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A SHORT EXTENSION OF TWO OF SPIRA'S RESULTS

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Abstract. Two inequalities concerning the symmetry of the zeta-function and the Ramanujan τ-function are improved through the use of some elementary considerations.

1. Introduction

The functional equation for the Riemann zeta-function $\zeta(s)$ is

$$\zeta(1-s) = g(s)\zeta(s), \qquad g(s) = 2^{1-s}\pi^{-s}\cos(\tfrac{1}{2}s\pi)\Gamma(s),$$

see [7, (2.1.8)]. Spira [5] proved that

$$|\zeta(1-s)| > |\zeta(s)|, \quad \tfrac{1}{2} < \sigma < 1, \quad t \geqslant 10, \quad \text{when } \zeta(s) \neq 0. \tag{1}$$

Dixon and Schoenfeld [2] gave a simpler and sharper proof of (1) for $|t| \geqslant 6.8$ and for all $\sigma > \frac{1}{2}$. Saidak and Zvengrowski [4] proved (1) for $|t| \geqslant 2\pi + 1$, and, in fact, their proof is valid for $|t| \geqslant 7$. Recently, Nazardonyavi and Yakubovich[1] [3] gave an alternative proof of (1) in the range $|t| \geqslant 12$. They remark that this result may be extended to $|t| \geqslant 6.5$ by a computer simulation. Spira [*op. cit.*] notes that (1) 'fails for t around 2π'. Indeed, for $t^* = 6.2898$ one may compute

$$\frac{|\zeta(0.48 - it^*)|}{|\zeta(0.52 + it^*)|} - 1 < -8 \times 10^{-8}. \tag{2}$$

The purpose of this short article is to examine the proof given by Dixon and Schoenfeld and to prove

THEOREM 1. *$|\zeta(1-s)| > |\zeta(s)|$ except at the zeroes of $\zeta(s)$, where $|t| \geqslant 6.29073$ and $\sigma > \frac{1}{2}$.*

By the functional equation, $\zeta(1-s)$ and $\zeta(s)$ have the same zeroes when $0 < \sigma < 1$. This means that Theorem 1 gives rise to the following Corollary, which improves on Proposition 1 in [3].

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[1]The authors considered the equivalent problem of $|\zeta(1-s)| < |\zeta(s)|$ for $0 < \sigma < \frac{1}{2}$.

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COROLLARY 1. *A necessary and sufficient condition for the Riemann hypothesis is*

$$|\zeta(1-s)| > |\zeta(s)|, \quad \sigma > \tfrac{1}{2}, \quad |t| \geqslant 6.29073.$$

In light of (2) Theorem 1 and Corollary 1 are close to best possible. The purpose of this article is to show that the range of t in (1) can be extended relatively easily. In [3] the range is increased at the cost of significant computation. By contrast, almost no computation is required to establish Theorem 1 and Corollary 1.

Similarly, for $F(s) = \sum_{n=1}^{\infty} \tau(n)/n^s$, where $\tau(n)$ is the Ramanujan τ-function, Spira [6] proved that

$$|F(12-s)| > |F(s)|, \quad 6 < \sigma < \frac{13}{2}, \quad t \geqslant 4.35, \tag{3}$$

except at the zeroes of $F(s)$. This improved on a result of Berndt [1] who proved (3) for $t \geqslant 6.8$. At no extra charge, the proof of Theorem 1 gives

THEOREM 2. $|F(12-s)| > |F(s)|$ *except at the zeroes of* $F(s)$, *where* $t \geqslant 3.8085$ *and* $6 < \sigma < \frac{13}{2}$.

2. Proof of Theorems 1 and 2

Dixon and Schoenfeld consider the function $h(s) = \log|g(s)/g(\frac{1}{2}+it)|$. Since $|g(\frac{1}{2}+it)| = 1$, one may prove Theorem 1 by showing that $h(s) > 0$ for $\sigma > \frac{1}{2}$ and $t \geqslant 6.29073$.

Starting at [2, (1)] we have

$$\frac{h(s)}{\sigma - \frac{1}{2}} > \left\{ \frac{\partial}{\partial \sigma} \log|\Gamma(\sigma + it)| \right\}_{\sigma=\sigma_1} - 2\pi e^{-\pi t} - \log 2\pi,$$

for some number $\sigma_1 \in (1/2, \sigma)$. Using Stirling's formula we arrive at [2, (3)] which is

$$\frac{\partial}{\partial \sigma} \log|\Gamma(\sigma+it)| = \Re\left\{ \log s - \frac{1}{2s} - \frac{1}{12s^2} + 6\int_0^{\infty} \frac{P_3(x)}{(s+x)^4}\,dx \right\}, \tag{4}$$

where $P_3(x)$ is a Bernouilli polynomial of period 1, which is equal to $x(2x^2-3x+1)/12$ on $[0,1]$. Some simple calculus gives $\max_{x\in[0,1]} P_3(x) = \sqrt{3}/216$.

Rather than bound each term in (4) by its modulus, as in [2], we consider each real part separately. For $s = \sigma + it$ the right-hand side of (4) is bounded below by

$$\frac{1}{2}\log(\sigma^2+t^2) - \frac{\sigma}{2(\sigma^2+t^2)} - \frac{(\sigma^2-t^2)}{12(\sigma^2+t^2)^2} - \frac{\sqrt{3}}{36}\int_0^{\infty} \frac{dx}{\{(\sigma+x)^2+t^2\}^2}. \tag{5}$$

The integral in (5), denoted by I, is clearly decreasing in σ, whence we conclude

$$I \leqslant \int_0^{\infty} \frac{dx}{\{(\frac{1}{2}+x)^2+t^2\}^2} = \frac{\tan^{-1} 2t - \frac{2t}{4t^2+1}}{2t^3}.$$

Denote the first three terms in (5) by $J(\sigma,t)$. It is easy to show that

$$\frac{\partial J}{\partial \sigma} = \frac{\sigma^3(1+3\sigma+6\sigma^2)+3t^2(\sigma-\frac{1}{2})\{2t^2+4\sigma(\sigma+\frac{1}{2})\}}{6(\sigma^2+t^2)^3},$$

which is clearly positive for $\sigma \geqslant \frac{1}{2}$, whence

$$\frac{h(s)}{\sigma-\frac{1}{2}} > J(\tfrac{1}{2},t) - \frac{\sqrt{3}(\tan^{-1} 2t - \frac{2t}{4t^2+1})}{72t^3} - 2\pi e^{-\pi t} - \log 2\pi = G(t),$$

say. We should like to show that $G(t)$ is ultimately increasing. Every term in the derivative of $G(t)$ is positive with the possible exception of

$$80(9-\sqrt{3})t^4 - 32\sqrt{3}t^2 - 3\sqrt{3},$$

which has real roots at $t = \pm 0.3918\ldots$. Thus, $G(t)$ is increasing for all $t \geqslant 0.4$. A quick computational check shows that $G(6.29072) < 0 < G(6.29073)$, which proves Theorem 1. $\square$

To prove Theorem 2 we note that, by Spira [6, p. 384], it is sufficient to show that

$$\left\{\frac{\partial}{\partial \sigma} \log|\Gamma(\sigma+it)|\right\}_{\sigma=\sigma_1} - \log 2\pi > 0,$$

where $\sigma_1 \in [\frac{11}{2}, \frac{13}{2}]$. But for a small alteration in bounding the integral I in (5), the calculation proceeds as before. It is sufficient to show that

$$H(t) = J(\tfrac{11}{2},t) - \frac{\sqrt{3}(\tan^{-1}\frac{2t}{11} - \frac{22t}{121+4t^2})}{72t^3} > 0.$$

As in the case with $G(t)$, one can easily show that $H(t)$ is ultimately increasing; in fact, $H(t)$ is increasing for all $t \geqslant 0.8$. A computational check shows that $H(3.8024) < 0 < H(3.8085)$, which establishes Theorem 2. $\square$

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