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TWO–DIMENSIONAL ANALOGY OF THE KOROUS INEQUALITY

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Abstract. J. Korous reached an important and interesting result for general orthogonal polynomials in one variable. In the paper we generalize this result for orthogonal polynomials in two variables. The boundedness of two families of orthonormal polynomials associated with an arbitrary weight function $w(x,y)$ and its extension $W(x,y) = w(x,y)h(x,y)$, where $h(x,y)$ is a function satisfying certain conditions, is investigated.

1. Introduction

A generalization of the weight function and a study of relevant properties of generalized orthogonal polynomials are important parts of investigation in the theory of orthogonal polynomials. It presents a generalization in the following sense:

Let $\{p_n(x)\}_{n=0}^{\infty}$ be a system of polynomials orthonormal in $I \subset \mathbb{R}$ with respect to the weight function $w(x)$. We consider a system of orthonormal polynomials $\{q_n(x)\}_{n=0}^{\infty}$ on the same interval I associated with the weight function

$$\tilde{w}(x) = w(x)h(x), \tag{1}$$

where the factor $h(x)$ satisfies specific conditions. A typical example is the case when the weight function $w(x)$ is associated with so-called classical orthogonal polynomials. J. Korous reached an essential result for general orthogonal polynomials in one variable in [7] (or see [1], [15]). The theorem first published in that paper later became known as Korous theorem and it dealt with the boundedness and uniform boundedness of polynomials $\{q_n(x)\}_{n=0}^{\infty}$ orthonormal in a finite interval I. The Korous' estimation has the following form

$$|q_n(x)| \leqslant \frac{1}{\delta_0}|p_n(x)| + \frac{KM}{\delta_0^{3/2}}\left(|p_n(x)| + |p_{n-1}(x)|\right),$$

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where $\delta_0 = \min h(x)$, x belongs to the finite interval $I = (a,b)$ and $K = \max\{|a|,|b|\}$. The function $h(x)$ in (1) is bounded uniformly from zero $0 < m \leqslant h(x)$ and satisfies the Lipschitz condition

$$|h(x) - h(t)| \leqslant L|x-t|, \quad x,t \in I.$$

The expression of the reproducing kernel $K_n(x,t)$ (the Christoffel-Darboux formula for polynomials $p_k(x) = a_k^{(k)} x^k + a_{k-1}^{(k)} x^{k-1} + \ldots$)

$$K_n(x,t) = \sum_{k=0}^{n} p_k(x) p_k(t) = \frac{a_n^{(n)}}{a_{n+1}^{(n+1)}} \frac{p_{n+1}(x)p_n(t) - p_n(x)p_{n+1}(t)}{x-t} \tag{2}$$

plays a fundamental role in the original proof of the Korous theorem. Some other results about generalized orthogonal polynomials may be found in [2], [9], [10], [11], [13] and [14]. In our paper, we address this interesting topic to orthogonal polynomials in two variables. The boundedness of two families of orthonormal polynomials $\{P_{n,k}(x,y)\}$, $\{Q_{n,k}(x,y)\}$ associated with an arbitrary weight function $w(x,y)$ and its extension

$$W(x,y) = w(x,y)h(x,y),$$

where $h(x,y)$ is a function satisfying certain conditions, is investigated. Two of the authors studied special cases of the presented problem with some simplifying assumptions and by different methods in [3]. Namely, the factor $h(x,y)$ had a form $h(x,y) = R(x,y)$ (or $h(x,y) = R^{-1}(x,y)$) for some polynomial in two variables $R(x,y)$. The second special case is when the system of polynomials $\{P_{n,k}(x,y)\}$ has the separable weight function $w(x,y) = w_1(x)w_2(y)$ on the rectangle $(a,b) \times (c,d)$. Therefore, the result of this paper and the method of proof are more general.

1.1. Orthogonal polynomials in two variables

Many classical authors dealt with the issues associated with the orthogonal polynomials in two variables, e.g., Jackson (cf. [4]), Koornwinder (cf. [5], [6]), Krall, Sheffer (cf. [8]) and Sujetin (cf. [12]). We recall some properties of the algebraic polynomials in two variables given in [12].

Let $G \subset \mathbb{R}^2$ be a bounded or unbounded domain and let the weight function $w : G \to \langle 0, \infty)$ has all power moments

$$h_{n,k} = \iint_G w(x,y) x^{n-k} y^k dxdy$$

of finite values. Then we may define orthonormal polynomials in the form

$$P_{n,k}(x,y) = \sum_{m=0}^{n-1} \sum_{s=0}^{m} c_{m,s} x^{m-s} y^s + \sum_{s=0}^{k} c_{n,s} x^{n-s} y^s, \tag{3}$$

which satisfy the condition of orthonormality

$$\iint_G w(x,y) P_{n,k}(x,y) P_{u,v}(x,y) dxdy = \delta_{n,u}\delta_{k,v},$$

where $\delta_{i,j}$ is the Kronecker delta. It is obvious to assume that the principal coefficients $c_{n,k}$ are not equal to zero and $c_{n,k} > 0$. Index n in the formula (3) is the total degree of the polynomial with respect to the variables x and y. We define that the order (u,v) of the polynomial $P_{u,v}(x,y)$ is lower then the order (n,k) of (3), if either $u < n$ or $u = n$ and $v < k$. We denote it by $(u,v) < (n,k)$.

LEMMA 1. ([12, Theorem 2]) *Let a polynomial $P_{n,k}(x,y)$ of order (n,k) with the principal coefficient $c_{n,k} \neq 0$ be given. This polynomial is an orthogonal polynomial with a weight function $w(x,y)$ over a domain G if and only if for any polynomial $F_{u,v}(x,y)$ of lower order (u,v) the following condition*

$$\iint_G w(x,y) P_{n,k}(x,y) F_{u,v}(x,y)dxdy = 0 \tag{4}$$

holds.

Condition (4) is often called the first criterion of orthogonality. The sum

$$K_n(x,y,u,v) = \sum_{m=0}^{n} \sum_{s=0}^{m} P_{m,s}(x,y) P_{m,s}(u,v), \tag{5}$$

similarly to the case of orthonormal polynomials in one variable, there is called the kernel of order n of the orthonormal polynomials system $\{P_{n,k}(x,y)\}$. The function $K_n(x,y,u,v)$ is important for the integral representation of partial sums of the expansion of the function into the series of orthonormal polynomials. For the sake of brevity, we introduce the notation

$$L_n(x,y,u,v) = K_n(x,y,u,v) - K_{n-1}(x,y,u,v) = \sum_{k=0}^{n} P_{n,k}(x,y) P_{n,k}(u,v) \tag{6}$$

and

$$M_n(x,y,u,v,t,\tau) = L_n(x,y,t,\tau) L_{n+1}(u,v,t,\tau) - L_n(u,v,t,\tau) L_{n+1}(x,y,t,\tau). \tag{7}$$

The formula (for arbitrary constants *A, B*)

$$[(Au + Bv) - (Ax + By)] K_n(x,y,u,v) = \iint_G w(t,\tau)(At + B\tau) M_n(x,y,u,v,t,\tau) dt d\tau \tag{8}$$

is a generalization of the Christoffel-Darboux formula to the case of orthonormal polynomials in two variables (cf. [12]). This expression has a complex integral form, as opposed to the simple relation (2).

2. Main results

First, we need to estimate the kernel $K_n(x,y,u,v)$. We use the formula (8).

LEMMA 2. *Let* $\{P_{n,k}(x,y)\}$ *be a system of polynomials orthonormal on the domain* G *with respect to the weight function* $w(x,y)$. *Then the following estimation*

$$(|x-u|+|y-v|)|K_n(x,y,u,v)| \leqslant \iint_G w(t,\tau)(|t|+|\tau|)|M_n(x,y,u,v,t,\tau)|dtd\tau \tag{9}$$

holds.

Proof. If we put $A=B=-1$, then (8) arises by summing the two terms

$$(x-u)K_n(x,y,u,v) = -\iint_G w(t,\tau)tM_n(x,y,u,v,t,\tau)dtd\tau \tag{10}$$

and

$$(y-v)K_n(x,y,u,v) = -\iint_G w(t,\tau)\tau M_n(x,y,u,v,t,\tau)dtd\tau. \tag{11}$$

Obviously, from (10) and (11), we have estimations

$$|x-u||K_n(x,y,u,v)| \leqslant \iint_G w(t,\tau)|t||M_n(x,y,u,v,t,\tau)|dtd\tau,$$

$$|y-v||K_n(x,y,u,v)| \leqslant \iint_G w(t,\tau)|\tau||M_n(x,y,u,v,t,\tau)|dtd\tau.$$

Adding them term by term, we obtain (9). □

Now we can formulate the analogy of the Korous theorem.

THEOREM 1. *Let* $\{P_{n,k}(x,y)\}$, $\{Q_{n,k}(x,y)\}$ *be the systems of orthonormal polynomials, where* $P_{n,k}(x,y)$ *be polynomials orthonormal on the bounded domain* G *with respect to the weight function* $w(x,y)$ *and* $Q_{n,k}(x,y)$ *be polynomials orthonormal on the same domain* G *with respect to the generalized weight function*

$$W(x,y) = h(x,y)w(x,y),$$

where the function $h(x,y)$ *is bounded uniformly from zero, as well as it satisfies the Lipschitz condition*

$$0 < m \leqslant h(x,y), \tag{12}$$

$$|h(x,y)-h(\xi,\eta)| \leqslant L(|x-\xi|+|y-\eta|), \quad (x,y),\ (\xi,\eta)\in G. \tag{13}$$

Then the following inequality

$$|Q_{n,k}(x,y)| \leqslant \frac{L}{m^{\frac{3}{2}}}\left(\iint_G w(u,v)E_{n-1}^2(x,y,u,v)dudv\right)^{\frac{1}{2}} + \frac{1}{\sqrt{m}}\sum_{s=0}^{k}|P_{n,s}(x,y)| \tag{14}$$

holds on G. $E_{n-1}(x,y,u,v)$ *is determined by the formula (see (6) and (7))*

$$E_{n-1}(x,y,u,v) = \iint_G |M_{n-1}(x,y,u,v,t,\tau)|\big(|t|+|\tau|\big)w(t,\tau)dtd\tau. \tag{15}$$

Proof. The polynomial $Q_{n,k}(x,y)$ can be represented in the form

$$Q_{n,k}(x,y) = \sum_{m=0}^{n-1}\sum_{s=0}^{m} c_{m,s}P_{m,s}(x,y) + \sum_{s=0}^{k} c_{n,s}P_{n,s}(x,y), \tag{16}$$

where the coefficients $c_{i,j}$ are determined by the equalities

$$c_{i,j} = \iint_G w(u,v)Q_{n,k}(u,v)P_{i,j}(u,v)dudv. \tag{17}$$

Substituting (17) into the expansion (16), we obtain

$$\begin{aligned} Q_{n,k}(x,y) &= \iint_G w(u,v)Q_{n,k}(u,v)\Big[\sum_{m=0}^{n-1}\sum_{s=0}^{m} P_{m,s}(x,y)P_{m,s}(u,v)\Big]dudv \\ &\quad + \iint_G w(u,v)Q_{n,k}(u,v)\Big[\sum_{s=0}^{k} P_{n,s}(x,y)P_{n,s}(u,v)\Big]dudv. \end{aligned}$$

With regard to (5), we have

$$\begin{aligned} Q_{n,k}(x,y) &= \iint_G w(u,v)Q_{n,k}(u,v)K_{n-1}(x,y,u,v)dudv \\ &\quad + \iint_G w(u,v)Q_{n,k}(u,v)\Big[\sum_{s=0}^{k} P_{n,s}(x,y)P_{n,s}(u,v)\Big]dudv. \end{aligned} \tag{18}$$

First, let us estimate the first term in the right-hand side of this equality. We denote the first integral in the previous relation by

$$S_1 = \iint_G w(u,v)Q_{n,k}(u,v)K_{n-1}(x,y,u,v)dudv.$$

The expression $K_{n-1}(x,y,u,v)$ being a polynomial of total degree $n-1$ at most as a function of u and v, is orthogonal to $Q_{n,k}(u,v)$ with respect to the weight function $W(u,v) = h(u,v)w(u,v)$, for any values of x,y

$$\iint_G Q_{n,k}(u,v)K_{n-1}(x,y,u,v)h(u,v)w(u,v)dudv = 0.$$

Hence

$$\begin{aligned} h(x,y)S_1 &= h(x,y)\iint_G w(u,v)Q_{n,k}(u,v)K_{n-1}(x,y,u,v)dudv \\ &= \iint_G w(u,v)Q_{n,k}(u,v)K_{n-1}(x,y,u,v)\big(h(x,y)-h(u,v)\big)dudv. \end{aligned}$$

Consequently the integral S_1 is

$$S_1 = \frac{1}{h(x,y)}\iint_G w(u,v)Q_{n,k}(u,v)K_{n-1}(x,y,u,v)\big(h(x,y)-h(u,v)\big)dudv$$

and

$$|S_1| \leqslant \frac{1}{h(x,y)} \iint_G w(u,v)|Q_{n,k}(u,v)||K_{n-1}(x,y,u,v)||h(x,y)-h(u,v)|dudv.$$

Using Lemma 2, (12) and (13), we obtain the estimation

$$|S_1| \leqslant \frac{L}{m} \iint_G w(u,v)|Q_{n,k}(u,v)| \times \Big(\iint_G |M_{n-1}(x,y,u,v,t,\tau)|(|t|+|\tau|)w(t,\tau)dtd\tau \Big) dudv$$

and under the notation (15)

$$|S_1| \leqslant \frac{L}{m} \iint_G w(u,v)|Q_{n,k}(u,v)|E_{n-1}(x,y,u,v)dudv.$$

With the help of the Schwarz inequality and (12), we have

$$\begin{aligned} |S_1| &\leqslant \frac{L}{m}\frac{1}{\sqrt{m}} \iint_G w(u,v)\sqrt{h(u,v)}|Q_{n,k}(u,v)|E_{n-1}(x,y,u,v)dudv \\ &\leqslant \frac{L}{m^{\frac{3}{2}}} \Big(\iint_G w(u,v)h(u,v)Q^2_{n,k}(u,v)dudv \Big)^{\frac{1}{2}} \Big(\iint_G w(u,v)E^2_{n-1}(x,y,u,v)dudv \Big)^{\frac{1}{2}}. \end{aligned}$$

Note that by virtue of the orthonormality of the polynomial $Q_{n,k}(x,y)$ with the weight function $W(x,y)$, the following equality

$$\iint_G w(u,v)h(u,v)Q^2_{n,k}(u,v)dudv = 1$$

holds and we will have

$$|S_1| \leqslant \frac{L}{m^{\frac{3}{2}}} \Big(\iint_G w(u,v)E^2_{n-1}(x,y,u,v)dudv \Big)^{\frac{1}{2}}. \tag{19}$$

Now we pass to estimating the second term in the right-hand side of the equality (18). Denoting it by S_2

$$S_2 = \iint_G w(u,v)Q_{n,k}(u,v)\Big[\sum_{s=0}^{k} P_{n,s}(x,y)P_{n,s}(u,v) \Big] dudv.$$

Using the condition (12), we obtain

$$|S_2| \leqslant \frac{1}{\sqrt{m}} \sum_{s=0}^{k} |P_{n,s}(x,y)| \iint_G w(u,v)\sqrt{h(u,v)}|Q_{n,k}(u,v)||P_{n,s}(u,v)|dudv$$

and by Schwarz inequality

$$|S_2| \leqslant \frac{1}{\sqrt{m}} \sum_{s=0}^{k} |P_{n,s}(x,y)| \Big(\iint_G w(u,v)h(u,v)Q_{n,k}^2(u,v)dudv \Big)^{\frac{1}{2}}$$
$$\times \Big(\iint_G w(u,v)P_{n,s}^2(u,v)dudv \Big)^{\frac{1}{2}}.$$

From orthonormal properties of the systems $\{P_{n,k}(x,y)\}$ and $\{Q_{n,k}(x,y)\}$

$$|S_2| \leqslant \frac{1}{\sqrt{m}} \sum_{s=0}^{k} |P_{n,s}(x,y)| \tag{20}$$

follows. By virtue of (18), we get

$$|Q_{n,k}(x,y)| \leqslant |S_1| + |S_2|.$$

And finally from (19), (20), we have the statement of the theorem given by (14). □

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