A NOTE ON SOME INEQUALITIES FOR UNITARILY INVARIANT NORMS

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Abstract. In this paper, we obtain an improved inequality for unitarily invariant norms, which was established by Fu and He [J. Math. Inequal. 7 (4), (2013), 727–737].

1. Introduction

Let $M_{m,n}$ be the space of $m \times n$ complex matrices and $M_n = M_{n,n}$. A norm $\| \cdot \|$ is called unitarily invariant norm if $\|UAV\| = \|A\|$ for all $A \in M_n$ and for all unitary matrices $U, V \in M_n$. The Ky Fan $k$-norm $\| \cdot \|_{(k)}$ is defined as

$$\|A\|_{(k)} = \sum_{j=1}^{k} s_j(A), \quad k = 1, \ldots, n,$$

where $s_i(A) (i = 1, \ldots, n)$ are the singular values of $A$ with $s_1(A) \geq \cdots \geq s_n(A)$, which are the eigenvalues of the positive semidefinite matrix $|A| = (AA^*)^{1/2}$, arranged in decreasing order and repeated according to multiplicity. The Schatten $p$-norm $\| \cdot \|_p$ is defined as

$$\|A\|_p = \left( \sum_{j=1}^{n} s_j^p(A) \right)^{1/p} = (\text{tr}|A|^p)^{1/p}, \quad 1 \leq p < \infty.$$

It is known that these norms are unitarily invariant, and it is evident that each unitarily invariant norm is symmetric gauge function of singular values [1].

Let $A, B, X \in M_n$ such that $A$ and $B$ are positive semidefinite. Then, the function

$$\varphi (v) = \|A^vXB^{2-v} + A^{2-v}XB^v\|$$

is convex on $[0,2]$, attains its minimum at $v = 1$, consequently $\varphi (1) \leq \varphi (v)$, which implies that

$$2\|AXB\| \leq \|A^vXB^{2-v} + A^{2-v}XB^v\|, \quad 0 \leq v \leq 2. \quad (1)$$

Zhan proved in [2] that if $A, B, X \in M_n$ such that $A$ and $B$ are positive semidefinite, then

$$\|A^vXB^{2-v} + A^{2-v}XB^v\| \leq \frac{2}{t+2} \|A^2X + tAXB + XB^2\|, \quad (2)$$


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for $\frac{1}{2} \leq v \leq \frac{3}{2}$ and $-2 < t \leq 2$. So it follows from (1) and (2) that

$$2\|AXB\| \leq \|A^vXB^{2-v} + A^{2-v}XB^v\| \leq \frac{2}{t+2}\|A^2X + tAXB + XB^2\|.$$  \hspace{1cm} (3)

Recently, Fu and He [3] obtained an improvement of inequality (3) which can be stated as follows:

$$2\|AXB\| + 2\left(\int_{\frac{1}{2}}^{\frac{3}{2}} \|A^vXB^{2-v} + A^{2-v}XB^v\| \, dv - 2\|AXB\|\right) \leq \frac{2}{t+2}\|A^2X + tAXB + XB^2\|$$  \hspace{1cm} (4)

for $\frac{1}{2} \leq v \leq \frac{3}{2}$ and $-2 < t \leq 2$.

For more information on inequalities for unitarily invariant norms the reader is referred to [2-8].

In this paper, we will give a refinement of inequality (4).

2. Main results

We begin this section with two lemmas.

**Lemma 1.** [4, 5] Let $f$ be a real valued convex function on the interval $[a, b]$ which contains $(x_1, x_2)$. Then for $x_1 \leq x \leq x_2$, we have

$$f(x) \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1}x - \frac{x_1 f(x_2) - x_2 f(x_1)}{x_2 - x_1}. $$

**Lemma 2.** (Hermite-Hadamard Integral Inequality) [7] Let $f$ be a real valued convex function on the interval $[a, b]$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t) \, dt \leq \frac{f(a) + f(b)}{2}. $$

**Theorem 1.** Let $A, B, X \in M_n$ such that $A$ and $B$ are positive semidefinite. Then

$$2\|AXB\| + 4\left(\int_{\frac{1}{2}}^{\frac{3}{2}} \|A^vXB^{2-v} + A^{2-v}XB^v\| \, dv - \|AXB\| - \frac{1}{2}\|A^\frac{3}{2}XB^\frac{3}{2} + A^\frac{3}{2}XB^\frac{3}{2}\|\right) \leq \frac{2}{t+2}\|A^2X + tAXB + XB^2\|,$$

where $\frac{1}{2} \leq v \leq \frac{3}{2}$, $-2 < t \leq 2$.

**Proof.** It is known that if $A, B, X \in M_n$ such that $A$ and $B$ are positive semidefinite, then the function

$$\phi(v) = \|A^vXB^{2-v} + A^{2-v}XB^v\|, \quad 0 \leq v \leq 2$$
is convex on $[0, 2]$, attains its minimum at $v = 1$. It follows that if $\frac{1}{2} \leq v \leq \frac{3}{4}$, then by Lemma 1 and the convexity of function $\varphi$, we have

$$\varphi(v) \leq \frac{\varphi\left(\frac{3}{4}\right) - \varphi\left(\frac{1}{2}\right)}{\frac{3}{4} - 1} v - \frac{1}{\frac{3}{4} - 1} \varphi\left(\frac{3}{4}\right) - \frac{3}{4} \varphi\left(\frac{1}{2}\right),$$

that is,

$$\varphi(v) \leq (3 - 4v) \varphi\left(\frac{1}{2}\right) + 2(2v - 1) \varphi\left(\frac{3}{4}\right).$$

Thus

$$\int_{\frac{1}{2}}^{\frac{3}{4}} \varphi(v) \, dv \leq \varphi\left(\frac{1}{2}\right) \int_{\frac{1}{2}}^{\frac{3}{4}} (3 - 4v) \, dv + 2 \varphi\left(\frac{3}{4}\right) \int_{\frac{1}{2}}^{\frac{3}{4}} (2v - 1) \, dv,$$

which implies

$$\int_{\frac{1}{2}}^{\frac{3}{4}} \varphi(v) \, dv \leq \frac{1}{8} \left[ \varphi\left(\frac{1}{2}\right) + \varphi\left(\frac{3}{4}\right) \right]. \tag{5}$$

If $\frac{3}{4} \leq v \leq 1$, then by Lemma 1 and the convexity of function $\varphi$, we have

$$\varphi(v) \leq \frac{\varphi(1) - \varphi\left(\frac{3}{4}\right)}{1 - \frac{3}{4}} v - \frac{3}{1 - \frac{3}{4}} \varphi(1) - \varphi\left(\frac{3}{4}\right),$$

that is,

$$\varphi(v) \leq 4(1 - v) \varphi\left(\frac{3}{4}\right) + (4v - 3) \varphi(1).$$

Thus

$$\int_{\frac{3}{4}}^{1} \varphi(v) \, dv \leq 4 \varphi\left(\frac{3}{4}\right) \int_{\frac{3}{4}}^{1} (1 - v) \, dv + \varphi(1) \int_{\frac{3}{4}}^{1} (4v - 3) \, dv,$$

which implies

$$\int_{\frac{3}{4}}^{1} \varphi(v) \, dv \leq \frac{1}{8} \left[ \varphi\left(\frac{3}{4}\right) + \varphi(1) \right]. \tag{6}$$

If $1 \leq v \leq \frac{5}{4}$, similarly, we have

$$\varphi(v) \leq \frac{\varphi\left(\frac{5}{4}\right) - \varphi(1)}{\frac{5}{4} - 1} v - \frac{\varphi\left(\frac{5}{4}\right) - \varphi(1)}{\frac{5}{4} - 1},$$

that is,

$$\varphi(v) \leq (5 - 4v) \varphi(1) + 4(v - 1) \varphi\left(\frac{5}{4}\right).$$

Thus

$$\int_{1}^{\frac{5}{4}} \varphi(v) \, dv \leq \varphi(1) \int_{1}^{\frac{5}{4}} (5 - 4v) \, dv + 4 \varphi\left(\frac{5}{4}\right) \int_{1}^{\frac{5}{4}} (v - 1) \, dv,$$
which implies
\[ \int_{1}^{\frac{5}{4}} \varphi(v) dv \leq \frac{1}{8} \left[ \varphi(1) + \varphi\left(\frac{5}{4}\right)\right]. \tag{7} \]

If \( \frac{5}{4} \leq v \leq \frac{3}{2} \), similarly, we have
\[ \varphi(v) \leq \frac{\varphi\left(\frac{3}{2}\right) - \varphi\left(\frac{5}{4}\right)}{\frac{3}{2} - \frac{5}{4}} v - \frac{5}{4} \varphi\left(\frac{3}{2}\right) - \frac{3}{2} \varphi\left(\frac{5}{4}\right), \]
that is,
\[ \varphi(v) \leq (6 - 4v) \varphi\left(\frac{5}{4}\right) + (4v - 5) \varphi\left(\frac{3}{2}\right). \]

Thus
\[ \int_{\frac{5}{4}}^{\frac{3}{2}} \varphi(v) dv \leq \varphi\left(\frac{5}{4}\right) \int_{\frac{5}{4}}^{\frac{3}{2}} (6 - 4v) dv + \varphi\left(\frac{3}{2}\right) \int_{\frac{5}{4}}^{\frac{3}{2}} (4v - 5) dv, \]
which implies
\[ \int_{\frac{5}{4}}^{\frac{3}{2}} \varphi(v) dv \leq \frac{1}{8} \left[ \varphi\left(\frac{5}{4}\right) + \varphi\left(\frac{3}{2}\right)\right]. \tag{8} \]

It follows from (5), (6), (7), (8) and \( \varphi\left(\frac{1}{2}\right) = \varphi\left(\frac{3}{2}\right), \varphi\left(\frac{3}{4}\right) = \varphi\left(\frac{5}{4}\right) \) that
\[ 4 \int_{\frac{1}{2}}^{\frac{3}{2}} \varphi(v) dv \leq \varphi\left(\frac{1}{2}\right) + \varphi(1) + 2 \varphi\left(\frac{3}{4}\right), \]
which is equivalent to
\[ \varphi(1) + 4 \left[ \int_{\frac{1}{2}}^{\frac{3}{2}} \varphi(v) dv - \frac{1}{2} \varphi(1) - \frac{1}{2} \varphi\left(\frac{3}{4}\right) \right] \leq \varphi\left(\frac{1}{2}\right). \]

The last inequality is
\[ 2 \|AXB\| + 4 \left( \int_{\frac{1}{2}}^{\frac{3}{2}} \|A^vXB^{2-v} + A^{2-v}XB^v\| dv - \|AXB\| - \frac{1}{2} \left\| A^{\frac{5}{4}}XB^{\frac{1}{4}} + A^{\frac{3}{4}}XB^{\frac{1}{2}} \right\| \right) \]
\[ \leq \left\| A^{\frac{1}{2}}XB^{\frac{3}{2}} + A^{\frac{3}{4}}XB^{\frac{1}{2}} \right\|. \]

By (2), we get
\[ 2 \|AXB\| + 4 \left( \int_{\frac{1}{2}}^{\frac{3}{2}} \|A^vXB^{2-v} + A^{2-v}XB^v\| dv - \|AXB\| - \frac{1}{2} \left\| A^{\frac{3}{4}}XB^{\frac{1}{4}} + A^{\frac{5}{4}}XB^{\frac{1}{2}} \right\| \right) \]
\[ \leq \frac{2}{t+2} \|A^{2}X + tAXB + XB^{2}\|. \]

This completes the proof. \( \Box \)
Remark 1. Theorem 1 is better than inequality (4). In fact, by Lemma 2, we have
\[ \phi \left( \frac{3}{4} \right) \leq 2 \int_{1/2}^{1} \phi(v) \, dv, \tag{9} \]
\[ \phi \left( \frac{5}{4} \right) \leq 2 \int_{1}^{3/2} \phi(v) \, dv, \tag{10} \]
It follows from (9), (10) and \( \phi \left( \frac{3}{4} \right) = \phi \left( \frac{5}{4} \right) \) that
\[ 2 \phi \left( \frac{3}{4} \right) \leq 2 \int_{1/2}^{3/2} \phi(v) \, dv, \]
that is
\[ \int_{1/2}^{3/2} \phi(v) \, dv \geq \phi \left( \frac{3}{4} \right), \]
where \( \phi(v) = \| A^v X B^{2-v} + A^{2-v} X B^v \| \). Thus
\[ \int_{1/2}^{3/2} \| A^v X B^{2-v} + A^{2-v} X B^v \| \, dv \geq \| A^{3/2} X B^{5/4} + A^{5/4} X B^{3/4} \|. \]
So
\[ 4 \left( \int_{1/2}^{3/2} \| A^v X B^{2-v} + A^{2-v} X B^v \| \, dv - \| A X B \| - \frac{1}{2} \| A^{3/2} X B^{5/4} + A^{5/4} X B^{3/4} \| \right) \]
\[ -2 \left( \int_{1/2}^{3/2} \| A^v X B^{2-v} + A^{2-v} X B^v \| \, dv - 2 \| A X B \| \right) \]
\[ = 2 \left( \int_{1/2}^{3/2} \| A^v X B^{2-v} + A^{2-v} X B^v \| \, dv - \| A^{3/2} X B^{5/4} + A^{5/4} X B^{3/4} \| \right) \geq 0. \]

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