Journal of Mathematical Inequalities
Volume 9, Number 3 (2015), 727–738 doi:10.7153/jmi-09-60

# SOME INEQUALITIES FOR THE MULTIPLICATIVE SUM ZAGREB INDEX OF GRAPH OPERATIONS

MAHDIEH AZARI AND ALI IRANMANESH*

(*Communicated by A. Aglić Aljinović*)

*Abstract.* The multiplicative sum Zagreb index is defined for a simple graph $G$ as the product of the terms $d_G(u)+d_G(v)$ over all edges $uv \in E(G)$, where $d_G(u)$ denotes the degree of the vertex $u$ of $G$. In this paper, we present some lower bounds for the multiplicative sum Zagreb index of several graph operations such as union, join, corona product, composition, direct product, Cartesian product and strong product in terms of the multiplicative sum Zagreb index and the multiplicative Zagreb indices of their components.

## 1. Introduction

In theoretical Chemistry, the physico–chemical properties of chemical compounds are often modeled by means of molecular–graph–based structure–descriptors, which are also referred to as *topological indices* [12, 17]. The *Zagreb indices* are among the oldest topological indices, and were introduced as early as in 1972 [13]. These indices have since been used to study molecular complexity, chirality, ZE–isomerism and hetero–systems. For details on their theory and applications see [1, 4, 5, 6, 15, 21]. Let $G$ be a simple graph with the vertex set $V(G)$ and the edge set $E(G)$. The first and second Zagreb indices of $G$ are denoted by $M_1(G)$ and $M_2(G)$, respectively, and defined as:

$$M_1(G) = \sum_{u \in V(G)} d_G(u)^2 \quad \text{and} \quad M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v),$$

where $d_G(u)$ denotes the degree of the vertex $u$ of $G$. The first Zagreb index can also be expressed as a sum over edges of $G$:

$$M_1(G) = \sum_{\text{uv} \in E(G)} [d_G(u) + d_G(v)].$$

The multiplicative versions of Zagreb indices were introduced by Todeschini et al. in 2010 [16]. The first and second *multiplicative Zagreb indices* of $G$ are denoted by $\Pi_1(G)$ and $\Pi_2(G)$, respectively, and defined as:

$$\Pi_1(G) = \prod_{u \in V(G)} d_G(u)^2 \quad \text{and} \quad \Pi_2(G) = \prod_{uv \in E(G)} d_G(u) d_G(v).$$

*Mathematics subject classification* (2010): 05C35, 05C76, 05C07.
*Keywords and phrases*: Degree, lower bound, multiplicative sum Zagreb index, multiplicative Zagreb indices, graph operations.

* Corresponding author.

The second multiplicative Zagreb index can also be expressed as a product over vertices of $G$ [11]:

$$\Pi_2(G) = \prod_{u \in V(G)} d_G(u)^{d_G(u)}.$$

In 2012, Eliasi et al. introduced another multiplicative version of the first Zagreb index called *multiplicative sum Zagreb index* [9]. The multiplicative sum Zagreb index of $G$ is denoted by $\Pi_1^*(G)$ and defined as:

$$\Pi_1^*(G) = \prod_{uv \in E(G)} [d_G(u) + d_G(v)].$$

We refer the reader to [10, 18, 19] for mathematical properties and applications of the multiplicative Zagreb indices and multiplicative sum Zagreb index.

It is well-known that many graphs of general and in particular of chemical interest arise from simpler graphs via various *graph operations* sometimes known as *graph products*. It is, hence, important to understand how certain invariants of such graph operations are related to the corresponding invariants of their components. In [14], Khalifeh et al. presented some exact formulae for computing the Zagreb indices of some graph operations. In [7], Das et al. obtained some upper bounds for the first and second multiplicative Zagreb indices of various graph operations in terms of the Zagreb indices of their components. In this paper, we present some lower bounds for the multiplicative sum Zagreb index of several graph operations in terms of the multiplicative sum Zagreb index and the first and second multiplicative Zagreb indices of their components. Readers interested in more information on computing topological indices of graph operations can be referred to [2, 3, 8, 20].

## 2. Main results

In this section, we study the behavior of the multiplicative sum Zagreb index under several graph operations. All considered operations are binary. Hence, we will usually deal with two finite and simple graphs $G_1$ and $G_2$. For a given graph $G_i$, its vertex and edge sets will be denoted by $V(G_i)$ and $E(G_i)$, and its order and size by $n_i$ and $m_i$, respectively, where $i \in \{1,2\}$. Throughout the section, we assume that $G_1$ and $G_2$ have no isolated vertices.

At first, we recall two well-known inequalities.

LEMMA 2.1. (AM-GM inequality) *Let $x_1, x_2 \ldots, x_n$ be nonnegative numbers. Then*

$$\frac{x_1 + x_2 + \ldots + x_n}{n} \geqslant \sqrt[n]{x_1 x_2 \ldots x_n},$$

*with equality if and only if $x_1 = x_2 = \ldots = x_n$.*

LEMMA 2.2. *Let $x_1, x_2 \ldots, x_n$, $y_1, y_2, \ldots, y_n$ be positive numbers and for every $1 \leqslant i \leqslant n$, $x_i \geqslant y_i$. Then*

$$x_1 x_2 \ldots x_n \geqslant y_1 y_2 \ldots y_n,$$

*with equality if and only if for every $1 \leqslant i \leqslant n$, $x_i = y_i$.*

### 2.1. Union

The *union* $G_1 \cup G_2$ of graphs $G_1$ and $G_2$ with disjoint vertex sets $V(G_1)$ and $V(G_2)$ is a graph with the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2)$. The degree of a vertex $u$ of $G_1 \cup G_2$ is equal to the degree of $u$ in the component $G_i,\ i \in \{1,2\}$ that contains it.

In the following Theorem, an exact formula is obtained for the multiplicative sum Zagreb index of the union of $G_1$ and $G_2$. Its proof follows easily from the definition, so is omitted.

THEOREM 2.3. *The multiplicative sum Zagreb index of $G_1 \cup G_2$ is given by:*

$$\Pi_1^*(G_1 \cup G_2) = \Pi_1^*(G_1) \times \Pi_1^*(G_2).$$

It is clear from Theorem 2.3 that we can restrict our attention to connected graphs, since for a graph with several connected components its multiplicative sum Zagreb index is equal to the product of the indices of its components.

### 2.2. Join

The *join* $G_1 + G_2$ of graphs $G_1$ and $G_2$ with disjoint vertex sets $V(G_1)$ and $V(G_2)$ is a graph union $G_1 \cup G_2$ together with all the edges joining $V(G_1)$ and $V(G_2)$. The join of two graphs is also known as their *sum*. The degree of a vertex $u$ of $G_1 + G_2$ is given by:

$$d_{G_1+G_2}(u) = \begin{cases} d_{G_1}(u) + n_2 & u \in V(G_1) \\ d_{G_2}(u) + n_1 & u \in V(G_2) \end{cases}.$$

THEOREM 2.4. *The multiplicative sum Zagreb index of $G_1 + G_2$ satisfies the following inequality:*

$$\Pi_1^*(G_1+G_2) > (2\sqrt{2})^{m_1+m_2}\sqrt{{n_2}^{m_1}{n_1}^{m_2}}(3\sqrt[3]{n_1+n_2})^{n_1 n_2}\sqrt[6]{\Pi_1(G_1)^{n_2}\Pi_1(G_2)^{n_1}}$$
$$\sqrt{\Pi_1^*(G_1)\Pi_1^*(G_2)}.$$

*Proof.* Let $G = G_1 + G_2$. By definition of the multiplicative sum Zagreb index, we have:

$$\begin{aligned}
\Pi_1^*(G) &= \prod_{uv \in E(G)} \Big[d_G(u) + d_G(v)\Big] \\
&= \prod_{uv \in E(G_1)} \Big[\Big(d_{G_1}(u) + n_2\Big) + \Big(d_{G_1}(v) + n_2\Big)\Big] \\
&\quad \times \prod_{uv \in E(G_2)} \Big[\Big(d_{G_2}(u) + n_1\Big) + \Big(d_{G_2}(v) + n_1\Big)\Big] \\
&\quad \times \prod_{u \in V(G_1)} \prod_{v \in V(G_2)} \Big[\Big(d_{G_1}(u) + n_2\Big) + \Big(d_{G_2}(v) + n_1\Big)\Big]
\end{aligned}$$

$$
\begin{aligned}
&= \prod_{uv\in E(G_1)} \left[\left(d_{G_1}(u)+d_{G_1}(v)\right)+2n_2\right] \\
&\times \prod_{uv\in E(G_2)} \left[\left(d_{G_2}(u)+d_{G_2}(v)\right)+2n_1\right] \\
&\times \prod_{u\in V(G_1)} \prod_{v\in V(G_2)} \left[d_{G_1}(u)+d_{G_2}(v)+(n_1+n_2)\right].
\end{aligned}
$$

Now by Lemmas 2.1 and 2.2,

$$
\begin{aligned}
\Pi_1^*(G) > &\prod_{uv\in E(G_1)} 2\sqrt{\left(d_{G_1}(u)+d_{G_1}(v)\right)\times 2n_2} \prod_{uv\in E(G_2)} 2\sqrt{\left(d_{G_2}(u)+d_{G_2}(v)\right)\times 2n_1} \\
&\times \prod_{u\in V(G_1)} \prod_{v\in V(G_2)} 3\sqrt[3]{d_{G_1}(u)\times d_{G_2}(v)\times (n_1+n_2)} \\
=&(2\sqrt{2})^{m_1+m_2}\sqrt{{n_2}^{m_1}{n_1}^{m_2}}\,(3\sqrt[3]{n_1+n_2})^{n_1n_2} \\
&\times \sqrt[6]{\Pi_1(G_1)^{n_2}\Pi_1(G_2)^{n_1}}\sqrt{\Pi_1^*(G_1)\Pi_1^*(G_2)}.
\end{aligned}
$$

Note that the above inequality is strict. Since if the equality holds then by Lemma 2.2, for every $u\in V(G_1)$ and $v\in V(G_2)$,

$$
d_{G_1}(u)+d_{G_2}(v)+(n_1+n_2) = 3\sqrt[3]{d_{G_1}(u)\times d_{G_2}(v)\times (n_1+n_2)}.
$$

By Lemma 2.1, this implies that for every $u\in V(G_1)$ and $v\in V(G_2)$, $d_{G_1}(u) = d_{G_2}(v) = n_1+n_2$, which is a contradiction. □

For a given graph $G$, the graph $K_1+G$ is called the *suspension* of $G$, where $K_1$ denotes the single vertex graph. In the following Corollary, we obtain a lower bound for the multiplicative sum Zagreb index of the suspension of graphs. Note that in Theorem 2.4, the graphs $G_1$ and $G_2$ are assumed to have no isolated vertices.

COROLLARY 2.5. *Let $G$ be a finite simple graph of order $n$ and size $m$. The multiplicative sum Zagreb index of $K_1+G$ satisfies the following inequality:*

$$
\Pi_1^*(K_1+G) > (\sqrt{2})^{2n+3m}(\sqrt{n+1})^n\sqrt[4]{\Pi_1(G)}\sqrt{\Pi_1^*(G)}.
$$

### 2.3. Corona product

The *corona product* $G_1\circ G_2$ of graphs $G_1$ and $G_2$ is a graph obtained by taking one copy of $G_1$ and $n_1$ copies of $G_2$ and joining the $i-$th vertex of $G_1$ to every vertex in $i-$th copy of $G_2$ for $1\leqslant i\leqslant n_1$. We denote the $i-$th copy of $G_2$ by $G_{2,i}$, $1\leqslant i\leqslant n_1$. The degree of a vertex $u$ of $G_1\circ G_2$ is given by:

$$
d_{G_1\circ G_2}(u) = \begin{cases} d_{G_1}(u)+n_2 & u\in V(G_1) \\ d_{G_2}(u)+1 & u\in V(G_{2,i}) \end{cases}.
$$

In the following Theorem, a lower bound is obtained for the multiplicative sum Zagreb index of the corona product $G_1 \circ G_2$ in terms of the order, size, the first multiplicative Zagreb index and the multiplicative sum Zagreb index of $G_1$ and $G_2$. Its proof is similar to the proof of Theorem 2.4, so is omitted.

THEOREM 2.6. *The multiplicative sum Zagreb index of $G_1 \circ G_2$ satisfies the following inequality:*

$$\Pi_1^*(G_1 \circ G_2) > (2\sqrt{2})^{m_1+n_1m_2}(\sqrt{n_2})^{m_1}(3\sqrt[3]{n_2+1})^{n_1n_2}\sqrt[6]{\Pi_1(G_1)^{n_2}\Pi_1(G_2)^{n_1}} \times \sqrt{\Pi_1^*(G_1)\Pi_1^*(G_2)^{n_1}}.$$

## 2.4. Composition

The *composition* $G_1[G_2]$ of graphs $G_1$ and $G_2$ is a graph with the vertex set $V(G_1) \times V(G_2)$ and two vertices $(u_1,u_2)$ and $(v_1,v_2)$ are adjacent if and only if $u_1v_1 \in E(G_1)$ or $[u_1 = v_1$ and $u_2v_2 \in E(G_2)]$. The composition of two graphs is also known as their *lexicographic product*. The degree of a vertex $u = (u_1,u_2)$ of $G_1[G_2]$ is given by:

$$d_{G_1[G_2]}(u) = n_2 d_{G_1}(u_1) + d_{G_2}(u_2).$$

THEOREM 2.7. *The multiplicative sum Zagreb index of $G_1[G_2]$ satisfies the following inequality:*

$$\Pi_1^*(G_1[G_2]) > 3^{n_2^2m_1}(2\sqrt{2})^{n_1m_2}(\sqrt[6]{n_2})^{3n_1m_2+2n_2^2m_1}\ \sqrt[12]{\Pi_1(G_1)^{3m_2}\Pi_1(G_2)^{4m_1n_2}} \times \sqrt[6]{\Pi_1^*(G_1)^{2n_2^2}\ \Pi_1^*(G_2)^{3n_1}}.$$

*Proof.* Let $G = G_1[G_2]$. By definition of the multiplicative sum Zagreb index, we have:

$$\begin{aligned}
\Pi_1^*(G) &= \prod_{(u_1,u_2)(v_1,v_2)\in E(G)} \Big[d_G((u_1,u_2)) + d_G((v_1,v_2))\Big] \\
&= \prod_{u_1v_1\in E(G_1)} \prod_{u_2\in V(G_2)} \prod_{v_2\in V(G_2)} \Big[\Big(n_2 d_{G_1}(u_1) + d_{G_2}(u_2)\Big) + \Big(n_2 d_{G_1}(v_1) + d_{G_2}(v_2)\Big)\Big] \\
&\quad \times \prod_{u_1\in V(G_1)} \prod_{u_2v_2\in E(G_2)} \Big[\Big(n_2 d_{G_1}(u_1) + d_{G_2}(u_2)\Big) + \Big(n_2 d_{G_1}(u_1) + d_{G_2}(v_2)\Big)\Big] \\
&= \prod_{u_1v_1\in E(G_1)} \prod_{u_2\in V(G_2)} \prod_{v_2\in V(G_2)} \Big[n_2\Big(d_{G_1}(u_1) + d_{G_1}(v_1)\Big) + d_{G_2}(u_2) + d_{G_2}(v_2)\Big] \\
&\quad \times \prod_{u_1\in V(G_1)} \prod_{u_2v_2\in E(G_2)} \Big[2n_2 d_{G_1}(u_1) + \Big(d_{G_2}(u_2) + d_{G_2}(v_2)\Big)\Big].
\end{aligned}$$

Now by Lemmas 2.1 and 2.2,

$$\begin{aligned}
\Pi_1^*(G) > & \prod_{u_1v_1\in E(G_1)}\prod_{u_2\in V(G_2)}\prod_{v_2\in V(G_2)} 3\sqrt[3]{n_2\Big(d_{G_1}(u_1)+d_{G_1}(v_1)\Big)\times d_{G_2}(u_2)\times d_{G_2}(v_2)} \\
& \times \prod_{u_1\in V(G_1)}\prod_{u_2v_2\in E(G_2)} 2\sqrt{2n_2 d_{G_1}(u_1)\times\Big(d_{G_2}(u_2)+d_{G_2}(v_2)\Big)} \\
=&(3\sqrt[3]{n_2}\,)^{n_2^2m_1}\sqrt[3]{\Pi_1^*(G_1)^{n_2^2}}\sqrt[6]{\Pi_1(G_2)^{m_1n_2}}\sqrt[6]{\Pi_1(G_2)^{m_1n_2}} \\
& \times(2\sqrt{2n_2})^{n_1m_2}\sqrt[4]{\Pi_1(G_1)^{m_2}}\ \sqrt{\Pi_1^*(G_2)^{n_1}} \\
=&3^{n_2^2m_1}(2\sqrt{2})^{n_1m_2}(\sqrt[6]{n_2})^{3n_1m_2+2n_2^2m_1}\sqrt[12]{\Pi_1(G_1)^{3m_2}\Pi_1(G_2)^{4m_1n_2}} \\
& \times\sqrt[6]{\Pi_1^*(G_1)^{2n_2^2}\Pi_1^*(G_2)^{3n_1}}.
\end{aligned}$$

The above inequality is strict. Since if the equality holds then by Lemma 2.2, for every $u_1v_1\in E(G_1)$ and $u_2,v_2\in V(G_2)$,

$$\begin{aligned}
&n_2\Big(d_{G_1}(u_1)+d_{G_1}(v_1)\Big)+d_{G_2}(u_2)+d_{G_2}(v_2) \\
=&3\sqrt[3]{n_2\Big(d_{G_1}(u_1)+d_{G_1}(v_1)\Big)\times d_{G_2}(u_2)\times d_{G_2}(v_2)}.
\end{aligned}$$

By Lemma 2.1, this implies that for every $u_1v_1\in E(G_1)$ and $u_2,v_2\in V(G_2)$,

$$n_2\Big(d_{G_1}(u_1)+d_{G_1}(v_1)\Big)=d_{G_2}(u_2)=d_{G_2}(v_2).$$

So for every $u_2\in V(G_2)$, $d_{G_2}(u_2)\geqslant 2n_2$, which is a contradiction. □

## 2.5. Direct product

The *direct product* $G_1\times G_2$ of graphs $G_1$ and $G_2$ has the vertex set $V(G_1)\times V(G_2)$ and two vertices $(u_1,u_2)$ and $(v_1,v_2)$ are adjacent if and only if $u_1v_1\in E(G_1)$ and $u_2v_2\in E(G_2)$. The direct product of two graphs is also known as their *tensor product*, *Kronecker product*, *categorical product*, *cardinal product*, *relational product* or *conjunction*. The degree of a vertex $u=(u_1,u_2)$ of $G_1\times G_2$ is given by:

$$d_{G_1\times G_2}(u)=d_{G_1}(u_1)d_{G_2}(u_2).$$

THEOREM 2.8. *The multiplicative sum Zagreb index of $G_1\times G_2$ satisfies the following inequality:*

$$\Pi_1^*(G_1\times G_2)\geqslant 4^{m_1m_2}\Pi_2(G_1)^{m_2}\Pi_2(G_2)^{m_1},$$

*with equality if and only if $G_1$ and $G_2$ are regular graphs.*

*Proof.* Let $G = G_1 \times G_2$. By definition of the multiplicative sum Zagreb index, we have:

$$\Pi_1^*(G) = \prod_{(u_1,u_2)(v_1,v_2)\in E(G)} \Big[d_G((u_1,u_2)) + d_G((v_1,v_2))\Big].$$

Note that if $(u_1,u_2)(v_1,v_2) \in E(G)$, then $(u_1,v_2)(v_1,u_2) \in E(G)$. Hence,

$$\Pi_1^*(G) = \prod_{u_1v_1\in E(G_1)} \prod_{u_2v_2\in E(G_2)} \Big[\Big(d_{G_1}(u_1)d_{G_2}(u_2) + d_{G_1}(v_1)d_{G_2}(v_2)\Big) \times \Big(d_{G_1}(u_1)d_{G_2}(v_2) + d_{G_1}(v_1)d_{G_2}(u_2)\Big)\Big].$$

Now by Lemmas 2.1 and 2.2,

$$\begin{aligned}\Pi_1^*(G) &\geqslant \prod_{u_1v_1\in E(G_1)} \prod_{u_2v_2\in E(G_2)} \Big[2\sqrt{d_{G_1}(u_1)d_{G_2}(u_2) \times d_{G_1}(v_1)d_{G_2}(v_2)} \\ &\qquad \times 2\sqrt{d_{G_1}(u_1)d_{G_2}(v_2) \times d_{G_1}(v_1)d_{G_2}(u_2)}\ \Big] \\ &= \prod_{u_1v_1\in E(G_1)} \prod_{u_2v_2\in E(G_2)} 4\Big[d_{G_1}(u_1)d_{G_1}(v_1) \times d_{G_2}(u_2)d_{G_2}(v_2)\Big] \\ &= 4^{m_1m_2}\Pi_2(G_1)^{m_2}\Pi_2(G_2)^{m_1}.\end{aligned}$$

By Lemma 2.2, the above equality holds if and only if for every $u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)$,

$$d_{G_1}(u_1)d_{G_2}(u_2) + d_{G_1}(v_1)d_{G_2}(v_2) = 2\sqrt{d_{G_1}(u_1)d_{G_2}(u_2) \times d_{G_1}(v_1)d_{G_2}(v_2)},$$

and

$$d_{G_1}(u_1)d_{G_2}(v_2) + d_{G_1}(v_1)d_{G_2}(u_2) = 2\sqrt{d_{G_1}(u_1)d_{G_2}(v_2) \times d_{G_1}(v_1)d_{G_2}(u_2)}.$$

By Lemma 2.1, this implies that for every $u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)$,

$$d_{G_1}(u_1)\,d_{G_2}(u_2) = d_{G_1}(v_1)\,d_{G_2}(v_2),$$

and

$$d_{G_1}(u_1)\,d_{G_2}(v_2) = d_{G_1}(v_1)\,d_{G_2}(u_2),$$

and this clearly implies that $G_1$ and $G_2$ must be regular graphs. $\square$

### 2.6. Cartesian product

The *Cartesian product* $G_1\Box G_2$ of graphs $G_1$ and $G_2$ has the vertex set $V(G_1)\times V(G_2)$ and two vertices $(u_1,u_2)$ and $(v_1,v_2)$ are adjacent if and only if [$u_1 = v_1$ and $u_2v_2 \in E(G_2)$] or [$u_2 = v_2$ and $u_1v_1 \in E(G_1)$]. The degree of a vertex $u = (u_1,u_2)$ of $G_1\Box G_2$ is given by:

$$d_{G_1\Box G_2}(u) = d_{G_1}(u_1) + d_{G_2}(u_2).$$

THEOREM 2.9. *The multiplicative sum Zagreb index of $G_1 \square G_2$ satisfies the following inequality:*

$$\Pi_1^*(G_1 \square G_2) \geqslant (2\sqrt{2})^{n_1 m_2 + n_2 m_1} \sqrt[4]{\Pi_1(G_1)^{m_2} \Pi_1(G_2)^{m_1}} \sqrt{\Pi_1^*(G_1)^{n_2} \Pi_1^*(G_2)^{n_1}},$$

*with equality if and only if $G_1$ and $G_2$ are regular graphs with the same regularities.*

*Proof.* Let $G = G_1 \square G_2$. By definition of the multiplicative sum Zagreb index, we have:

$$\begin{aligned}
\Pi_1^*(G) &= \prod_{(u_1,u_2)(v_1,v_2)\in E(G)} \Big[ d_G((u_1,u_2)) + d_G((v_1,v_2)) \Big] \\
&= \prod_{u_1 \in V(G_1)} \prod_{u_2 v_2 \in E(G_2)} \Big[ \Big( d_{G_1}(u_1) + d_{G_2}(u_2) \Big) + \Big( d_{G_1}(u_1) + d_{G_2}(v_2) \Big) \Big] \\
&\quad \times \prod_{u_2 \in V(G_2)} \prod_{u_1 v_1 \in E(G_1)} \Big[ \Big( d_{G_1}(u_1) + d_{G_2}(u_2) \Big) + \Big( d_{G_1}(v_1) + d_{G_2}(u_2) \Big) \Big] \\
&= \prod_{u_1 \in V(G_1)} \prod_{u_2 v_2 \in E(G_2)} \Big[ 2d_{G_1}(u_1) + \Big( d_{G_2}(u_2) + d_{G_2}(v_2) \Big) \Big] \\
&\quad \times \prod_{u_2 \in V(G_2)} \prod_{u_1 v_1 \in E(G_1)} \Big[ 2d_{G_2}(u_2) + \Big( d_{G_1}(u_1) + d_{G_1}(v_1) \Big) \Big].
\end{aligned}$$

Now by Lemmas 2.1 and 2.2,

$$\begin{aligned}
\Pi_1^*(G) &\geqslant \prod_{u_1 \in V(G_1)} \prod_{u_2 v_2 \in E(G_2)} 2\sqrt{2d_{G_1}(u_1) \times \Big( d_{G_2}(u_2) + d_{G_2}(v_2) \Big)} \\
&\quad \times \prod_{u_2 \in V(G_2)} \prod_{u_1 v_1 \in E(G_1)} 2\sqrt{2d_{G_2}(u_2) \times \Big( d_{G_1}(u_1) + d_{G_1}(v_1) \Big)} \\
&= (2\sqrt{2})^{n_1 m_2} \sqrt[4]{\Pi_1(G_1)^{m_2}} \sqrt{\Pi_1^*(G_2)^{n_1}} \times (2\sqrt{2})^{n_2 m_1} \sqrt[4]{\Pi_1(G_2)^{m_1}} \sqrt{\Pi_1^*(G_1)^{n_2}} \\
&= (2\sqrt{2})^{n_1 m_2 + n_2 m_1} \sqrt[4]{\Pi_1(G_1)^{m_2} \Pi_1(G_2)^{m_1}} \sqrt{\Pi_1^*(G_1)^{n_2} \Pi_1^*(G_2)^{n_1}}.
\end{aligned}$$

By Lemma 2.2, the above equality holds if and only if for every $u_1 \in V(G_1)$ and $u_2 v_2 \in E(G_2)$,

$$2d_{G_1}(u_1) + \Big( d_{G_2}(u_2) + d_{G_2}(v_2) \Big) = 2\sqrt{2d_{G_1}(u_1) \times \Big( d_{G_2}(u_2) + d_{G_2}(v_2) \Big)},$$

and for every $u_2 \in V(G_2)$ and $u_1 v_1 \in E(G_1)$,

$$2d_{G_2}(u_2) + \Big( d_{G_1}(u_1) + d_{G_1}(v_1) \Big) = 2\sqrt{2d_{G_2}(u_2) \times \Big( d_{G_1}(u_1) + d_{G_1}(v_1) \Big)}.$$

By Lemma 2.1, this implies that for every $u_1 \in V(G_1)$ and $u_2 v_2 \in E(G_2)$, $2d_{G_1}(u_1) = d_{G_2}(u_2) + d_{G_2}(v_2)$, and for every $u_2 \in V(G_2)$ and $u_1 v_1 \in E(G_1)$, $2d_{G_2}(u_2) = d_{G_1}(u_1) + d_{G_1}(v_1)$. This clearly implies that $G_1$ and $G_2$ must be regular graphs with the same regularities. $\square$

## 2.7. Strong product

The *strong product* $G_1 \boxtimes G_2$ of graphs $G_1$ and $G_2$ has the vertex set $V(G_1) \times V(G_2)$ and two vertices $(u_1, u_2)$ and $(v_1, v_2)$ are adjacent if and only if $[u_1 = v_1$ and $u_2v_2 \in E(G_2)]$ or $[u_2 = v_2$ and $u_1v_1 \in E(G_1)]$ or $[u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)]$. The degree of a vertex $u = (u_1, u_2)$ of $G_1 \boxtimes G_2$ is given by:

$$d_{G_1 \boxtimes G_2}(u) = d_{G_1}(u_1) + d_{G_2}(u_2) + d_{G_1}(u_1)d_{G_2}(u_2).$$

THEOREM 2.10. *The multiplicative sum Zagreb index of $G_1 \boxtimes G_2$ satisfies the following inequality:*

$$\Pi_1^*(G_1 \boxtimes G_2) > 16^{m_1 m_2} \left(3\sqrt[3]{2}\right)^{n_1 m_2 + n_2 m_1} \sqrt[3]{\Pi_1(G_1)^{m_2} \Pi_1(G_2)^{m_1}} \sqrt{\Pi_2(G_1)^{m_2} \Pi_2(G_2)^{m_1}}$$
$$\sqrt[6]{\Pi_1^*(G_1)^{4n_2+3m_2} \Pi_1^*(G_2)^{4n_1+3m_1}}.$$

*Proof.* Let $G = G_1 \boxtimes G_2$. By definition of the multiplicative sum Zagreb index, we have:

$$\Pi_1^*(G) = \prod_{(u_1,u_2)(v_1,v_2) \in E(G)} \Big[d_G((u_1,u_2)) + d_G((v_1,v_2))\Big].$$

By definition of the strong product, we can partition the above product into three products as follows:

The first product $P_1$ is taken over all edges $(u_1, u_2)(v_1, v_2) \in E(G)$ such that $u_1 = v_1$ and $u_2v_2 \in E(G_2)$. The calculation of $P_1$ is as follows:

$$\begin{aligned}
P_1 = & \prod_{u_1 \in V(G_1)} \prod_{u_2v_2 \in E(G_2)} \Big[\Big(d_{G_1}(u_1) + d_{G_2}(u_2) + d_{G_1}(u_1)d_{G_2}(u_2)\Big) \\
& \qquad + \Big(d_{G_1}(u_1) + d_{G_2}(v_2) + d_{G_1}(u_1)d_{G_2}(v_2)\Big)\Big] \\
= & \prod_{u_1 \in V(G_1)} \prod_{u_2v_2 \in E(G_2)} \Big[2d_{G_1}(u_1) + \Big(d_{G_2}(u_2) + d_{G_2}(v_2)\Big) + d_{G_1}(u_1)\Big(d_{G_2}(u_2) + d_{G_2}(v_2)\Big)\Big].
\end{aligned}$$

Now by Lemmas 2.1 and 2.2,

$$\begin{aligned}
P_1 \geqslant & \prod_{u_1 \in V(G_1)} \prod_{u_2v_2 \in E(G_2)} 3\sqrt[3]{2d_{G_1}(u_1) \times \Big(d_{G_2}(u_2) + d_{G_2}(v_2)\Big) \times d_{G_1}(u_1)\Big(d_{G_2}(u_2) + d_{G_2}(v_2)\Big)} \\
= & \left(3\sqrt[3]{2}\right)^{n_1 m_2} \sqrt[3]{\Pi_1(G_1)^{m_2}} \sqrt[3]{\Pi_1^*(G_2)^{2n_1}}.
\end{aligned}$$

By Lemma 2.2, the above equality holds if and only if for every $u_1 \in V(G_1)$ and $u_2v_2 \in E(G_2)$,

$$\begin{aligned}
& 2d_{G_1}(u_1) + \Big(d_{G_2}(u_2) + d_{G_2}(v_2)\Big) + d_{G_1}(u_1)\Big(d_{G_2}(u_2) + d_{G_2}(v_2)\Big) \\
= & 3\sqrt[3]{2d_{G_1}(u_1) \times \Big(d_{G_2}(u_2) + d_{G_2}(v_2)\Big) \times d_{G_1}(u_1)\Big(d_{G_2}(u_2) + d_{G_2}(v_2)\Big)}.
\end{aligned}$$

By Lemma 2.1, this implies that for every $u_1 \in V(G_1)$ and $u_2v_2 \in E(G_2)$,

$$2d_{G_1}(u_1) = d_{G_2}(u_2) + d_{G_2}(v_2) = d_{G_1}(u_1)\Big(d_{G_2}(u_2) + d_{G_2}(v_2)\Big).$$

So the equality holds if and only if for every $u_1 \in V(G_1)$, $d_{G_1}(u_1) = 1$ and for every $u_2v_2 \in E(G_2)$, $d_{G_2}(u_2) + d_{G_2}(v_2) = 2$. That is $G_1$ and $G_2$ are 1-regular graphs.

The second product $P_2$ is taken over all edges $(u_1,u_2)(v_1,v_2) \in E(G)$ such that $u_1v_1 \in E(G_1)$ and $u_2 = v_2$. So,

$$\begin{aligned} P_2 = \prod_{u_1v_1 \in E(G_1)} \prod_{u_2 \in V(G_2)} \Big[ & \Big(d_{G_1}(u_1) + d_{G_2}(u_2) + d_{G_1}(u_1)d_{G_2}(u_2)\Big) \\ & + \Big(d_{G_1}(v_1) + d_{G_2}(u_2) + d_{G_1}(v_1)d_{G_2}(u_2)\Big)\Big]. \end{aligned}$$

By symmetry,

$$P_2 \geqslant (3\sqrt[3]{2})^{n_2m_1} \sqrt[3]{\Pi_1(G_2)^{m_1}} \sqrt[3]{\Pi_1^*(G_1)^{2n_2}},$$

with equality if and only if $G_1$ and $G_2$ are 1-regular graphs.

The third product $P_3$ is taken over all edges $(u_1,u_2)(v_1,v_2) \in E(G)$ such that $u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)$. The calculation of $P_3$ is as follows:

$$\begin{aligned} P_3 = \prod_{u_1v_1 \in E(G_1)} \prod_{u_2v_2 \in E(G_2)} & \Big[\Big(d_{G_1}(u_1) + d_{G_2}(u_2) + d_{G_1}(u_1)d_{G_2}(u_2)\Big) \\ & + \Big(d_{G_1}(v_1) + d_{G_2}(v_2) + d_{G_1}(v_1)d_{G_2}(v_2)\Big)\Big] \\ & \times \Big[\Big(d_{G_1}(u_1) + d_{G_2}(v_2) + d_{G_1}(u_1)d_{G_2}(v_2)\Big) \\ & + \Big(d_{G_1}(v_1) + d_{G_2}(u_2) + d_{G_1}(v_1)d_{G_2}(u_2)\Big)\Big] \\ = \prod_{u_1v_1 \in E(G_1)} \prod_{u_2v_2 \in E(G_2)} & \Big[\Big(d_{G_1}(u_1) + d_{G_1}(v_1)\Big) + \Big(d_{G_2}(u_2) + d_{G_2}(v_2)\Big) \\ & + d_{G_1}(u_1)d_{G_2}(u_2) + d_{G_1}(v_1)d_{G_2}(v_2)\Big] \\ & \times \Big[\Big(d_{G_1}(u_1) + d_{G_1}(v_1)\Big) + \Big(d_{G_2}(u_2) + d_{G_2}(v_2)\Big) \\ & + d_{G_1}(u_1)d_{G_2}(v_2) + d_{G_1}(v_1)d_{G_2}(u_2)\Big]. \end{aligned}$$

Now by Lemmas 2.1 and 2.2,

$$\begin{aligned} P_3 \geqslant & \prod_{u_1v_1 \in E(G_1)} \prod_{u_2v_2 \in E(G_2)} 16\sqrt{\Big(d_{G_1}(u_1) + d_{G_1}(v_1)\Big) \times \Big(d_{G_2}(u_2) + d_{G_2}(v_2)\Big)} \\ & \times \sqrt{d_{G_1}(u_1)d_{G_1}(v_1) \times d_{G_2}(u_2)d_{G_2}(v_2)} \\ = & 16^{m_1m_2}\sqrt{\Big(\Pi_1^*(G_1)\Pi_2(G_1)\Big)^{m_2}\Big(\Pi_1^*(G_2)\Pi_2(G_2)\Big)^{m_1}}\,. \end{aligned}$$

By Lemma 2.2, the above equality holds if and only if for every $u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)$,

$$\Big(d_{G_1}(u_1)+d_{G_1}(v_1)\Big)+\Big(d_{G_2}(u_2)+d_{G_2}(v_2)\Big)+d_{G_1}(u_1)d_{G_2}(u_2)+d_{G_1}(v_1)d_{G_2}(v_2)$$
$$=4\sqrt[4]{\Big(d_{G_1}(u_1)+d_{G_1}(v_1)\Big)\times\Big(d_{G_2}(u_2)+d_{G_2}(v_2)\Big)\times d_{G_1}(u_1)d_{G_2}(u_2)\times d_{G_1}(v_1)d_{G_2}(v_2)},$$

and

$$\Big(d_{G_1}(u_1)+d_{G_1}(v_1)\Big)+\Big(d_{G_2}(u_2)+d_{G_2}(v_2)\Big)+d_{G_1}(u_1)d_{G_2}(v_2)+d_{G_1}(v_1)d_{G_2}(u_2)$$
$$=4\sqrt[4]{\Big(d_{G_1}(u_1)+d_{G_1}(v_1)\Big)\times\Big(d_{G_2}(u_2)+d_{G_2}(v_2)\Big)\times d_{G_1}(u_1)d_{G_2}(v_2)\times d_{G_1}(v_1)d_{G_2}(u_2)}.$$

By Lemma 2.1, this implies that for every $u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)$,

$$\begin{aligned}d_{G_1}(u_1)+d_{G_1}(v_1)&=d_{G_2}(u_2)+d_{G_2}(v_2)=d_{G_1}(u_1)d_{G_2}(u_2)=d_{G_1}(v_1)d_{G_2}(v_2)\\&=d_{G_1}(u_1)d_{G_2}(v_2)=d_{G_1}(v_1)d_{G_2}(u_2).\end{aligned}$$

So, $G_1$ and $G_2$ must be 2-regular graphs.

Hence,

$$\begin{aligned}\Pi_1^*(G)=P_1P_2P_3 >&(3\sqrt[3]{2})^{n_1m_2}\sqrt[3]{\Pi_1(G_1)^{m_2}}\sqrt[3]{\Pi_1^*(G_2)^{2n_1}}\\&\times(3\sqrt[3]{2})^{n_2m_1}\sqrt[3]{\Pi_1(G_2)^{m_1}}\sqrt[3]{\Pi_1^*(G_1)^{2n_2}}\\&\times 16^{m_1m_2}\sqrt{\Big(\Pi_1^*(G_1)\Pi_2(G_1)\Big)^{m_2}\Big(\Pi_1^*(G_2)\Pi_2(G_2)\Big)^{m_1}}\\=&16^{m_1m_2}\,(3\sqrt[3]{2})^{n_1m_2+n_2m_1}\sqrt[3]{\Pi_1(G_1)^{m_2}\Pi_1(G_2)^{m_1}}\\&\times\sqrt{\Pi_2(G_1)^{m_2}\Pi_2(G_2)^{m_1}}\sqrt[6]{\Pi_1^*(G_1)^{4n_2+3m_2}\Pi_1^*(G_2)^{4n_1+3m_1}}.\quad\square\end{aligned}$$

*Acknowledgement.* The authors would like to thank the referee for his/her careful reading and useful suggestions which led us to improve the paper.

This work was partially supported by the Center of Excellence of Algebraic Hyperstructures and its Applications of Tarbiat Modares University (CEAHA).

REFERENCES

[1] M. AZARI AND A. IRANMANESH, *Chemical graphs constructed from rooted product and their Zagreb indices*, MATCH Commun. Math. Comput. Chem. **70** (2013), 901–919.

[2] M. AZARI AND A. IRANMANESH, *Computation of the edge Wiener indices of the sum of graphs*, Ars Combin. **100** (2011), 113–128.

[3] M. AZARI AND A. IRANMANESH, *Computing the eccentric-distance sum for graph operations*, Discrete Appl. Math. **161** (18) (2013), 2827–2840.

[4] M. AZARI AND A. IRANMANESH, *Generalized Zagreb index of graphs*, Studia Univ. Babes Bolyai Chem. **56**, 3 (2011), 59–70.

[5] M. AZARI, A. IRANMANESH AND I. GUTMAN, *Zagreb indices of bridge and chain graphs*, MATCH Commun. Math. Comput. Chem. **70** (2013), 921–938.

[6] J. BRAUN, A. KERBER AND C. RUCKER, *Similarity of molecular descriptors: the equivalence of Zagreb index and walk counts*, MATCH Commun. Math. Comput. Chem. **54** (2005), 163–176.

[7] K. C. DAS, A. YURTTAS, M. TOGAN, A. S. CEVIK AND I. N. CANGUL, *The multiplicative Zagreb indices of graph operations*, J. Inequal. Appl. 2013:90 (2013).

[8] T. DOŠLIĆ, *Splices, links and their degree-weighted Wiener polynomials*, Graph theory Notes New York. **48** (2005), 47–55.

[9] M. ELIASI, A. IRANMANESH AND I. GUTMAN, *Multiplicative Versions of first Zagreb index*, MATCH Commun. Math. Comput. Chem. **68** (2012), 217–230.

[10] M. ELIASI AND D. VUKIČEVIĆ, *Comparing the Multiplicative Zagreb indices*, MATCH Commun. Math. Comput. Chem. **69** (2013), 765–773.

[11] I. GUTMAN, *Multiplicative Zagreb indices of trees*, Bulletin of International Mathematical Virtual Institute. **1** (2011), 13–19.

[12] I. GUTMAN AND O. E. POLANSKY, *Mathematical Concepts in Organic Chemistry*, Springer, Berlin, 1986.

[13] I. GUTMAN AND N. TRINAJSTIĆ, *Graph theory and molecular orbitals, Total $\pi$ electron energy of alternant hydrocarbons*, Chem. Phys. Lett. **17** (1972), 535–538.

[14] M. H. KHALIFEH, H. YUSEFI AZARI AND A. R. ASHRAFI, *The first and second Zagreb indices of some graph operations*, Discrete Appl. Math. **157** (2009), 804–811.

[15] S. NIKOLIĆ, G. KOVAČEVIĆ, A. MILIČEVIĆ AND N. TRINAJSTIĆ, *The Zagreb indices 30 years after*, Croat. Chem. Acta. **76** (2003), 113–124.

[16] R. TODESCHINI AND V. CONSONNI, *New local vertex invariants and molecular descriptors based on functions of the vertex degrees*, MATCH Commun. Math. Comput. Chem. **64** (2010), 359–372.

[17] N. TRINAJSTIĆ, *Chemical Graph Theory*, CRC Press, Boca Raton, FL, 1992.

[18] K. XU AND K. C. DAS, *Trees, unicyclic and bicyclic graphs extremal with respect to multiplicative sum Zagreb index*, MATCH Commun. Math. Comput. Chem. **68** (2012), 257–272.

[19] K. XU AND H. HUA, *A unified approach to extremal multiplicative Zagreb indices for trees, unicyclic and bicyclic graphs*, MATCH Commun. Math. Comput. Chem. **68** (2012), 241–256.

[20] Y. N. YEH AND I. GUTMAN, *On the sum of all distances in composite graphs*, Discrete Math. **135** (1–3) (1994), 359–365.

[21] B. ZHOU AND I. GUTMAN, *Further properties of Zagreb indices*, MATCH Commun. Math. Comput. Chem. **54** (2005), 233–239.

(Received April 18, 2014)

*Mahdieh Azari*
*Department of Pure Mathematics*
*Faculty of Mathematical Sciences*
*Tarbiat Modares University*
*P. O. Box 14115–137, Tehran, Iran*
*e-mail:* mahdie.azari@gmail.com

*Ali Iranmanesh*
*Department of Pure Mathematics*
*Faculty of Mathematical Sciences*
*Tarbiat Modares University*
*P. O. Box 14115–137, Tehran, Iran*
*e-mail:* iranmanesh@modares.ac.ir