QUANTUM INTEGRAL INEQUALITIES
FOR CONVEX FUNCTIONS

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Abstract. In this paper we establish some new quantum integral inequalities for convex functions.

1. Introduction

A function $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on I if inequality

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y),$$

holds for all $x, y \in I$ and $t \in [0,1]$. Many inequalities have been established for convex functions but the most famous is the Hermite-Hadamard's inequality [1], due to its rich geometrical significance and applications, which is stated as follows:

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a} \int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}. \qquad (1.1)$$

Both the inequalities hold in reversed direction if f is concave. Since its discovery, Hermite-Hadamard's inequality has been considered the most useful inequality in mathematical analysis. This inequality has been extended in a number of ways and a number of papers have been written. The main aim of this paper is to establish some new quantum integral inequalities for convex functions. Many consequences of Hermite-Hadamard type inequalities are obtained as special cases when $q \to 1$.

2. Preliminaries

Let $J := [a,b] \subset \mathbb{R}$, $J^o := (a,b)$ be interval and $0 < q < 1$ be a constant. We define q-derivative of a function $f : J \to \mathbb{R}$ at a point $x \in J$ on $[a,b]$ as follows.

DEFINITION 2.1. Assume $f : J \to \mathbb{R}$ is a continuous function and let $x \in J$. Then the expression

$$_aD_qf(x) = \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)}, \quad x \neq a, \quad _aD_qf(a) = \lim_{x \to a} {_aD_qf(x)}, \quad (2.1)$$

is called the q-derivative on J of function f at x.

We say that f is q-differentiable on J provided $_aD_qf(x)$ exists for all $x \in J$. Note that if $a = 0$ in (2.1), then $_0D_qf = \mathscr{D}_qf$, where $\mathscr{D}_q$ is the well-known q-derivative of the function $f(x)$ defined by

$$\mathscr{D}_qf(x) = \frac{f(x) - f(qx)}{(1-q)x}. \quad (2.2)$$

For more details, see [2].

LEMMA 2.1. [3] *Let* $\alpha \in \mathbb{R}$, *then we have*

$$_aD_q(x-a)^\alpha = \left(\frac{1-q^\alpha}{1-q}\right)(x-a)^{\alpha-1}. \quad (2.3)$$

DEFINITION 2.2. Assume $f : J \to \mathbb{R}$ is a continuous function. Then the q-integral on J is defined by

$$\int_a^x f(t)_a d_q t = (1-q)(x-a)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a), \quad (2.4)$$

for $x \in J$. Moreover, if $c \in (a,x)$ then the definite q-integral on J is defined by

$$\int_c^x f(t)_a d_q t = \int_a^x f(t)_a d_q t - \int_a^c f(t)_a d_q t$$

$$= (1-q)(x-a)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a)$$

$$-(1-q)(c-a)\sum_{n=0}^{\infty} q^n f(q^n c + (1-q^n)a).$$

Note that if $a = 0$, then (2.4) reduces to the classical q-integral of a function $f(x)$, defined by $\int_0^x f(t)_0 d_q t = (1-q)x \sum_{n=0}^{\infty} q^n f(q^n x)$ for $x \in [0,\infty)$. For more details, see [2].

THEOREM 2.1. [3] *Let* $f : J \to \mathbb{R}$ *be a continuous function. Then we have*

(i) $_aD_q \int_a^x f(t)_a d_q t = f(x)$;

(ii) $\int_c^x {}_aD_q f(t)_a d_q t = f(x) - f(c)$ *for* $c \in (a,x)$.

THEOREM 2.2. [3] *Assume* $f, g : J \to \mathbb{R}$ *are continuous functions,* $\alpha \in \mathbb{R}$. *Then, for* $x \in J$,

(i) $\int_a^x [f(t) + g(t)]_a d_q t = \int_a^x f(t)_a d_q t + \int_a^x g(t)_a d_q t$;

(ii) $\int_a^x (\alpha f)(t)_a d_q t = \alpha \int_a^x f(t)_a d_q t$;

(iii) $\int_c^x f(t)_a D_q g(t)_a d_q t = (fg)|_c^x - \int_c^x g(qt + (1-q)a)_a D_q f(t)_a d_q t$, *for* $c \in (a,x)$.

LEMMA 2.2. [4] *For* $\alpha \in \mathbb{R} \backslash \{-1\}$, *the following formula holds:*

$$\int_a^x (t-a)^\alpha {}_a d_q t = \left(\frac{1-q}{1-q^{\alpha+1}} \right) (x-a)^{\alpha+1}. \tag{2.5}$$

THEOREM 2.3. [4] (*q-Hermite-Hadamard*) *Let* $f : J \to \mathbb{R}$ *be a convex continuous function on* J *and* $0 < q < 1$. *Then we have*

$$f\left(\frac{a+b}{2} \right) \leqslant \frac{1}{b-a} \int_a^b f(t)_a d_q t \leqslant \frac{qf(a) + f(b)}{1+q}. \tag{2.6}$$

3. Auxiliary results

In this section, we present some auxiliary results which are used throughout this article.

LEMMA 3.1. *Let* $f : J \to \mathbb{R}$ *be a continuous function and* $0 < q < 1$. *If* $_aD_q f$ *is an integrable function on* J^o, *then the following equality holds:*

$$\begin{aligned} &\frac{1}{b-a} \int_a^b f(s)_a d_q s - \frac{qf(a) + f(b)}{1+q} \\ &= \frac{q(b-a)}{1+q} \int_0^1 (1 - (1+q)s)_a D_q f(sb + (1-s)a)_0 d_q s. \end{aligned} \tag{3.1}$$

Proof. Using Definitions 2.1 and 2.2, we have

$$\int_0^1 (1 - (1+q)s)_a D_q f(sb + (1-s)a)_0 d_q s$$

$$= \int_0^1 \left(\frac{f(sb + (1-s)a) - f(sqb + (1-sq)a)}{(1-q)(b-a)s} \right)_0 d_q s$$

$$- (1+q) \int_0^1 s \left(\frac{f(sb + (1-s)a) - f(sqb + (1-sq)a)}{(1-q)(b-a)s} \right)_0 d_q s$$

$$= \frac{1}{b-a}\left[\sum_{n=0}^{\infty} f(q^n b+(1-q^n)a) - \sum_{n=0}^{\infty} f(q^{n+1}b+(1-q^{n+1})a)\right]$$

$$-\frac{(1+q)}{b-a}\left[\sum_{n=0}^{\infty} q^n f(q^n b+(1-q^n)a) - \sum_{n=0}^{\infty} q^n f(q^{n+1}b+(1-q^{n+1})a)\right]$$

$$= \frac{f(b)-f(a)}{b-a} - \frac{(1+q)}{b-a}\sum_{n=0}^{\infty} q^n f(q^n b+(1-q^n)a)$$

$$+\frac{(1+q)}{q(b-a)}\sum_{n=1}^{\infty} q^n f(q^n b+(1-q^n)a)$$

$$= \frac{f(b)-f(a)}{b-a} - \frac{(1+q)}{b-a}\sum_{n=0}^{\infty} q^n f(q^n b+(1-q^n)a)$$

$$+\frac{(1+q)}{q(b-a)}\left[f(b)-f(b)+\sum_{n=1}^{\infty} q^n f(q^n b+(1-q^n)a)\right]$$

$$= \frac{f(b)-f(a)}{b-a} - \frac{(1+q)}{q(b-a)}f(b) + \frac{(1+q)}{q(b-a)}\sum_{n=0}^{\infty} q^n f(q^n b+(1-q^n)a)$$

$$-\frac{(1+q)}{b-a}\sum_{n=0}^{\infty} q^n f(q^n b+(1-q^n)a)$$

$$= -\frac{qf(a)+f(b)}{q(b-a)} + \frac{(1+q)(1-q)}{q(b-a)}\frac{(b-a)}{(b-a)}\sum_{n=0}^{\infty} q^n f(q^n b+(1-q^n)a)$$

$$= -\frac{qf(a)+f(b)}{q(b-a)} + \frac{(1+q)}{q(b-a)^2}\int_a^b f(s)\,_a d_q s.$$

Therefore, we obtain the desired result in (3.1) as required. The proof is completed. □

REMARK 3.1. If $q \to 1$, then (3.1) reduces to

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(s)ds = \frac{b-a}{2}\int_0^1 (1-2s)f'(sb+(1-s)a)ds.$$

See also [5, Lemma 2.1, page 91].

LEMMA 3.2. *Let $0 < q < 1$ be a constant. Then the following equality holds:*

$$\int_0^1 s|1-(1+q)s|_0 d_q s = \frac{q(1+4q+q^2)}{(1+q+q^2)(1+q)^3}. \tag{3.2}$$

Proof. By computing directly, we have

$$\int_0^1 s|1-(1+q)s|_0 d_q s$$

$$= \int_0^{\frac{1}{1+q}} s(1-(1+q)s)_0 d_q s + \int_{\frac{1}{1+q}}^1 s((1+q)s-1)_0 d_q s$$

$$= \int_0^{\frac{1}{1+q}} s_0 d_q s - (1+q) \int_0^{\frac{1}{1+q}} s^2 {}_0 d_q s + (1+q) \int_{\frac{1}{1+q}}^1 s^2 {}_0 d_q s - \int_{\frac{1}{1+q}}^1 s_0 d_q s$$

$$= \frac{1}{(1+q)^3} - \frac{1+q}{(1+q+q^2)(1+q)^3} + \frac{(1+q)(3q+3q^2+q^3)}{(1+q+q^2)(1+q)^3} - \frac{2q+q^2}{(1+q)^3}$$

$$= \frac{q(1+4q+q^2)}{(1+q+q^2)(1+q)^3}.$$

The proof is completed. $\square$

LEMMA 3.3. *Let $0 < q < 1$ be a constant. Then the following equality holds:*

$$\int_0^1 (1-s)|1-(1+q)s|_0 d_q s = \frac{q(1+3q^2+2q^3)}{(1+q+q^2)(1+q)^3}. \tag{3.3}$$

Proof. Taking into account Lemma 3.2, we have

$$\int_0^1 (1-s)|1-(1+q)s|_0 d_q s$$

$$= \int_0^1 |1-(1+q)s|_0 d_q s - \int_0^1 s|1-(1+q)s|_0 d_q s$$

$$= \int_0^{\frac{1}{1+q}} (1-(1+q)s)_0 d_q s + \int_{\frac{1}{1+q}}^1 ((1+q)s-1)_0 d_q s - \int_0^1 s|1-(1+q)s|_0 d_q s$$

$$= \int_0^{\frac{1}{1+q}} 1_0 d_q s - (1+q) \int_0^{\frac{1}{1+q}} s_0 d_q s + (1+q) \int_{\frac{1}{1+q}}^1 s_0 d_q s - \int_{\frac{1}{1+q}}^1 1_0 d_q s - \int_0^1 s|1-(1+q)s|_0 d_q s$$

$$= \frac{1}{1+q} - \frac{1+q}{(1+q)^3} + \frac{(1+q)(2q+q^2)}{(1+q)^3} - \frac{q}{1+q} - \frac{q(1+4q+q^2)}{(1+q+q^2)(1+q)^3}$$

$$= \frac{q(1+3q^2+2q^3)}{(1+q+q^2)(1+q)^3}.$$

The proof is completed. $\square$

4. Main results

In this section, we present some q-integral inequalities for convex functions on $[a,b]$.

THEOREM 4.1. *Let $f : J \to \mathbb{R}$ be a continuous function. If $|_a D_q f|$ is convex and integrable on J^o, then the following inequality holds:*

$$\left| \frac{q f(a) + f(b)}{1+q} - \frac{1}{b-a} \int_a^b f(s)_a d_q s \right|$$

$$\leqslant \frac{q^2(b-a)}{(1+q+q^2)(1+q)^4} \left((1+4q+q^2)|_a D_q f(b)| + q(1+3q^2+2q^3)|_a D_q f(a)| \right). \tag{4.1}$$

Proof. Using Lemma 3.1 and the convexity of $_aD_qf$ on J^o, we have

$$\left| \frac{qf(a)+f(b)}{1+q} - \frac{1}{b-a} \int_a^b f(s)_a d_q s \right|$$

$$= \left| \frac{q(b-a)}{1+q} \int_0^1 (1-(1+q)s)_a D_q f(sb+(1-s)a)_0 d_q s \right|$$

$$\leqslant \frac{q(b-a)}{1+q} \int_0^1 |1-(1+q)s| \left[s \left| _aD_q f(b) \right| + (1-s) \left| _aD_q f(a) \right| \right] {}_0 d_q s$$

$$= \frac{q(b-a)}{1+q} \left[\left| _aD_q f(b) \right| \int_0^1 |1-(1+q)s|s_0 d_q s + \left| _aD_q f(a) \right| \int_0^1 |1-(1+q)s|(1-s)_0 d_q s \right].$$

Applying Lemmas 3.2 and 3.3, we have

$$\left| \frac{qf(a)+f(b)}{1+q} - \frac{1}{b-a} \int_a^b f(s)_a d_q s \right|$$

$$\leqslant \frac{q(b-a)}{1+q} \left[\left(\frac{q(1+4q+q^2)}{(1+q+q^2)(1+q)^3} \right) \left| _aD_q f(b) \right| + \left(\frac{q(1+3q^2+2q^3)}{(1+q+q^2)(1+q)^3} \left| _aD_q f(a) \right| \right) \right]$$

$$= \frac{q^2(b-a)}{(1+q+q^2)(1+q)^4} \left[(1+4q+q^2) \left| _aD_q f(b) \right| + ((1+3q^2+2q^3) \left| _aD_q f(a) \right|) \right].$$

The proof is completed. $\square$

REMARK 4.1. If $q \to 1$, then the inequality (4.1) reduces to the integral inequality for convex functions

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(s)ds \right| \leqslant \frac{(b-a)(|f'(a)|+|f'(b)|)}{8}.$$

See also [5, Theorem 2.2, page 92].

Next, we present the second result of q integral inequality for convex functions on $[a,b]$.

THEOREM 4.2. *Let* $f : J \to \mathbb{R}$ *be a continuous function. If* $|_aD_qf|^r$ *is convex and integrable on* J^o *and* $r \geqslant 1$ *then the following inequality holds:*

$$\left| \frac{qf(a)+f(b)}{1+q} - \frac{1}{b-a} \int_a^b f(s)_a d_q s \right|$$

$$\leqslant \frac{q^2(2+q+q^2)(b-a)}{(1+q)^4} \left[\frac{(1+4q+q^2)|_aD_q f(b)|^r + (1+3q^2+2q^3)|_aD_q f(a)|^r}{(1+q+q^2)(2+q+q^3)} \right]^{1/r}.$$

$$(4.2)$$

Proof. From Lemma 3.1, we have

$$\left| \frac{qf(a)+f(b)}{1+q} - \frac{1}{b-a} \int_a^b f(s)_a d_q s \right|$$
$$\leqslant \frac{q(b-a)}{1+q} \int_0^1 |1-(1+q)s||_a D_q f(sb+(1-s)a)|_0 d_q s \tag{4.3}$$

and by the power-mean inequality

$$\int_0^1 |1-(1+q)s||_a D_q f(sb+(1-s)a)|_0 d_q s \leqslant \left(\int_0^1 |1-(1+q)s|_0 d_q s \right)^{1-\frac{1}{r}}$$
$$\times \left(\int_0^1 |1-(1+q)s||_a D_q f(sb+(1-s)a)|^r_0 d_q s \right)^{\frac{1}{r}}. \tag{4.4}$$

Using convexity of $|_a D_q f|^r$ and applying Lemmas 3.2 and 3.3, it follows that

$$\int_0^1 |1-(1+q)s||_a D_q f(sb+(1-s)a)|^r_0 d_q s$$
$$\leqslant \int_0^1 |1-(1+q)s|[s|_a D_q f(b)|^r + (1-s)|_a D_q f(a)|^r]_0 d_q s$$
$$\leqslant |_a D_q f(b)|^r \int_0^1 s|1-(1+q)s|_0 d_q s + |_a D_q f(a)|^r \int_0^1 (1-s)|1-(1+q)s|_0 d_q s$$
$$= \frac{q(1+4q+q^2)}{(1+q+q^2)(1+q)^3} |_a D_q f(b)|^r + \frac{q(1+3q^2+2q^3)}{(1+q+q^2)(1+q)^3} |_a D_q f(a)|^r$$
$$= \frac{q}{(1+q+q^2)(1+q)^3} \left[(1+4q+q^2)|_a D_q f(b)|^r + (1+3q^2+2q^3)|_a D_q f(a)|^r \right]. \tag{4.5}$$

Applying the fact that $\int_0^1 |1-(1+q)s|_0 d_q s = (q(2+q+q^3))/(1+q)^3$ and substituting (4.4) into (4.5) we get

$$\int_0^1 |1-(1+q)s||_a D_q f(sb+(1-s)a)|_0 d_q s \leqslant \left(\frac{q(2+q+q^3)}{(1+q)^3} \right)^{1-\frac{1}{r}}$$
$$\times \left(\frac{q}{(1+q+q^2)(1+q)^3} \left((1+4q+q^2)|_a D_q f(b)|^q + (1+3q^2+2q^3)|_a D_q f(a)|^q \right) \right)^{\frac{1}{r}},$$

which yield the desired inequality in (4.2) as requested. This completes the proof. □

REMARK 4.2. If $q \to 1$, then the inequality (4.2) reduces to integral inequality for convex function

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(s)ds \right| \leqslant \frac{(b-a)}{4} \left[\frac{|f'(a)|^r + |f'(b)|^r}{2} \right]^{\frac{1}{r}}.$$

See also [6, Theorem 1, page 52].

Next, two more q-integral inequalities for convex functions are established.

THEOREM 4.3. *Let f and g be real-valued, nonnegative and convex functions on J. Then the following inequalities hold:*

(i) $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)g(x)_a d_q x \leqslant \dfrac{f(a)g(a)}{1+q+q^2} + \dfrac{q(1+q^2)f(b)g(b)+q^2 N(a,b)}{(1+q)(1+q+q^2)},$ (4.6)

(ii) $2f\left(\dfrac{a+b}{2}\right) g\left(\dfrac{a+b}{2}\right) \leqslant \dfrac{1}{b-a}\displaystyle\int_a^b f(x)g(x)_a d_q x$

$$+\dfrac{2q^2 M(a,b)+(1+2q+q^2)N(a,b)}{2(1+q)(1+q+q^2)}.$$ (4.7)

where $M(a,b) = f(a)g(a)+f(b)g(b)$ *and* $N(a,b) = f(a)g(b)+f(b)g(a)$.

Proof. (i) Using the convexity of f and g, for all $s \in [0,1]$, we have

$$f(sb+(1-s)a) \leqslant sf(b)+(1-s)f(a),$$ (4.8)

$$g(sb+(1-s)a) \leqslant sg(b)+(1-s)g(a).$$ (4.9)

Multiplying (4.8) with (4.9), we get

$$f(sb+(1-s)a)g(sb+(1-s)a)$$
$$\leqslant s^2 f(b)g(b)+(1-s)^2 f(a)g(a)+s(1-s)[f(a)g(b)+f(b)g(a)].$$ (4.10)

Taking q-integral for (4.10) with respect to s on $(0,1)$, we obtain

$$\int_0^1 f(sb+(1-s)a)g(sb+(1-s)a)_0 d_q s$$
$$\leqslant f(b)g(b)\int_0^1 s^2 {}_0 d_q s + f(a)g(a)\int_0^1 (1-s)^2 {}_0 d_q s$$
$$+[f(a)g(b)+f(b)g(a)]\int_0^1 s(1-s)_0 d_q s$$
$$= \frac{f(b)g(b)}{1+q+q^2} + \frac{q(1+q^2)f(a)g(a)}{(1+q)(1+q+q^2)} + \frac{q^2\,(f(a)g(b)+f(b)g(a))}{(1+q)(1+q+q^2)}.$$

By substituting $sa+(1-s)b = x$, we deduce the desired inequality in (4.6).

(ii) Since f and g are convex functions on $[a,b]$, for $s \in [0,1]$, we get

$$f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right)$$
$$= f\left(\frac{sb+(1-s)a}{2} + \frac{(1-s)b+sa}{2}\right) g\left(\frac{sb+(1-s)a}{2} + \frac{(1-s)b+sa}{2}\right)$$

$$\leqslant \left(\frac{1}{2}f(sb+(1-s)a)+\frac{1}{2}f((1-s)b+sa)\right)$$

$$\times \left(\frac{1}{2}g(sb+(1-s)a)+\frac{1}{2}g((1-s)b+sa)\right)$$

$$\leqslant \frac{1}{4}\left[f(sb+(1-s)a)g(sb+(1-s)a)+f((1-s)b+sa)g((1-s)b+sa)\right]$$

$$+\frac{1}{4}[(sf(b)+(1-s)f(a))((1-s)g(b)+sg(a))+((1-s)f(b)+sf(a))$$

$$\times (sg(b)+(1-s)g(a))]$$

$$= \frac{1}{4}\left(f(sb+(1-s)a)g(sb+(1-s)a)+f((1-s)b+sa)g((1-s)b+sa)\right)$$

$$+\frac{1}{4}[2s(1-s)(f(a)g(a)+f(b)g(b))+(s^2+(1-s)^2)(f(a)g(b)+f(b)g(a))].$$

Taking q-integral with respect to s on $[0,1]$, we obtain

$$f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)$$

$$\leqslant \frac{1}{4}\int_0^1 (f(sb+(1-s)a)g(sb+(1-s)a)+f((1-s)b+sa)g((1-s)b+sa))_0d_qs$$

$$+\frac{1}{4}[2(f(a)g(a)+f(b)g(b))\int_0^1 s(1-s)_0d_qs+(f(a)g(b)+f(b)g(a))$$

$$\times \int_0^1 (s^2+(1-s)^2)_0d_qs]$$

$$= \frac{2}{4(b-a)}\int_a^b f(x)g(x)_ad_qx+\frac{1}{4}\left[2(f(a)g(a)+f(b)g(b))\left(\frac{1}{1+q}-\frac{1}{1+q+q^2}\right)\right.$$

$$\left.+(f(a)g(b)+f(b)g(a))\left(\frac{1}{1+q+q^2}+1-\frac{2}{1+q}+\frac{1}{1+q+q^2}\right)\right]$$

$$= \frac{1}{2(b-a)}\int_a^b f(x)g(x)_ad_qx+\frac{1}{4}\left[\frac{2q^2(f(a)g(a)+f(b)g(b))}{(1+q)(1+q+q^2)}\right.$$

$$\left.+\frac{(1+2q+q^3)(f(a)g(b)+f(b)g(a))}{(1+q)(1+q+q^2)}\right],$$

which leads to the estimate (4.7). This complete the proof. $\quad\square$

REMARK 4.3. If $q \to 1$, then the inequalities (4.6) and (4.7) reduce to the integral inequalities for convex functions

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leqslant \frac{1}{3}M(a,b)+\frac{1}{6}N(a,b),$$

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)g(x)dx+\frac{1}{6}M(a,b)+\frac{1}{3}N(a,b),$$

respectively. See also [7, Theorem 5.2.1, pages 250-251].

Now, the last two q-integral inequalities for convex functions are established.

THEOREM 4.4. *Let f and g be real-valued, nonnegative and convex functions on J. Then the following inequalities hold:*

(i)
$$\frac{(1+q)(1+q+q^2)}{(b-a)^2} \int_a^b \int_a^b \int_0^1 f(sy+(1-s)x)g(sy+(1-s)x)_0d_qsad_qxad_qy$$

$$\leqslant \frac{1+2q+q^2}{b-a} \int_a^b f(x)g(x)_ad_qx + \frac{2q^2}{(1+q)^2}((q^2f(a)g(a)+f(b)g(b))+qN(a,b)),$$
(4.11)

(ii)
$$\frac{1+q+q^2}{b-a} \int_a^b \int_0^1 f\left(sy+(1-s)\frac{a+b}{2}\right)g\left(sy+(1-s)\frac{a+b}{2}\right)_0d_qsad_qy$$

$$\leqslant \frac{1}{b-a} \int_a^b f(x)g(x)_ad_qx + \frac{q(1+q^2)}{4(1+q)}(M(a,b)+N(a,b))$$

$$+ \frac{q^2}{2(1+q)^2}(2(qf(a)g(a)+f(b)g(b))+(1+q)N(a,b)).$$
(4.12)

where $M(a,b) = f(a)g(a)+f(b)g(b)$ and $N(a,b) = f(a)g(b)+f(b)g(a)$.

Proof. (i) From the convexity of f and g, for all $s \in [0,1]$, $x,y \in J$, we have

$$f(sy+(1-s)x) \leqslant sf(y)+(1-s)f(x),$$
(4.13)

$$g(sy+(1-s)x) \leqslant sg(y)+(1-s)g(x).$$
(4.14)

Multiplying (4.13) with (4.14), we get

$$f(sy+(1-s)x)g(sy+(1-s)x)$$
$$\leqslant s^2f(y)g(y)+(1-s)^2f(x)g(x)+s(1-s)[f(x)g(y)+f(y)g(x)].$$
(4.15)

Taking q-integral for (4.15) with respect to s on $(0,1)$, one has

$$\int_0^1 f(sy+(1-s)x)g(sy+(1-s)x)_0d_qs$$

$$\leqslant f(y)g(x) \int_0^1 s^2_0d_qs + f(x)g(x) \int_0^1 (1-s)^2_0d_qs$$

$$+ [f(x)g(y)+f(y)g(x)] \int_0^1 s(1-s)_0d_qs$$

$$= \frac{f(y)g(y)}{1+q+q^2} + \frac{q(1+q^2)f(x)g(x)}{(1+q)(1+q+q^2)} + \frac{q^2(f(x)g(y)+f(y)g(x))}{(1+q)(1+q+q^2)}.$$
(4.16)

Next, taking double q-integral to both sides of (4.16) with respect to x,y on (a,b), we obtain

$$\int_a^b \int_a^b \int_0^1 f(sy+(1-s)x)g(sy+(1-s)x)_0 d_q s_a d_q x_a d_q y$$

$$\leqslant \frac{(b-a)}{1+q+q^2}\int_a^b f(y)g(y)_a d_q y + \frac{q(1+q^2)(b-a)}{(1+q)(1+q+q^2)}\int_a^b f(x)g(x)_a d_q x$$

$$+\frac{q^2}{(1+q)(1+q+q^2)}\left(\int_a^b g(y)_a d_q y \int_a^b f(x)_a d_q x + \int_a^b f(y)_a d_q y \int_a^b (g(x))_a d_q x\right).$$

$$(4.17)$$

By applying Theorem 2.3 on the right hand side of (4.17), we have

$$\int_a^b \int_a^b \int_0^1 f(sy+(1-s)x)g(sy+(1-s)x)_0 d_q s_a d_q x_a d_q y$$

$$\leqslant \left(\frac{1}{1+q+q^2}+\frac{q(1+q^2)}{(1+q)(1+q+q^2)}\right)(b-a)\int_a^b f(x)g(x)_a d_q x$$

$$+\frac{2q^2(b-a)^2}{(1+q)^3(1+q+q^2)}\left((q^2 f(a)g(a)+f(b(g(b)))+qN(a,b)\right). \quad (4.18)$$

Multiplying both sides of (4.18) by $((1+q)(1+q+q^2))/(b-a)^2$, we deduce the desired inequality in (4.11).

(ii) Since f and g are convex functions on $[a,b]$, for $s \in [0,1]$, we get

$$f\left(sy+(1-s)\left(\frac{a+b}{2}\right)\right) \leqslant sf(y)+(1-s)f\left(\frac{a+b}{2}\right), \quad (4.19)$$

$$g\left(sy+(1-s)\left(\frac{a+b}{2}\right)\right) \leqslant sg(y)+(1-s)g\left(\frac{a+b}{2}\right). \quad (4.20)$$

Multiplying (4.19) with (4.20), we obtain

$$f\left(sy+(1-s)\left(\frac{a+b}{2}\right)\right)g\left(sy+(1-s)\left(\frac{a+b}{2}\right)\right)$$

$$\leqslant s^2 f(y)g(y)+(1-s)^2 f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \quad (4.21)$$

$$+s(1-s)\left[f(y)g\left(\frac{a+b}{2}\right)+f\left(\frac{a+b}{2}\right)g(y)\right].$$

Taking q-integral for (4.21) with respect to s on $(0,1)$, it follows that

$$\int_0^1 f\left(sy+(1-s)\left(\frac{a+b}{2}\right)\right)g\left(sy+(1-s)\left(\frac{a+b}{2}\right)\right)_0 d_q s$$

$$\leqslant f(y)g(y)\int_0^1 s^2{}_0 d_q s + f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\int_0^1 (1-s)^2{}_0 d_q s$$

$$+\left[f(y)g\left(\frac{a+b}{2}\right)+f\left(\frac{a+b}{2}\right)g(y)\right]\int_0^1 s(1-s)_0 d_q s$$

$$= \frac{f(y)g(y)}{1+q+q^2} + f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\left[1 - \frac{2}{1+q} + \frac{1}{1+q+q^2}\right]$$
$$+ \left[f(y)g\left(\frac{a+b}{2}\right) + f\left(\frac{a+b}{2}\right)g(y)\right]\left[\frac{1}{1+q} - \frac{1}{1+q+q^2}\right]$$
$$= \frac{f(y)g(y)}{1+q+q^2} + \left[\frac{q+q^3}{(1+q)(1+q+q^2)}\right]f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)$$
$$+ \left[\frac{q^2}{(1+q)(1+q+q^2)}\right]\left[f(y)g\left(\frac{a+b}{2}\right) + f\left(\frac{a+b}{2}\right)g(y)\right]$$

Applying q-integral with respect to y on $[a,b]$ and using Theorem 2.3 with the convexity of the functions f,g, we observe that

$$\int_a^b \int_0^1 f\left(sy + (1-s)\left(\frac{a+b}{2}\right)\right)g\left(sy + (1-s)\left(\frac{a+b}{2}\right)\right){}_0 d_q s\, _a d_q y$$
$$\leqslant \frac{1}{1+q+q^2}\int_a^b f(y)g(y)\, _a d_q y + \frac{q+q^3}{(1+q)(1+q+q^2)}\int_a^b f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right){}_a d_q y$$
$$+ \frac{q^2}{(1+q)(1+q+q^2)}\left(g\left(\frac{a+b}{2}\right)\int_a^b f(y)\, _a d_q y + f\left(\frac{a+b}{2}\right)\int_a^b g(y)\, _a d_q y\right)$$
$$\leqslant \frac{1}{1+q+q^2}\int_a^b f(y)g(y)\, _a d_q y + \frac{(q+q^3)(b-a)}{4(1+q)(1+q+q^2)}(f(a)+f(b))(g(a(+g(b))$$
$$+ \frac{q^2(b-a)}{(1+q)(1+q+q^2)}\left(\left(\frac{g(a)+g(b)}{2}\right)\left(\frac{qf(a)+f(b)}{1+q}\right)\right.$$
$$+ \left.\left(\frac{f(a)+f(b)}{2}\right)\left(\frac{qg(a)+g(b)}{1+q}\right)\right)$$
$$= \frac{1}{1+q+q^2}\int_a^b f(y)g(y)\, _a d_q y + \frac{q(1+q^2)(b-a)}{4(1+q)(1+q+q^2)}(M(a,b)+N(a,b))$$
$$+ \frac{q^2(b-a)}{2(1+q)^2(1+q+q^2)}(2(qf(a)g(a)+f(b)g(b))+(1+q)N(a,b)). \qquad (4.22)$$

Multiplying both sides of (4.22) by $((1+q)(1+q+q^2))/(b-a)$, we deduce the desired inequality in (4.12). This complete the proof. $\square$

REMARK 4.4. If $q \to 1$, then the inequalities (4.11) and (4.12) reduce to the integral inequalities for convex functions

$$\frac{3}{2(b-a)^2}\int_a^b \int_a^b \int_0^1 f(sy + (1-s)x)g(sy + (1-s)x){}_0 d_q s\, dx\, dy$$
$$\leqslant \frac{1}{b-a}\int_a^b f(x)g(x)\, dx + \frac{1}{8}(M(a,b)+N(a,b)),$$

$$\frac{3}{b-a}\int_a^b \int_0^1 f\left(sy+(1-s)\frac{a+b}{2}\right) g\left(sy+(1-s)\frac{a+b}{2}\right) ds\, dy$$

$$\leqslant \frac{1}{b-a}\int_a^b f(x)g(x)dx + \frac{1}{2}(M(a,b)+N(a,b)),$$

respectively. See also [7, Theorem 5.2.2, pages 252–253].

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