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ON THE EXTREMAL ENERGY OF BICYCLIC DIGRAPHS

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Abstract. The eigenvalues of a graph are the eigenvalues of its adjacency matrix. The energy of a graph is the sum of absolute values of its eigenvalues. Recently, the concept of energy of graphs is extended to digraphs. Minimal and maximal energy among n-vertex unicyclic digraphs is known, where $n \geqslant 2$. In this paper, we address the problem of finding minimal and maximal energy among n-vertex bicyclic digraphs which contain vertex-disjoint directed cycles, where $n \geqslant 4$.

1. Introduction

The eigenvalues of a graph are the eigenvalues of its adjacency matrix. The eigenvalues of a graph form the spectrum of this graph. The energy of a graph is defined to be the sum of the absolute values of its eigenvalues. The concept of energy of simple graphs was first introduced by Gutman [3]. This topic has stimulated extensive research due to its close links to Chemistry. Several extensions and variations of the energy of graph have been studied in the literature. The reader is referred to [1, 6, 5, 4, 11] for a comprehensive study on the bounds for the energy of bipartite graphs, trees and benzenoids. Hou [7] addressed the problem of finding the unicyclic graphs with minimal energy. He showed that the unicyclic graphs S_n^3 has the minimum energy among all unicyclic graphs with n vertices. Here S_n^3 denotes the graph obtained from the star graph with n vertices by joining two pendent vertices by an edge. Hou et al. [8] considered the problem of finding unicyclic graphs with maximal energy. They find unicyclic graphs with maximum energy among all unicyclic graphs with fixed number of vertices and with fixed length of cycles. Moreover, the authors show that the energy of any unicyclic bipartite graph on n vertices is always less than the energy of P_n^6. Here P_n^6 denotes the unicyclic graph obtained by connecting a vertex of the cycle C_6 of length $6 \leqslant n$ with a terminal vertex of the path P_{n-6} of length $n-6$.

Pena and Rada [9] extended the concept of energy to digraphs. In case of digraphs, the eigenvalues may be complex numbers since the adjacency matrix of a digraph is not symmetric. The energy of a digraph is defined to be the sum of the absolute values of the real parts of its eigenvalues. The authors find the unicyclic digraphs which have minimal and maximal energy among all unicyclic digraphs with fixed number of vertices. Furthermore, the increasing property of the energy of digraphs is discussed.

In this paper, we continue the study of finding extremal energy for digraphs. We find minimal and maximal energy among all those n-vertex bicyclic digraphs which contain vertex-disjoint directed cycles, where $n \geqslant 4$.

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2. Preliminaries

First, we give few definitions and terminologies. A directed graph (or digraph) is a pair $G = (\mathscr{V}, \mathscr{A})$ of disjoint finite sets $\mathscr{V}$ and $\mathscr{A}$ where $\mathscr{A} \subseteq \mathscr{V} \times \mathscr{V}$. The elements of $\mathscr{V}$ are called vertices and the elements of $\mathscr{A}$ are called arcs. If there is an arc from a vertex u to a vertex v, we denote it by uv. A directed path P_n of length $n-1$ $(n \geqslant 2)$ is a digraph with vertex set $\{v_i \mid i = 1, \ldots, n\}$ of n elements and arc set $\{v_i v_{i+1} \mid i = 1, \ldots, n-1\}$ of $n-1$ elements. A directed cycle C_n of length n $(n \geqslant 2)$ is a digraph with vertex set $\{v_i \mid i = 1, \ldots, n\}$ of n elements and arc set $\{v_i v_{i+1} \mid i = 1, \ldots, n-1\} \cup \{v_n v_1\}$ of n elements. A unicyclic digraph is a connected digraph which contains a unique directed cycle. A bicyclic digraph is a connected digraph which contains exactly two directed cycles.

The adjacency matrix $A(G) = [a_{ij}]_{n \times n}$ of an n-vertex digraph $G = (\mathscr{V}, \mathscr{A})$ is defined as

$$a_{ij} = \begin{cases} 1 \text{ if } v_i v_j \in \mathscr{A} \\ 0 \text{ otherwise} \end{cases} \quad (\forall v_i, v_j \in \mathscr{V}).$$

Let C_n be a directed cycle of length n. Then Pena and Rada [9] show that the spectrum of C_n consists of the values $\exp\{2k\pi\imath\}$, where $k = 0, 1, \ldots, n-1$. Therefore, the energy of C_n is given by

$$E(C_n) = \sum_{k=0}^{n-1} \left| \cos \frac{2k\pi}{n} \right|. \tag{1}$$

It is readily seen that

$$E(C_k) = 2 \quad \text{for } k = 2, 3, 4. \tag{2}$$

Moreover, if G is an n-vertex unicyclic digraph with a unique directed cycle C_r of length r $(2 \leqslant r \leqslant n)$, then it is shown in [9] that

$$E(G) = E(C_r) = \sum_{k=0}^{r-1} \left| \cos \frac{2k\pi}{r} \right|. \tag{3}$$

The following theorem gives the extremal energy among n-vertex unicyclic digraphs.

THEOREM 1. (Pena and Rada [9]) *Among all n-vertex unicyclic digraphs, the minimal energy is attained in digraphs which contain a cycle of length 2, 3 or 4. The maximal energy is attained in the cycle C_n on length n.*

From the proof of Theorem 1, we derive the following inequalities:

$$E(C_r) > 2 \quad \text{for } r \geqslant 5 \tag{4}$$

$$E(C_{r_1}) \geqslant E(C_{r_2}) \quad \text{for } r_1 \geqslant r_2 \geqslant 5. \tag{5}$$

The inequality in (5) is strict if $r_1 > r_2$.

A digraph $G = (\mathscr{V}, \mathscr{A})$ is strongly connected if for each $u, v \in \mathscr{V}$, there is a directed path from u to v and one from v to u. The strong components of a digraph are its maximal strong subdigraphs. Next theorem gives the energy of a digraph which contains k strong components, $k \geqslant 1$.

THEOREM 2. (Pena and Rada [9]) *Let $H_1,\dots,H_k$ be the strong components of a digraph G. Then*

$$E(G)=\sum_{i=1}^{k}E(H_i).$$

We know that any positive integer n has one of the forms: $n\equiv 0(\bmod 4)$ or $n\equiv 2(\bmod 4)$ or $n\equiv 1(\bmod 2)$. Pirzada and Bhat [10] derived the following formulae to calculate the energy of a directed cycle C_n:

$$E(C_n)=\begin{cases} 2\cot\frac{\pi}{n} & \text{if } n\equiv 0(\bmod 4)\\ 2\csc\frac{\pi}{n} & \text{if } n\equiv 2(\bmod 4)\\ \csc\frac{\pi}{2n} & \text{if } n\equiv 1(\bmod 2). \end{cases} \tag{6}$$

Next lemma will be used in finding extremal energy of bicyclic digraphs.

LEMMA 1. *Let x,a,b be real numbers such that $x\geqslant a>0$ and $b>0$. Then*

$$\frac{x\pi}{bx^2-\pi^2}\leqslant\frac{a\pi}{ba^2-\pi^2}. \tag{7}$$

Proof. Let x,a,b be real numbers such that $x\geqslant a>0$ and $b>0$. Then we have

$$bx\geqslant ba, \tag{8}$$

$$-\frac{\pi^2}{x}\geqslant-\frac{\pi^2}{a}. \tag{9}$$

Inequalities (8) and (9) give

$$bx-\frac{\pi^2}{x}\geqslant ba-\frac{\pi^2}{a}.$$

That is,

$$\frac{\pi x}{bx^2-\pi^2}\leqslant\frac{\pi a}{ba^2-\pi^2}. \qquad \square$$

For any real number x with $0\leqslant x\leqslant\frac{\pi}{2}$, the following inequalities hold:

$$\sin x\leqslant x,\quad \sin x\geqslant x-\frac{x^3}{3!},\quad \cos x\geqslant 1-\frac{x^2}{2} \tag{10}$$

$$\cot x\leqslant\frac{1}{x},\quad \cot x\geqslant\frac{1}{x}-\frac{x}{2}\quad \text{if } x\neq 0. \tag{11}$$

3. Energy of bicyclic digraphs

We consider the set $\mathscr{D}_n$ consisting of n-vertex bicyclic digraphs such that directed cycles in a digraph are vertex-disjoint, where $n\geqslant 4$. Let $G\in\mathscr{D}_n$ be a digraph with two cycles C_{r_1} and C_{r_2} of lengths r_1 and r_2 respectively, where $2\leqslant r_1,r_2\leqslant n-2$. Then Theorem 2 gives

$$E(G)=E(C_{r_1})+E(C_{r_2}).$$

Next lemma is easily seen from (2) and (5).

LEMMA 2. *Let n and m be positive integers where $n \geqslant 4$ such that $2 \leqslant m \leqslant 4$ or $2 \leqslant n-m \leqslant 4$. Then*

$$E(C_{n-2}) + E(C_2) \geqslant E(C_{n-m}) + E(C_m),$$

where C_{n-2}, C_2, C_{n-m} and C_m are vertex-disjoint directed cycles.

Next lemma gives lower bounds for the sum of energies of two vertex-disjoint directed cycles C_{n-2} and C_2, where $n \geqslant 4$.

LEMMA 3. *Let C_{n-2} and C_2 be two vertex-disjoint directed cycles, $n \geqslant 4$. Then we have the following inequalities:*

$$E(C_{n-2}) + E(C_2) \geqslant \begin{cases} \frac{2n}{\pi} - \frac{4}{\pi} - \frac{\pi^2}{3(n-2)^2} + 2 & \text{if } n \equiv 0 \pmod 4 \\ \frac{2n}{\pi} - \frac{4}{\pi} - \frac{\pi}{(n-2)} + 2 & \text{if } n \equiv 2 \pmod 4 \\ \frac{2n}{\pi} - \frac{4}{\pi} - \frac{\pi^2}{12(n-2)^2} + 2 & \text{if } n \equiv 1 \pmod 2. \end{cases}$$

Proof. By (2), $E(C_2) = 2$. If $n \equiv 0 \pmod 4$ then (6) and (10) yield

$$\begin{aligned} E(C_{n-2}) + E(C_2) &= 2\left(\csc\frac{\pi}{n-2} + 1\right) \\ &= 2\left(\frac{1+\sin\frac{\pi}{n-2}}{\sin\frac{\pi}{n-2}}\right) \\ &\geqslant 2\left(\frac{1+\frac{\pi}{n-2} - \frac{\pi^3}{6(n-2)^3}}{\frac{\pi}{n-2}}\right) \\ &= \frac{2n}{\pi} - \frac{4}{\pi} - \frac{\pi^2}{3(n-2)^2} + 2. \end{aligned}$$

Next, we consider the case when $n \equiv 2 \pmod 4$. From (6) and (10), we get

$$\begin{aligned} E(C_{n-2}) + E(C_2) &= 2\left(\cot\frac{\pi}{n-2} + 1\right) \\ &\geqslant 2\left(\frac{n-2}{\pi} - \frac{\pi}{2(n-2)} + 1\right) \\ &= \frac{2n}{\pi} - \frac{4}{\pi} - \frac{\pi}{(n-2)} + 2. \end{aligned}$$

Finally, we consider the case when $n \equiv 1(\bmod 2)$. In this case, (6) and (10) give

$$\begin{aligned} E(C_{n-2})+E(C_2) &= \csc\frac{\pi}{2(n-2)}+2 \\ &= \left(\frac{1+2\sin\frac{\pi}{2(n-2)}}{\sin\frac{\pi}{2(n-2)}}\right) \\ &\geqslant \frac{1+2\left(\frac{\pi}{2(n-2)}-\frac{\pi^3}{48(n-2)^3}\right)}{\frac{\pi}{2(n-2)}} \\ &= \frac{2n}{\pi}-\frac{4}{\pi}-\frac{\pi^2}{12(n-2)^2}+2. \end{aligned}$$

This completes the proof. □

Lemmas 4–6 give different upper bounds for the sum of energies of vertex-disjoint directed cycles C_{n-m} and C_m, where $m,\ n-m \geqslant 2$.

LEMMA 4. *If $n \equiv 0(\bmod 4)$, $m,\ n-m \geqslant 2$ then the following holds:*

$$E(C_{n-m})+E(C_m) \leqslant \begin{cases} \frac{2n}{\pi} & \text{if } m \equiv 0(\bmod 4) \\ \frac{2n}{\pi}+\frac{2(n-m)\pi}{6(n-m)^2-\pi^2}+\frac{2m\pi}{6m^2-\pi^2} & \text{if } m \equiv 2(\bmod 4) \\ \frac{2n}{\pi}+\frac{2(n-m)\pi}{24(n-m)^2-\pi^2}+\frac{2m\pi}{24m^2-\pi^2} & \text{if } m \equiv 1(\bmod 2), \end{cases}$$

where C_{n-m} and C_m are vertex-disjoint directed cycles

Proof. We consider three cases. First, let $m \equiv 0(\bmod 4)$. By (6) and (11), we get

$$\begin{aligned} E(C_{n-m})+E(C_m) &= 2\left(\cot\frac{\pi}{n-m}+\cot\frac{\pi}{m}\right) \\ &\leqslant 2\left(\frac{n-m}{\pi}+\frac{m}{\pi}\right) \\ &= \frac{2n}{\pi}. \end{aligned}$$

Next, we take $m \equiv 2(\bmod 4)$. By (6) and (10), we get

$$\begin{aligned} E(C_{n-m})+E(C_m) &= 2\left(\csc\frac{\pi}{n-m}+\csc\frac{\pi}{m}\right) \\ &\leqslant 2\left(\frac{1}{(\frac{\pi}{n-m})(1-\frac{\pi^2}{6(n-m)^2})}+\frac{1}{(\frac{\pi}{m})(1-\frac{\pi^2}{6m^2})}\right) \\ &= \frac{2n}{\pi}+2\left(\frac{(n-m)\pi}{6(n-m)^2-\pi^2}+\frac{m\pi}{6m^2-\pi^2}\right). \end{aligned}$$

If $m \equiv 1(\bmod 2)$ then one can analogously show that

$$\begin{aligned} E(C_{n-m})+E(C_m) &= \csc\frac{\pi}{2(n-m)}+\csc\frac{\pi}{2m} \\ &\leqslant \frac{2n}{\pi}+2\left(\frac{(n-m)\pi}{24(n-m)^2-\pi^2}+\frac{m\pi}{24m^2-\pi^2}\right). \quad \square \end{aligned}$$

LEMMA 5. *If* $n \equiv 2(\bmod 4)$, $m,\ n-m \geqslant 2$ *then the following holds:*

$$E(C_{n-m})+E(C_m) \leqslant \begin{cases} \frac{2n}{\pi}+\frac{2(n-m)\pi}{6(n-m)^2-\pi^2} & \text{if } m \equiv 0(\bmod 4) \\ \frac{2n}{\pi}+\frac{2m\pi}{6m^2-\pi^2} & \text{if } m \equiv 2(\bmod 4) \\ \frac{2n}{\pi}+\frac{2(n-m)\pi}{24(n-m)^2-\pi^2}+\frac{2m\pi}{24m^2-\pi^2} & \text{if } m \equiv 1(\bmod 2), \end{cases}$$

where C_{n-m} *and* C_m *are vertex-disjoint directed cycles.*

Proof. Firstly, consider the case when $m \equiv 0(\bmod 4)$. It follows from (6), (10) and (11) that

$$\begin{aligned} E(C_{n-m})+E(C_m) &= 2\left(\csc\frac{\pi}{n-m}+\cot\frac{\pi}{m}\right) \\ &\leqslant 2\left(\frac{1}{\left(\frac{\pi}{n-m}-\frac{\pi^3}{6(n-m)^3}\right)}+\frac{m}{\pi}\right) \\ &= \frac{2n}{\pi}+\frac{2(n-m)\pi}{6(n-m)^2-\pi^2}. \end{aligned}$$

Secondly, if $m \equiv 2(\bmod 4)$, then one can analogously prove that

$$\begin{aligned} E(C_{n-m})+E(C_m) &= 2\left(\cot\frac{\pi}{n-m}+\csc\frac{\pi}{m}\right) \\ &\leqslant \frac{2n}{\pi}+\frac{2m\pi}{6m^2-\pi^2}. \end{aligned}$$

Finally, we take $m \equiv 1(\bmod 2)$. By (6) and (10), we get

$$\begin{aligned} E(C_{n-m})+E(C_m) &= \csc\frac{\pi}{2(n-m)}+\csc\frac{\pi}{2m} \\ &\leqslant \frac{1}{\left(\frac{\pi}{2(n-m)}\right)\left(1-\frac{\pi^2}{24(n-m)^2}\right)}+\frac{1}{\left(\frac{\pi}{2m}\right)\left(1-\frac{\pi^2}{24m^2}\right)} \\ &= \frac{2n}{\pi}+2\left(\frac{(n-m)\pi}{24(n-m)^2-\pi^2}+\frac{m\pi}{24m^2-\pi^2}\right). \quad \square \end{aligned}$$

LEMMA 6. *If $n \equiv 1 (\bmod 2)$, m, $n-m \geqslant 2$ then the following holds:*

$$E(C_{n-m})+E(C_m) \leqslant \begin{cases} \frac{2n}{\pi}+\frac{2(n-m)\pi}{24(n-m)^2-\pi^2} & \text{if } m \equiv 0 (\bmod 4) \\ \frac{2n}{\pi}+\frac{2(n-m)\pi}{24(n-m)^2-\pi^2}+\frac{2m\pi}{6m^2-\pi^2} & \text{if } m \equiv 2 (\bmod 4) \\ \frac{2n}{\pi}+\frac{2m\pi}{24m^2-\pi^2} & \text{if } m \equiv 1 (\bmod 2) \text{ and } n-m \equiv 0 (\bmod 4) \\ \frac{2n}{\pi}+\frac{2(n-m)\pi}{6(n-m)^2-\pi^2}+\frac{2m\pi}{24m^2-\pi^2} & \text{if } m \equiv 1 (\bmod 2) \text{ and } n-m \equiv 2 (\bmod 4), \end{cases}$$

where C_{n-m} and C_m are vertex-disjoint directed cycles.

Proof. Let $m \equiv 0 (\bmod 4)$. Then by (6), (10) and (11), we get

$$\begin{aligned} E(C_{n-m})+E(C_m) &= \csc\frac{\pi}{2(n-m)}+2\cot\frac{\pi}{m} \\ &\leqslant \left(\frac{1}{\frac{\pi}{2(n-m)}\left(1-\frac{\pi^2}{24(n-m)^2}\right)}\right)+2\left(\frac{m}{\pi}\right) \\ &= \frac{2n}{\pi}+\frac{2(n-m)\pi}{24(n-m)^2-\pi^2}. \end{aligned}$$

Next, we consider the case when $m \equiv 2 (\bmod 4)$. By (6) and (10), we get

$$\begin{aligned} E(C_{n-m})+E(C_m) &= \csc\frac{\pi}{2(n-m)}+2\csc\frac{\pi}{m} \\ &\leqslant \left(\frac{1}{\frac{\pi}{2(n-m)}\left(1-\frac{\pi^2}{24(n-m)^2}\right)}\right)+2\left(\frac{1}{\frac{\pi}{m}\left(1-\frac{\pi^2}{6m^2}\right)}\right) \\ &= \frac{2n}{\pi}+\frac{2(n-m)\pi}{24(n-m)^2-\pi^2}+\frac{2m\pi}{6m^2-\pi^2}. \end{aligned}$$

Let $m \equiv 1 (\bmod 2)$ and $n-m \equiv 0 (\bmod 4)$. One can analogously prove that

$$\begin{aligned} E(C_{n-m})+E(C_m) &= 2\cot\frac{\pi}{n-m}+\csc\frac{\pi}{2m} \\ &\leqslant \frac{2n}{\pi}+\frac{2m\pi}{24m^2-\pi^2}. \end{aligned}$$

Finally, we let $m \equiv 1 (\bmod 2)$ and $n-m \equiv 2 (\bmod 4)$. By (6) and (10), it is obvious to show that

$$\begin{aligned} E(C_{n-m})+E(C_m) &= 2\csc\frac{\pi}{n-m}+\csc\frac{\pi}{2m} \\ &\leqslant \frac{2(n-m)}{\pi}+\frac{2(n-m)\pi}{6(n-m)^2-\pi^2}+\frac{2m}{\pi}+\frac{2m\pi}{24m^2-\pi^2}. \end{aligned}$$

This completes the proof. □

We see that if $m, n-m \geqslant 5$ then $n \geqslant 10$. Therefore, we prove the following lemma by using Lemmas 3 – 6.

LEMMA 7. *Let* $m, n-m \geqslant 5$. *Then*

$$E(C_{n-2}) + E(C_2) \geqslant E(C_{n-m}) + E(C_m), \tag{12}$$

where C_{n-2} *and* C_2 *are vertex-disjoint directed cycles.*

Proof. First note that $n \geqslant 10$ since $m, n-m \geqslant 5$. We consider three cases:

Case 1. When $n \equiv 0 (\bmod 4)$. In this case, $n-2 \geqslant 10$. This together Lemma 3 gives

$$\begin{aligned} E(C_{n-2}) + E(C_2) &\geqslant \frac{2n}{\pi} - \frac{4}{\pi} - \frac{\pi^2}{3(n-2)^2} + 2 \\ &\geqslant \frac{2n}{\pi} - \frac{4}{\pi} + 2 - \frac{\pi^2}{3(10)^2} \\ &\geqslant \frac{2n}{\pi} + 0.69. \end{aligned} \tag{13}$$

On the other hand, if $m \equiv 0 (\bmod 4)$ then Lemma 4 gives

$$E(C_{n-m}) + E(C_m) \leqslant \frac{2n}{\pi}. \tag{14}$$

The inequality (12) follows from (13) and (14).

If $m \equiv 2 (\bmod 4)$ then $m, n-m \geqslant 6$. By Lemma 1 and Lemma 4, we have

$$\begin{aligned} E(C_{n-m}) + E(C_m) &\leqslant \frac{2n}{\pi} + \frac{2(n-m)\pi}{6(n-m)^2 - \pi^2} + \frac{2m\pi}{6m^2 - \pi^2} \\ &\leqslant \frac{2n}{\pi} + \frac{12\pi}{6^3 - \pi^2} + \frac{12\pi}{6^3 - \pi^2} \\ &\leqslant \frac{2n}{\pi} + 0.37. \end{aligned} \tag{15}$$

The inequality (12) follows from (13) and (15).

If $m \equiv 1 (\bmod 2)$ then $m \geqslant 5$ and $n-m \geqslant 7$. Lemma 1 and Lemma 4 imply

$$\begin{aligned} E(C_{n-m}) + E(C_m) &\leqslant \frac{2n}{\pi} + \frac{2(n-m)\pi}{24(n-m)^2 - \pi^2} + \frac{2m\pi}{24m^2 - \pi^2} \\ &\leqslant \frac{2n}{\pi} + \frac{14\pi}{24(7)^2 - \pi^2} + \frac{10\pi}{24(5)^2 - \pi^2} \\ &\leqslant \frac{2n}{\pi} + 0.10. \end{aligned} \tag{16}$$

The inequality (12) follows from (13) and (16).

Case 2. When $n \equiv 2(\bmod 4)$. In this case, $n-2 \geqslant 8$. It follows from Lemma 3 that

$$\begin{aligned} E(C_{n-2})+E(C_2) &\geqslant \frac{2n}{\pi}-\frac{4}{\pi}+2-\frac{\pi}{(n-2)} \\ &\geqslant \frac{2n}{\pi}-\frac{4}{\pi}-\frac{\pi}{8}+2 \\ &\geqslant \frac{2n}{\pi}+0.33. \end{aligned} \tag{17}$$

If $m \equiv 0(\bmod 4)$ then $n-m \geqslant 6$. Lemma 1 and Lemma 5 imply

$$\begin{aligned} E(C_{n-m})+E(C_m) &\leqslant \frac{2n}{\pi}+\frac{2(n-m)\pi}{6(n-m)^2-\pi^2} \\ &\leqslant \frac{2n}{\pi}+\frac{12\pi}{6^3-\pi^2} \\ &\leqslant \frac{2n}{\pi}+0.19. \end{aligned} \tag{18}$$

Inequalities (17) and (18) give (12).

If $m \equiv 2(\bmod 4)$ then $m \geqslant 6$. Lemma 5 and Lemma 1 give

$$\begin{aligned} E(C_{n-m})+E(C_m) &\leqslant \frac{2n}{\pi}+\frac{2m\pi}{6m^2-\pi^2} \\ &\leqslant \frac{2n}{\pi}+\frac{12\pi}{6^3-\pi^2} \\ &\leqslant \frac{2n}{\pi}+0.19. \end{aligned} \tag{19}$$

Inequalities (17) and (19) imply (12).

If $m \equiv 1(\bmod 2)$ then $m, n-m \geqslant 5$. Lemma 5 and Lemma 1 give

$$\begin{aligned} E(C_{n-m})+E(C_m) &\leqslant \frac{2n}{\pi}+\frac{2(n-m)\pi}{24(n-m)^2-\pi^2}+\frac{2m\pi}{24m^2-\pi^2} \\ &\leqslant \frac{2n}{\pi}+\frac{10\pi}{24(5)^2-\pi^2} \\ &\leqslant \frac{2n}{\pi}+0.06. \end{aligned} \tag{20}$$

Inequalities (17) and (20) imply (12).

Case 3. When $n \equiv 1(\bmod 2)$. In this case, $n-2 \geqslant 9$. From Lemma 3, one can easily see that

$$
\begin{aligned}
E(C_{n-2})+E(C_2) &\geqslant \frac{2(n-2)}{\pi}-\frac{\pi^2}{12(n-2)^2}+2 \\
&\geqslant \frac{2n}{\pi}-\frac{4}{\pi}+2-\frac{\pi^2}{12(9)^2} \\
&\geqslant \frac{2n}{\pi}+0.71.
\end{aligned} \tag{21}
$$

On the other hand, if $m \equiv 0(\bmod 4)$ then $n-m \geqslant 5$. It follows from Lemma 1 and Lemma 6 that

$$
\begin{aligned}
E(C_{n-m})+E(C_m) &\leqslant \frac{2n}{\pi}+\frac{2(n-m)\pi}{24(n-m)^2-\pi^2} \\
&\leqslant \frac{2n}{\pi}+\frac{10\pi}{24(5)^2-\pi^2} \\
&\leqslant \frac{2n}{\pi}+0.06.
\end{aligned} \tag{22}
$$

Inequalities (21) and (22) imply (12).

If $m \equiv 2(\bmod 4)$ then $m \geqslant 6$ and $n-m \geqslant 5$. Lemma 1 and Lemma 6 imply

$$
\begin{aligned}
E(C_{n-m})+E(C_m) &\leqslant \frac{2n}{\pi}+\frac{2(n-m)\pi}{24(n-m)^2-\pi^2}+\frac{2m\pi}{6m^2-\pi^2} \\
&\leqslant \frac{2n}{\pi}+\frac{10\pi}{24(5)^2-\pi^2}+\frac{12\pi}{6^3-\pi^2} \\
&\leqslant \frac{2n}{\pi}+0.24.
\end{aligned} \tag{23}
$$

Inequalities (21) and (23) imply (12).

If $m \equiv 1(\bmod 2)$ and $n-m \equiv 0(\bmod 4)$ then $m \geqslant 5$. Lemma 1 and Lemma 6 give

$$
\begin{aligned}
E(C_{n-m})+E(C_m) &\leqslant \frac{2n}{\pi}+\frac{2m\pi}{24m^2-\pi^2} \\
&\leqslant \frac{2n}{\pi}+\frac{10\pi}{24(5)^2-\pi^2} \\
&\leqslant \frac{2n}{\pi}+0.06.
\end{aligned} \tag{24}
$$

Inequalities (21) and (24) imply (12).

If $m \equiv 1(\bmod 2)$ and $n-m \equiv 2(\bmod 4)$ then $m \geqslant 5$ and $n-m \geqslant 6$. Lemma 1 and Lemma 6 give

$$\begin{aligned} E(C_{n-m})+E(C_m) &\leqslant \frac{2n}{\pi}+\frac{2(n-m)\pi}{6(n-m)^2-\pi^2}+\frac{2m\pi}{24m^2-\pi^2} \\ &\leqslant \frac{2n}{\pi}+\frac{12\pi}{6^3-\pi^2}+\frac{10\pi}{24(5)^2-\pi^2} \\ &\leqslant \frac{2n}{\pi}+0.24. \end{aligned} \tag{25}$$

Inequalities (21) and (25) imply (12). □

Combining Lemma 2 and Lemma 7, we have the following theorem.

THEOREM 3. *Let $m, n-m \geqslant 2$. Then the following holds:*

$$E(C_{n-2})+E(C_2) \geqslant E(C_{n-m})+E(C_m),$$

where C_{n-2}, C_2, C_{n-m} and C_m are vertex-disjoint directed cycles.

Proof. The proof follows from Lemma 2 and Lemma 7. □

Pena and Rada [9] found the unicyclic digraphs which have minimal and maximal energy among all unicyclic digraphs with fixed number of vertices. We address the same problem for the set $\mathscr{D}_n$. The following theorem gives the graphs in $\mathscr{D}_n$ with minimal and maximal energy.

THEOREM 4. *Let $G \in \mathscr{D}_n$ with directed cycles C_{r_1} and C_{r_2}, where $2 \leqslant r_1, r_2 \leqslant n-2$. Then G has minimal energy if $2 \leqslant r_1, r_2 \leqslant 4$ and maximal energy if $r_1 = n-2$ and $r_2 = 2$ or the vice versa.*

Proof. Let $G \in \mathscr{D}_n$ with directed cycles C_{r_1} and C_{r_2}, where $2 \leqslant r_1, r_2 \leqslant n-2$. It follows from (1) and Theorem 2 that

$$E(G) = E(C_{r_1})+E(C_{r_2}) = \sum_{k=0}^{r_1-1}\Big|\cos\frac{2k\pi}{r_1}\Big| + \sum_{k=0}^{r_2-1}\Big|\cos\frac{2k\pi}{r_2}\Big|.$$

If $2 \leqslant r_1, r_2 \leqslant 4$ then $E(C_{r_1}) = E(C_{r_2}) = 2$, that is, $E(G) = 4$. It follows from (4) that $E(G) = 4$ is the minimal energy among all digraphs of $\mathscr{D}_n$.

Next, we let $r_1 = n-2$ and $r_2 = 2$. Take any digraph $H \in \mathscr{D}_n$ with directed cycles C_{s_1} and C_{s_2}, where $2 \leqslant s_1, s_2 \leqslant n-2$. It follows from Theorem 3 that

$$E(G) = E(C_{n-2})+E(C_2) \geqslant E(C_{n-s_1})+E(C_{s_1}).$$

As $n-s_1 \geqslant s_2$, it follows from (5) that

$$E(C_{n-s_1})+E(C_{s_1}) \geqslant E(C_{s_2})+E(C_{s_1}) = E(H).$$

This shows that G has the maximal energy among all digraphs of $\mathscr{D}_n$ if $r_1 = n-2$ and $r_2 = 2$ or the vice versa. □

Conclusion

In this paper, we introduced the set $\mathscr{D}_n$ consisting of n-vertex bicyclic digraphs such that directed cycles in a digraph are vertex-disjoint, where $n \geqslant 4$. We succeeded in finding minimal and maximal energy among bicyclic digraphs in the set $\mathscr{D}_n$. It will be interesting to consider a more general class of bicyclic digraphs and finding the extremal energy.

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