,q}^{-1} y = \sin_{p,q}^{-1}\left((1 - y^p)^{1/q}\right), \qquad y \in I. \tag{2.5}$$

Corresponding formulas for the (p,q)-hyperbolic functions read as follows [3, 12]:

$$\cosh_{p,q} x = \frac{d}{dx} \sinh_{p,q} x, \qquad x \in \mathbb{R}, \tag{2.6}$$

$$|\cosh_{p,q} x|^p - |\sinh_{p,q} x|^q = 1, \qquad x \in \mathbb{R} \tag{2.7}$$

and

$$\cosh_{p,q}^{-1} y = \sinh_{p,q}^{-1}\left((y^p - 1)^{1/q}\right), \qquad y \geqslant 1. \tag{2.8}$$

We shall also utilize functions

$$\tan_{p,q} x = \frac{\sin_{p,q} x}{\cos_{p,q} x}, \tag{2.9}$$

and

$$\tanh_{p,q} x = \frac{\sinh_{p,q} x}{\cosh_{p,q} x} \tag{2.10}$$

(see, e.g., [3, 12]).

It is obvious that the functions under discussion become classical trigonometric and hyperbolic functions when $p = q = 2$. The (p,q)- trigonometric and hyperbolic functions, have been first studied by P. Drábek and R. Manásevich (see [11]). For more details concerning properties of functions under discussion and the inequalities involving these functions the interested reader is referred to [1, 2, 3, 4, 12, 14, 15, 16, 17, 30, 41, 42].

It is worth mentioning that the $(2,4)$-trigonometric and hyperbolic functions also appear in mathematical literature where they are called Gaussian lemniscate functions. The last two families of higher transcendental functions have been studied extensively in [5, 9, 19, 25, 26, 28].

For the sake of completeness of presentation and reader's convenience as well, we recall now some formulas involving generalized Jacobian elliptic functions. Let $k, x \in [0,1]$. Parameter k is called the modulus. The inverse function of the (p,q)-Jacobian elliptic sine function $sn_{p,q}$ is defined as follows

$$sn_{p,q}^{-1}(x,k) \equiv sn_{p,q}^{-1}(x) = \int_0^x [(1-t^q)(1-k^q t^q)]^{-1/p} dt \tag{2.11}$$

(see [41, 42]). Also, let

$$K_{p,q}(k) \equiv K_{p,q} = sn_{p,q}^{-1}(1,k). \tag{2.12}$$

Function $sn_{p,q}^{-1} : [0,1] \to [0, K_{p,q}]$ is strictly increasing on the stated domain. Three other functions subordinated to $sn_{p,q}$ are defined as follows [41, 42]:

$$cn_{p,q}(x) = (1 - sn_{p,q}^q(x))^{1/p},$$

$$dn_{p,q}(x) = (1 - k^q sn_{p,q}^q(x))^{1/p}$$

and

$$sc_{p,q}(x) = \frac{sn_{p,q}(x)}{cn_{p,q}(x)},$$

where $x \in [0, K_{p,q}]$. It is known that

$$\frac{d}{dx} sn_{p,q}(x) = cn_{p,q}(x) dn_{p,q}(x). \tag{2.13}$$

Also, we will deal with the amplitude function $am_{p,q}$ where

$$am_{p,q}(x,k) \equiv am_{p,q}(x) := \sin_{p,q}^{-1}(sn_{p,q}(x)).$$

This yields

$$sn_{p,q}(x) = \sin_{p,q}(am_{p,q}(x)). \tag{2.14}$$

Clearly $am_{p,q} : [0, K_{p,q}] \to [0, \pi_{p,q}/2]$. Making use of formula (2.14) we obtain

$$sc_{p,q}(x) = \tan_{p,q}(am_{p,q}(x)). \tag{2.15}$$

3. The R-hypergeometric functions of two variables

In this section we give the definition of the bivariate R-hypergeometric functions which are used in the sequel. Some results for these functions are also included here.

In what follows the symbol $\mathbb{R}_+$ will stand for the set of positive numbers. Let $b=(b_1,b_2)\in\mathbb{R}^2_+$. By μ_b, where

$$\mu_b(t)=\frac{\Gamma(b_1+b_2)}{\Gamma(b_1)\Gamma(b_2)}t^{b_1-1}(1-t)^{b_2-1}$$

we will denote the Dirichlet measure on the interval $[0,1]$. It is well-known that μ_b is the probability measure on its domain.

Also, let $X=(x,y)\in\mathbb{R}^2_+$. In [8, 10] the R-hypergeometric function $R_\alpha(b;X)$ ($\alpha\in\mathbb{R}$) is defined as follows

$$R_\alpha(b;X)=\int_0^1 (u\cdot X)^\alpha \mu_b(t)dt, \tag{3.1}$$

where $u=(t,1-t)$ and $u\cdot X=tx+(1-t)y$ is the dot product of u and X. Many of the important special functions, including Gauss' hypergeometric function F and some elliptic integrals admit the integral representation (3.1). For more details, the interested reader is referred to Carlson's monograph [10].

A nice feature of the R-hypergeometric function is its permutation symmetry in both parameters and variables, i.e.,

$$R_\alpha(b_1,b_2;x,y)=R_\alpha(b_2,b_1;y,x). \tag{3.2}$$

Another remarkable property of R_α is homogeneity of degree α in its variables:

$$R_\alpha(b_1,b_2;\gamma x,\gamma y)=\gamma^\alpha R_\alpha(b_1,b_2;x,y) \tag{3.3}$$

($\gamma>0$).

For the later use, let us also record Carlson's inequality [7, Thm. 3]

$$\Big[R_\alpha(b;X)\Big]^{1/\alpha}\leqslant\Big[R_\beta(b;X)\Big]^{1/\beta} \tag{3.4}$$

($\alpha,\beta\neq 0,\ \alpha\leqslant\beta$).

We will also need the following result which appears in [36, Prop. 2.1]. Let $\alpha<0$, $b\in\mathbb{R}^2_+$, and let $X,Y\in\mathbb{R}^2_+$. Then the following inequality

$$R_\alpha\Big(b;\lambda X+(1-\lambda)Y\Big)\leqslant\Big[R_\alpha(b;X)\Big]^\lambda\Big[R_\alpha(b;Y)\Big]^{1-\lambda} \tag{3.5}$$

holds true for all $0\leqslant\lambda\leqslant 1$.

4. Definition and basic properties of the (p,q)-version of SB

Throughout the sequel we will assume that $p,q>1$ and also, that $x,y\in\mathbb{R}_+$. For the sake of presentation we recall first a formula for the mean SB in terms of the R-hypergeometric function:

$$SB(x,y)=R_{-\frac{1}{2}}\left(\frac{1}{2},1;x^2,y^2\right)^{-1}$$

(see [6, 10]).

We define the (p,q)-version of the mean SB as follows

$$SB_{p,q}(x,y)=y^{1-q/p}R_{-\frac{1}{p}}\left(\frac{1}{q},1;x^p,y^p\right)^{-q/p}. \tag{4.1}$$

The right member of (4.1) is a special case of what is called in mathematical literature the R-hypergeometric mean (see [7, 9, 10]). Using elementary properties of the R-hypergeometric functions we see that $SB_{p,q}(x,y)$ is the mean value of x and y. Moreover, this mean is nonsymmetric and homogeneous of degree 1 in its variables. The well known results on the R-hypergeometric means lead to the conclusion that $SB_{p,q}$ is strictly increases with and increase in p and/or in q.

For the brevity of notation let us introduce a particular R-hypergeometric function

$$R_T(x,y)=R_{-\frac{1}{p}}\left(\frac{1}{q},1;x,y\right). \tag{4.2}$$

Clearly function R_T is nonsymmetric and homogeneous of degree $-1/p$ in its variables. Comparison with (4.1) yields

$$SB_{p,q}(x,y)=y^{1-q/p}R_T(x^p,y^p)^{-q/p}. \tag{4.3}$$

We shall demonstrate now that $SB_{p,q}$ can also be expressed in terms of $\cos^{-1}_{p,q}$ and $\cosh^{-1}_{p,q}$:

$$SB_{p,q}(x,y)=\begin{cases}\dfrac{(y^p-x^p)^{1/p}}{[\cos^{-1}_{p,q}(x/y)]^{q/p}}, & x<y\\[2ex] \dfrac{(x^p-y^p)^{1/p}}{[\cosh^{-1}_{p,q}(x/y)]^{q/p}}, & y<x\\[2ex] x, & x=y.\end{cases} \tag{4.4}$$

For the proof of the first part of (4.4) we let $u=x/y$. Also, we record a formula which shows that the Gauss hypergeometric function F can be expressed in terms of of the bivariate R-hypergeometric function:

$$F(\alpha,\beta;\gamma;z)=R_{-\alpha}(\beta,\gamma-\beta;1-z,1)$$

(see, e.g., [10, (5.9-12)]). Application of the last formula to (2.1) yields

$$\sin^{-1}_{p,q}u=uR_T(1-u^q,1),$$

where $0 < u < 1$. This in conjunction with (2.5) gives

$$\cos_{p,q}^{-1} u = (1-u^p)^{1/q} R_T(u^p, 1).$$

Letting above $u = x/y$ and utilizing homogeneity of the function R_T we obtain

$$\cos_{p,q}^{-1}(x/y) = y^{1-p/q}(y^p - x^p)^{1/q} R_T(x^p, y^p).$$

Raising both sides to the power of $-q/p$ and applying (4.3) we obtain

$$[\cos_{p,q}^{-1}(x/y)]^{-q/p} = (y^p - x^p)^{-1/p} SB_{p,q}(x,y).$$

This completes proof of the first part of (4.4). The second part can be established in an analogous manner. A key formula needed here reads as follows

$$\cosh_{p,q}^{-1} u = (u^p - 1)^{1/q} R_T(u^p, 1), \tag{4.5}$$

$u > 1$. We omit further details.

Function R_T admits an integral representation:

$$R_T(x,y) = \frac{1}{B(\frac{1}{p}, 1 + \frac{1}{q} - \frac{1}{p})} \int_0^\infty t^{1/q-1/p}(t+x)^{-1/q}(t+y)^{-1} dt, \tag{4.6}$$

where B stands for the beta function. This follows from the known result [39, 19.16.9]

$$R_{-\alpha}(\beta_1, \beta_2; x, y) = \frac{1}{B(\alpha, \alpha')} \int_0^\infty t^{\alpha'-1}(t+x)^{-\beta_1}(t+y)^{-\beta_2} dt,$$

where $\alpha' = \beta_1 + \beta_2 - \alpha$. Letting $\alpha = 1/p$, $\beta_1 = 1/q$ and $\beta_2 = 1$ we obtain formula (4.6).

5. Inequalities involving $SB_{p,q}$

This section deals with inequalities involving the $SB_{p,q}$ mean. Our first result reads as follows.

THEOREM 5.1. *Let $p, q > 1$. Then the two-sided inequality*

$$x^{\frac{q}{p(q+1)}} y^{1-\frac{q}{p(q+1)}} < SB_{p,q}(x,y) < y^{1-q/p}\left(\frac{x^p + qy^p}{q+1}\right)^{q/p^2}. \tag{5.1}$$

holds true for all positive and unequal numbers x and y.

Proof. The following lower and upper bounds for the bivariate R-hypergeometric function $R_{-\alpha}(b_1, b_2; u, v)$ can be found in [8, (2.11), (2.15)]

$$(w_1 u + w_2 v)^{-\alpha} < R_{-\alpha}(b_1, b_2; u, v) < \left(u^{w_1} v^{w_2}\right)^{-\alpha}, \tag{5.2}$$

where $0 < \alpha < c := b_1 + b_2$, $w_1 = b_1/c$ and $w_2 = b_2/c$. Letting in (5.2) $\alpha = 1/p$, $b_1 = 1/q$, $b_2 = 1$, $u = x^p$ and $v = y^p$ and next raising each member of the resulting inequality to the power of $-q/p$ we obtain

$$x^{\frac{q}{p(q+1)}} y^{\frac{q^2}{p(q+1)}} < R_{-1/p}(1/q, 1; x^p, y^p)^{-q/p} < \left(\frac{x^p + qy^p}{q+1}\right)^{q/p^2}.$$

Multiplying each member of the last two-sided inequality by $y^{1-q/p}$ and next applying formula (4.1) we obtain the asserted result. □

THEOREM 5.2. *Let the positive numbers x and y be such that $x > y$. Then*

$$SB_{p,q}(x,y) < SB_{p,q}(y,x). \tag{5.3}$$

Proof. We shall prove the assertion using integral formula (4.6) and formula (4.3). Let $u > 1$ and let $t > 0$. Then $u^q > 1$ and

$$(t+u^q)^{1-1/q} > (t+1)^{1-1/q}$$

or what is the same that

$$t^{1/q-1/p}(t+u^q)^{-1/q}(t+1)^{-1} > t^{1/q-1/p}(t+u^q)^{-1}(t+1)^{-1/q}$$

because $1 - 1/q > 0$. Integration yields

$$\begin{aligned} &\frac{1}{B(\alpha,\alpha')}\int_0^\infty t^{1/q-1/p}(t+u^q)^{-1/q}(t+1)^{-1}dt \\ &> \frac{1}{B(\alpha,\alpha')}\int_0^\infty t^{1/q-1/p}(t+u^q)^{-1}(t+1)^{-1/q}dt, \end{aligned}$$

where $\alpha = 1/p$ and $\alpha' = 1 + 1/q - 1/p$. Using (4.6) we see that the last inequality can be written in the form

$$R_T(u^q, 1) > R_T(1, u^q).$$

Raising both sides to the power of $-q/p$ and next applying formula (4.3) we obtain

$$SB_{p,q}(u,1) < SB_{p,q}(1,u).$$

Letting $u = x/y$ and next utilizing homogeneity of $SB_{p,q}$ we obtain the desired result. □

Another inequality for the $SB_{p,q}$ mean is contained in the following.

THEOREM 5.3. *Let $x_1, x_2, y_1, y_2 > 0$. Then*

$$SB_{p,q}(x_1,y_1)SB_{p,q}(x_2,y_2) \leqslant SB_{p,q}^2\Big[A_p(x_1,x_2), A_p(y_1,y_2)\Big]. \tag{5.4}$$

Proof. First we make the following substitutions in (3.5)

$$\alpha = -\frac{1}{p}, \quad b = \left(\frac{1}{q}, 1\right), \quad X = (x_1^p, y_1^p), \quad Y = (x_2^p, y_2^p), \quad \lambda = \frac{1}{2}$$

next we use (4.2), and finally we raise both sides of the resulting inequality to the power of -2. This gives

$$R_T(x_1^p, y_1^p)^{-1} R_T(x_2^p, y_2^p)^{-1} \leqslant \left[R_T(A_p^p(x_1, x_2), A_p^p(y_1, y_2))^{-1}\right]^2.$$

Raising both sides to the power of p/q and next multiplying both sides of the resulting inequality by $y^{2(1-q/p)}$ and using (4.3) we obtain the desired inequality (5.4). □

6. Main results

In this section we shall prove, among other things, the Wilker-type and Huygens-type inequalities which involve the (p,q)-trigonometric, hyperbolic and Jacobian elliptic functions.

In the sequel we will utilize the following result [24, 30]:

THEOREM A. *Let u, v, λ, μ be positive numbers. Assume that u and v satisfy the separation condition*

$$u < 1 < v. \tag{6.1}$$

Then the inequality

$$1 < \frac{\lambda}{\lambda+\mu} u^r + \frac{\mu}{\lambda+\mu} v^s \tag{6.2}$$

holds true if either

$$1 < u^\gamma v^\delta, \qquad s > 0 \qquad \text{and} \qquad r\lambda \leqslant s\mu\gamma/\delta \tag{6.3}$$

or if

$$u^\gamma v^\delta < 1, \qquad s < 0 \qquad \text{and} \qquad r\lambda \leqslant s\mu\gamma/\delta, \tag{6.4}$$

for some $\gamma, \delta \geqslant 0$ with $\gamma + \delta = 1$. If u and v satisfy the separation condition (6.1) together with

$$1 < \gamma\frac{1}{u} + \delta\frac{1}{v}, \tag{6.5}$$

then the inequality (6.2) is also valid if

$$r \leqslant s \leqslant -1 \qquad \text{and} \qquad \mu\gamma \leqslant \lambda\delta. \tag{6.6}$$

We are in a position to prove the following.

THEOREM 6.1. *Let $t \in (0, \pi_{p,q}/2)$. Then*

$$(\cos_{p,q} t)^{\frac{1}{q+1}} < \frac{\sin_{p,q} t}{t} < \left(\frac{q + \cos_{p,q}^p t}{q+1}\right)^{\frac{1}{p}}. \tag{6.7}$$

Proof. First we let $x = \cos_{p,q} t$ and $y = 1$ in (4.4) to obtain

$$SB_{p,q}(\cos_{p,q} t, 1) = \left(\frac{\sin_{p,q} t}{t}\right)^{\frac{q}{p}} \tag{6.8}$$

and next apply the two-sided inequality (5.1) to obtain

$$(\cos_{p,q} t)^{\frac{q}{p(q+1)}} < \left(\frac{\sin_{p,q} t}{t}\right)^{\frac{q}{p}} < \left(\frac{q + \cos_{p,q}^{p} t}{q+1}\right)^{\frac{q}{p^2}}.$$

Hence the desired inequality (6.8) follows. □

A special case of the left inequality in (6.7) when $p = q = 2$ is commonly referred to in mathematical literature as the Adamović-Mitrinovć inequality (see [18]). When $p = 2$ and $q = 4$, then the left inequality in (6.8) appears in [19] while the case when $p = q > 1$ is obtained in [16].

A result, similar to (6.7), can be obtained for the (p,q)-hyperbolic functions

$$(\cosh_{p,q} t)^{\frac{1}{q+1}} < \frac{\sinh_{p,q} t}{t} < \left(\frac{q + \cosh_{p,q}^{p} t}{q+1}\right)^{\frac{1}{p}}, \qquad t > 0. \tag{6.9}$$

We omit further details.

The Wilker and Huygens-type inequality for the (p,q)-trigonometric functions reads as follows.

THEOREM 6.2. *Let $t \in (0, \pi_{p,q}/2)$ let $\lambda, \mu, s > 0$ and let $r\lambda \leqslant s\mu q$. Then the following inequality*

$$1 < \frac{\lambda}{\lambda+\mu}\left(\frac{\sin_{p,q} t}{t}\right)^{r} + \frac{\mu}{\lambda+\mu}\left(\frac{\tan_{p,q} t}{t}\right)^{s} \tag{6.10}$$

holds true for all $p, q > 1$.

Proof. We shall prove this result utilizing Theorem A with

$$u = \frac{\sin_{p,q} t}{t} \qquad \text{and} \qquad v = \frac{\tan_{p,q} t}{t}. \tag{6.11}$$

To show that u and v satisfy the separation condition (6.1), i.e., that $u < 1 < v$ we use (6.8)

$$SB_{p,q}(\cos_{p,q} t, 1) = \left(\frac{\sin_{p,q} t}{t}\right)^{\frac{q}{p}} < 1$$

where the last inequality is a consequence of the fact that $SB_{p,q}(\cos_{p,q} t, 1) < 1$. Thus we have shown that $u < 1$. Now we utilize the left hand side of (6.7) and write it as

$$(\cos_{p,q} t)^{\frac{-q}{q+1}} < \frac{\tan_{p,q} t}{t}.$$

Taking into account that $\cos_{p,q} t < 1$ we see that the following inequality $1 < (\cos_{p,q} t)^{\frac{-q}{q+1}}$ is valid. This in turn yields $1 < \dfrac{\tan_{p,q} t}{t}$ or what is the same that $1 < v$. We shall show now that the first part of (6.3), i.e., $1 < u^{\gamma} v^{\delta}$ is satisfied with $\gamma = q/(q+1)$ and $\delta = 1/(q+1)$. To this aim we write the left hand side of (6.7) as

$$1 < \left(\frac{\sin_{p,q} t}{t}\right)^{\gamma} \left(\frac{\tan_{p,q} t}{t}\right)^{\delta}$$

or what is the same that

$$1 < u^{\gamma} v^{\delta}. \tag{6.12}$$

Application of Theorem A yields the assertion. □

A similar result is valid for the (p,q)-hyperbolic functions. We omit further details.

COROLLARY 6.3. *Let $p,q > 1$ and let $t \in (0, \pi_{p,q}/2)$. Then*

$$2 < \left(\frac{\sin_{p,q} t}{t}\right)^{q} + \frac{\tan_{p,q} t}{t} \tag{6.13}$$

and

$$q + 1 < q\frac{\sin_{p,q} t}{t} + \frac{\tan_{p,q} t}{t}. \tag{6.14}$$

Proof. Inequality (6.13) follows immediately from Theorem 6.2. For, put $\lambda = \mu = s = 1$ and $r = q$. Similarly, (6.14) follows from the same theorem with $\lambda = q$ and $r = s = \mu = 1$. □

When $p = q = 2$ inequality (6.13) becomes the Wilker inequality (see [43]), while (6.14) is called the Huygens inequality (see [13]) for circular functions. The Wilker and Huygens inequalities for hyperbolic functions are also published in mathematical literature. There is a large number of research papers devoted to the study of these inequalities. An interested reader is referred to [20, 24, 37, 44, 45, 46, 47, 48] and the references therein. Generalizations of these results when $p = q > 1$ have been discussed recently by several researchers. For details see [16, 32, 34].

COROLLARY 6.4. *Under the assumptions of Corollary 6.3 the following inequality*

$$\left(\frac{t}{\sin_{p,q} t}\right)^{q} + \frac{t}{\tan_{p,q} t} < \left(\frac{\sin_{p,q} t}{t}\right)^{q} + \frac{\tan_{p,q} t}{t} \tag{6.15}$$

holds true.

Proof. The desired result can be obtained by using the following observation made in [37]: if positive numbers a and b satisfy the condition $1 < ab$, then

$$\frac{1}{a} + \frac{1}{b} < a + b.$$

We let $a = u^q$ and $b = v$. Making use of (6.12) we see that $1 < ab$. The assertion now follows. $\square$

Our next result reads as follows.

THEOREM 6.5. *Let $x \in (0,1)$. Then*

$$1 < \frac{\lambda}{\lambda+\mu}\Big(\frac{\sinh_{p,q}^{-1} x}{x}\Big)^r + \frac{\mu}{\lambda+\mu}\Big(\frac{\sin_{p,q}^{-1} x}{x}\Big)^s \tag{6.16}$$

provided

$$\lambda, \mu, s > 0 \qquad \textit{and} \qquad r\lambda \leqslant s\mu. \tag{6.17}$$

Proof. For the brevity of notation let

$$u = \frac{\sinh_{p,q}^{-1}}{x} \qquad \text{and} \qquad v = \frac{\sin_{p,q}^{-1} x}{x}.$$

We shall demonstrate first that u and v satisfy the separation condition (6.1). To this aim we utilize the $SB_{p,q}$ mean twice to obtain

$$SB_{p,q}((1-x^q)^{1/p},1) = \frac{x^{q/p}}{[\cos_{p,q}^{-1}((1-x^q)^{1/p})]^{q/p}} = \Big(\frac{x}{\sin_{p,q}^{-1} x}\Big)^{q/p} = \Big(\frac{1}{v}\Big)^{q/p} < 1$$

and

$$SB_{p,q}((1+x^q)^{1/p},1) = \frac{x^{q/p}}{[\cosh_{p,q}^{-1}((1+x^q)^{1/p})]^{q/p}} = \Big(\frac{x}{\sinh_{p,q}^{-1} x}\Big)^{q/p} = \Big(\frac{1}{u}\Big)^{q/p} > 1.$$

This yields the condition (6.1). We shall prove now that $1 < uv$, i.e., that the first inequality in (6.3) is satisfied with $\gamma = \delta = 1/2$. To this aim we shall employ Theorem 5.3 with $x_1 = (1+x^q)^{1/p}$, $x_2 = (1-x^q)^{1/p}$ and $y_1 = y_2 = 1$. This yields $A_p(x_1,x_2) = A_p(y_1,y_2) = 1$. Thus

$$SB_{p,q}\Big[A_p(x_1,x_2), A_p(y_1,y_2)\Big] = SB_{p,q}(1,1) = 1.$$

It follows from the above computations that

$$SB_{p,q}(x_1,y_1)SB_{p,q}(x_2,y_2) = \Big(\frac{1}{uv}\Big)^{q/p}.$$

Making use of Theorem 5.3 we obtain $1 < uv$. Application of Theorem A completes the proof. $\square$

We shall establish now the following

THEOREM 6.6. *Let* $x \in (0,1)$. *Then*

$$\frac{\sin_{p,q} x}{x} > \frac{x}{\sin_{p,q}^{-1} x} \tag{6.18}$$

and

$$\frac{\sinh_{p,q} x}{x} > \frac{x}{\sinh_{p,q}^{-1} x}. \tag{6.19}$$

Proof. We shall prove that the inequality (6.18) is valid. Using (2.4) and the known inequality $x > \sin_{p,q} x$ we get

$$\cos_{p,q} x = (1 - \sin_{p,q}^q x)^{1/p} > (1 - x^q)^{1/p}.$$

Monotonicity of $SB_{p,q}$ in its first variable together with (6.8) and the formula

$$SB_{p,q}((1-x^q)^{1/p}, 1) = \left(\frac{x}{\sin_{p,q}^{-1} x}\right)^{q/p}$$

established in the proof of Theorem 6.5 yield

$$\left(\frac{\sin_{p,q} x}{x}\right)^{q/p} = SB_{p,q}(\cos_{p,q} x, 1) > SB_{p,q}((1-x^q)^{1/p}, 1) = \left(\frac{x}{\sin_{p,q}^{-1} x}\right)^{q/p}.$$

Inequality (6.18) now follows. We leave the proof of the inequality (6.19) to the interested reader. □

For $p = q = 2$ inequalities (6.18) and (6.19) have been obtained in [15]. For $p = q > 1$ these inequalities have been established in [16]. Their counterparts for the Jacobian elliptic functions are proven in [38].

Our next result reads as follows.

THEOREM 6.7. *Let* $x \in (0,1)$. *If* $p_1 > p_2 > 1$ *and* $q > 1$, *then*

$$\left(\frac{\sin_{p_1,q}^{-1} x}{x}\right)^{p_1} < \left(\frac{\sin_{p_2,q}^{-1} x}{x}\right)^{p_2}. \tag{6.20}$$

Also, if $a > 1$, $h > 0$ *and* $p > a + 2h$ *then*

$$(\sin_{p/(a+h),q}^{-1} x)^2 < (\sin_{p/a,q}^{-1} x)(\sin_{p/(a+2h),q}^{-1} x). \tag{6.21}$$

Proof. Inequality (6.20) is an immediate consequence of the monotonicity property of the Gauss function F. It has been noticed in [22, (4.33)] that if $a > 0$ and $c > b > 0$, then for $x \in I$ function $a \to F(a,b;c,x)^{1/a}$ is strictly increasing. This in conjunction with (2.1) gives the desired result. Inequality (6.21) follows from the Lapunov's inequality for integrals (see, e.g., [40]). For the reader's convenience we recall here this result. Let $f \in C[0,1]$ and let τ be a probability measure on $(0,1)$. If $0 < r < s < t$, then

$$\left[\int_0^1 f^s(y)\tau(y)dy\right]^{t-r} < \left[\int_0^1 f^r(y)\tau(y)dy\right]^{t-s}\left[\int_0^1 f^t(y)\tau(y)dy\right]^{s-r}. \tag{6.22}$$

Utilizing a well known integral formula for the Gauss function F (see, e.g., [39, 10]):

$$F(\alpha,\beta;\gamma;z) = \frac{1}{B(\beta,\gamma-\beta)}\int_0^1 (1-zy)^{-\alpha}y^{\beta-1}(1-y)^{\gamma-\beta-1}dy$$

we obtain using (2.1)

$$\frac{\sin_{p,q}^{-1} x}{x} = \int_0^1 f_p(y)\tau(y)dy, \tag{6.23}$$

where

$$f_p(y) = (1-x^q y)^{-1/p} \qquad \text{and} \qquad \tau(y) = \frac{1}{q}y^{1/q-1}.$$

To complete the proof we utilize (6.23) three times with $p := p/a, p := p/(a+h)$ and $p := p/(a+2h)$ and next apply inequality (6.22) with $r = a, s = a+h$ and $t = a+2h$ to obtain the desired result. □

By the same means one can demonstrate that the inequalities (6.20) and (6.21) are satisfied with $\sin_{p,q}^{-1}$ replaced by $\sinh_{p,q}^{-1}$. We omit further details.

We close this section with inequalities for the generalized Jacobian elliptic functions.

THEOREM 6.8. *Let $x \in (0, K_{p,q})$ let $\lambda, \mu, s > 0$ and let $r\lambda \leqslant s\mu q$. Then the following inequality*

$$1 < \frac{\lambda}{\lambda+\mu}\left(\frac{sn_{p,q}(x)}{am_{p,q}(x)}\right)^r + \frac{\mu}{\lambda+\mu}\left(\frac{sc_{p,q}(x)}{am_{p,q}(x)}\right)^s \tag{6.24}$$

holds true for all $p,q > 1$.

Proof. In the inequality (6.10) (see Theorem 6.2) we put $t = am_{p,q}(x)$. Making use of (2.14) and (2.15) we obtain the desired result. □

For special cases of the last result the interested reader is referred to [26, 27, 30, 31, 32].

To obtain more inequalities for the generalized Jacobian elliptic functions we can use inequalities established in this section (see, e.g., (6.7), (6.13), (6.14), etc.). Application of (2.14) and (2.15) will be needed to obtain new inequalities for family of transcendental functions under discussion.

We will close this section with a few inequalities involving function $sn_{p,q}$. Let $D = (0, K_{p,q})$ and let

$$h(x) = \frac{1}{sn_{p,q}(x)}, \qquad x \in D. \tag{6.25}$$

We need the following

PROPOSITION 6.9. *Function $h(x)$ is strictly log-convex on the the stated domain.*

Proof. Let $g(x) = \ln(h(x))$. Then

$$g'(x) = -\frac{sn'_{p,q}(x)}{sn_{p,q}(x)} = -\frac{cn_{p,q}(x)dn_{p,q}(x)}{sn_{p,q}(x)},$$

where the last equality follows immediately from (2.13). Repeated application of (2.13) gives

$$g''(x) = (\phi(x))^2 + (q/p)sn^{q-2}_{p,q}(x)[cn^{2-p}_{p,q}(x)dn^2_{p,q}(x) + cn^2_{p,q}(x)dn^{2-p}_{p,q}(x)],$$

where

$$\phi(x) = \frac{cn_{p,q}(x)dn_{p,q}(x)}{sn_{p,q}(x)}. \tag{6.26}$$

Taking into account that the functions $sn_{p,q}, cn_{p,q}$ and $d_{p,q}$ are strictly positive on D we have $g''(x) > 0$. The desired result now follows. □

In what follows, letter J will stand for the domain of some generic function f.
We shall employ now the following result ([40]):

THEOREM B. *Let $f : J \to \mathbb{R}_+$ be a strictly log-convex function and let $x, y, z, w \in J$. Assume that $x < z$ and $y < w$. If $x \neq y$ and $z \neq w$, then*

$$\left(\frac{f(y)}{f(x)}\right)^{1/(y-x)} < \left(\frac{f(w)}{f(z)}\right)^{1/(w-z)}. \tag{6.27}$$

In particular, if $x < y < z$, then

$$f(y)^{z-x} < f(x)^{z-y} f(z)^{y-x}. \tag{6.28}$$

We are in a position to prove the following

THEOREM 6.10. *Let the numbers $x, y, z, w \in D$ satisfy assumptions of Theorem B. Then the following inequality*

$$\left(\frac{sn_{p,q}(x)}{sn_{p,q}(y)}\right)^{1/(y-x)} < \left(\frac{sn_{p,q}(z)}{sn_{p,q}(w)}\right)^{1/(w-z)} \tag{6.29}$$

holds true. Moreover, if $0 < x < y < z < K_{p,q}$, then

$$sn_{p,q}(y)^{z-x} > sn_{p,q}(x)^{z-y} sn_{p,q}(z)^{y-x}. \tag{6.30}$$

Proof. To obtain the inequalities (6.29) and (6.30) it suffices to employ Theorem B and Proposition 6.9. □

In order to prove the last result of this section we recall the following result (see [23]):

THEOREM C. *Let $f : J \to \mathbb{R}_+$ be a continuously differentiable log-convex-function and let $x, y, t \in J$, $x \neq y$. Then*

$$(x-y)\frac{f'(y+t)}{f(y+t)} < \ln\left(\frac{f(x+t)}{f(y+t)}\right) < (x-y)\frac{f'(x+t)}{f(x+t)}. \tag{6.31}$$

The last result of this section reads as follows.

THEOREM 6.11. *Let $x, y \in D$. If $x \neq y$, then*

$$(y-x)\phi(y) < \ln\left(\frac{sn_{p,q}(y)}{sn_{p,q}(x)}\right) < (y-x)\phi(x), \tag{6.32}$$

where ϕ is defined in (6.26).

Proof. We shall utilize Theorem C with $f(x) = 1/sn_{p,q}(x)$. Then $\ln f = -\ln(sn_{p,q})$. It follows from the proof of Proposition 6.9 that $f'/f = -\phi$. Application of Theorem C yields the assertion. □

If $p = q = 2$, then the inequalities obtained in Theorems 6.10 and 6.11, reduce to those established in [29].

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