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A NOTE ON SOME INEQUALITIES FOR UNITARILY INVARIANT NORMS

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Abstract. In this paper, we obtain an improved inequality for unitarily invariant norms, which was established by Fu and He [J. Math. Inequal. 7 (4), (2013), 727–737].

1. Introduction

Let $M_{m,n}$ be the space of $m\times n$ complex matrices and $M_n = M_{n,n}$. A norm $\|\cdot\|$ is called unitarily invariant norm if $\|UAV\| = \|A\|$ for all $A\in M_n$ and for all unitary matrices $U,V\in M_n$. The *Ky Fan k-norm* $\|\cdot\|_{(k)}$ is defined as

$$\|A\|_{(k)} = \sum_{j=1}^{k} s_j(A), \quad k=1,\cdots,n,$$

where $s_i(A)\,(i=1,\cdots,n)$ are the singular values of A with $s_1(A)\geqslant\cdots\geqslant s_n(A)$, which are the eigenvalues of the positive semidefinite matrix $|A| = (AA^*)^{\frac{1}{2}}$, arranged in decreasing order and repeated according to multiplicity. The *Schatten p-norm* $\|\cdot\|_p$ is defined as

$$\|A\|_p = \left(\sum_{j=1}^{n} s_j^p(A)\right)^{1/p} = \left(\operatorname{tr}|A|^p\right)^{1/p}, \quad 1\leqslant p<\infty.$$

It is known that these norms are unitarily invariant, and it is evident that each unitarily invariant norm is symmetric gauge function of singular values [1].

Let $A,B,X\in M_n$ such that A and B are positive semidefinite. Then, the function

$$\varphi(v) = \left\|A^vXB^{2-v} + A^{2-v}XB^v\right\|$$

is convex on $[0,2]$, attains its minimum at $v=1$, consequently $\varphi(1)\leqslant\varphi(v)$, which implies that

$$2\|AXB\| \leqslant \left\|A^vXB^{2-v} + A^{2-v}XB^v\right\|, \quad 0\leqslant v\leqslant 2. \tag{1}$$

Zhan proved in [2] that if $A,B,X\in M_n$ such that A and B are positive semidefinite, then

$$\left\|A^vXB^{2-v} + A^{2-v}XB^v\right\| \leqslant \frac{2}{t+2}\left\|A^2X + tAXB + XB^2\right\|, \tag{2}$$

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Paper JMI-09-69

for $\frac{1}{2} \leqslant v \leqslant \frac{3}{2}$ and $-2 < t \leqslant 2$. So it follows from (1) and (2) that

$$2\|AXB\| \leqslant \left\|A^vXB^{2-v} + A^{2-v}XB^v\right\| \leqslant \frac{2}{t+2}\left\|A^2X + tAXB + XB^2\right\|. \quad (3)$$

Recently, Fu and He [3] obtained an improvement of inequality (3) which can be stated as follows:

$$2\|AXB\| + 2\left(\int_{\frac{1}{2}}^{\frac{3}{2}} \left\|A^vXB^{2-v} + A^{2-v}XB^v\right\| dv - 2\|AXB\|\right) \leqslant \frac{2}{t+2}\left\|A^2X + tAXB + XB^2\right\| \quad (4)$$

for $\frac{1}{2} \leqslant v \leqslant \frac{3}{2}$ and $-2 < t \leqslant 2$.

For more information on inequalities for unitarily invariant norms the reader is referred to [2-8].

In this paper, we will give a refinement of inequality (4).

2. Main results

We begin this section with two lemmas.

LEMMA 1. [4, 5] *Let f be a real valued convex function on the interval $[a,b]$ which contains (x_1, x_2). Then for $x_1 \leqslant x \leqslant x_2$, we have*

$$f(x) \leqslant \frac{f(x_2) - f(x_1)}{x_2 - x_1}x - \frac{x_1 f(x_2) - x_2 f(x_1)}{x_2 - x_1}.$$

LEMMA 2. (Hermite-Hadamard Integral Inequality) [7] *Let f be a real valued convex function on the interval $[a,b]$. Then*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(t)\,dt \leqslant \frac{f(a)+f(b)}{2}.$$

THEOREM 1. *Let $A, B, X \in M_n$ such that A and B are positive semidefinite. Then*

$$2\|AXB\| + 4\left(\int_{\frac{1}{2}}^{\frac{3}{2}} \left\|A^vXB^{2-v} + A^{2-v}XB^v\right\| dv - \|AXB\| - \frac{1}{2}\left\|A^{\frac{3}{4}}XB^{\frac{5}{4}} + A^{\frac{5}{4}}XB^{\frac{3}{4}}\right\|\right)$$

$$\leqslant \frac{2}{t+2}\left\|A^2X + tAXB + XB^2\right\|,$$

where $\frac{1}{2} \leqslant v \leqslant \frac{3}{2}$, $-2 < t \leqslant 2$.

Proof. It is known that if $A, B, X \in M_n$ such that A and B are positive semidefinite, then the function

$$\varphi(v) = \left\|A^vXB^{2-v} + A^{2-v}XB^v\right\|, \quad 0 \leqslant v \leqslant 2$$

is convex on $[0,2]$, attains its minimum at $v=1$. It follows that if $\frac{1}{2}\leqslant v\leqslant\frac{3}{4}$, then by Lemma 1 and the convexity of function φ, we have

$$\varphi(v)\leqslant\frac{\varphi\left(\frac{3}{4}\right)-\varphi\left(\frac{1}{2}\right)}{\frac{3}{4}-\frac{1}{2}}v-\frac{\frac{1}{2}\varphi\left(\frac{3}{4}\right)-\frac{3}{4}\varphi\left(\frac{1}{2}\right)}{\frac{3}{4}-\frac{1}{2}},$$

that is,

$$\varphi(v)\leqslant(3-4v)\varphi\left(\frac{1}{2}\right)+2(2v-1)\varphi\left(\frac{3}{4}\right).$$

Thus

$$\int_{\frac{1}{2}}^{\frac{3}{4}}\varphi(v)\,dv\leqslant\varphi\left(\frac{1}{2}\right)\int_{\frac{1}{2}}^{\frac{3}{4}}(3-4v)\,dv+2\varphi\left(\frac{3}{4}\right)\int_{\frac{1}{2}}^{\frac{3}{4}}(2v-1)\,dv,$$

which implies

$$\int_{\frac{1}{2}}^{\frac{3}{4}}\varphi(v)\,dv\leqslant\frac{1}{8}\left[\varphi\left(\frac{1}{2}\right)+\varphi\left(\frac{3}{4}\right)\right].\tag{5}$$

If $\frac{3}{4}\leqslant v\leqslant 1$, then by Lemma 1 and the convexity of function φ, we have

$$\varphi(v)\leqslant\frac{\varphi(1)-\varphi\left(\frac{3}{4}\right)}{1-\frac{3}{4}}v-\frac{\frac{3}{4}\varphi(1)-\varphi\left(\frac{3}{4}\right)}{1-\frac{3}{4}},$$

that is,

$$\varphi(v)\leqslant 4(1-v)\varphi\left(\frac{3}{4}\right)+(4v-3)\varphi(1).$$

Thus

$$\int_{\frac{3}{4}}^{1}\varphi(v)\,dv\leqslant 4\varphi\left(\frac{3}{4}\right)\int_{\frac{3}{4}}^{1}(1-v)\,dv+\varphi(1)\int_{\frac{3}{4}}^{1}(4v-3)\,dv,$$

which implies

$$\int_{\frac{3}{4}}^{1}\varphi(v)\,dv\leqslant\frac{1}{8}\left[\varphi\left(\frac{3}{4}\right)+\varphi(1)\right].\tag{6}$$

If $1\leqslant v\leqslant\frac{5}{4}$, similarly, we have

$$\varphi(v)\leqslant\frac{\varphi\left(\frac{5}{4}\right)-\varphi(1)}{\frac{5}{4}-1}v-\frac{\varphi\left(\frac{5}{4}\right)-\frac{5}{4}\varphi(1)}{\frac{5}{4}-1},$$

that is,

$$\varphi(v)\leqslant(5-4v)\varphi(1)+4(v-1)\varphi\left(\frac{5}{4}\right).$$

Thus

$$\int_{1}^{\frac{5}{4}}\varphi(v)\,dv\leqslant\varphi(1)\int_{1}^{\frac{5}{4}}(5-4v)\,dv+4\varphi\left(\frac{5}{4}\right)\int_{1}^{\frac{5}{4}}(v-1)\,dv,$$

which implies

$$\int_1^{\frac{5}{4}} \varphi(v)\,dv \leqslant \frac{1}{8}\left[\varphi(1)+\varphi\left(\frac{5}{4}\right)\right]. \tag{7}$$

If $\frac{5}{4} \leqslant v \leqslant \frac{3}{2}$, similarly, we have

$$\varphi(v) \leqslant \frac{\varphi\left(\frac{3}{2}\right)-\varphi\left(\frac{5}{4}\right)}{\frac{3}{2}-\frac{5}{4}}v - \frac{\frac{5}{4}\varphi\left(\frac{3}{2}\right)-\frac{3}{2}\varphi\left(\frac{5}{4}\right)}{\frac{3}{2}-\frac{5}{4}},$$

that is,

$$\varphi(v) \leqslant (6-4v)\,\varphi\left(\frac{5}{4}\right)+(4v-5)\,\varphi\left(\frac{3}{2}\right).$$

Thus

$$\int_{\frac{5}{4}}^{\frac{3}{2}} \varphi(v)\,dv \leqslant \varphi\left(\frac{5}{4}\right)\int_{\frac{5}{4}}^{\frac{3}{2}}(6-4v)\,dv+\varphi\left(\frac{3}{2}\right)\int_{\frac{5}{4}}^{\frac{3}{2}}(4v-5)\,dv,$$

which implies

$$\int_{\frac{5}{4}}^{\frac{3}{2}} \varphi(v)\,dv \leqslant \frac{1}{8}\left[\varphi\left(\frac{5}{4}\right)+\varphi\left(\frac{3}{2}\right)\right]. \tag{8}$$

It follows from (5), (6), (7), (8) and $\varphi\left(\frac{1}{2}\right)=\varphi\left(\frac{3}{2}\right)$, $\varphi\left(\frac{3}{4}\right)=\varphi\left(\frac{5}{4}\right)$ that

$$4\int_{\frac{1}{2}}^{\frac{3}{2}} \varphi(v)\,dv \leqslant \varphi\left(\frac{1}{2}\right)+\varphi(1)+2\varphi\left(\frac{3}{4}\right),$$

which is equivalent to

$$\varphi(1)+4\left[\int_{\frac{1}{2}}^{\frac{3}{2}} \varphi(v)\,dv-\frac{1}{2}\varphi(1)-\frac{1}{2}\varphi\left(\frac{3}{4}\right)\right] \leqslant \varphi\left(\frac{1}{2}\right).$$

The last inequality is

$$2\|AXB\|+4\left(\int_{\frac{1}{2}}^{\frac{3}{2}}\left\|A^vXB^{2-v}+A^{2-v}XB^v\right\|dv-\|AXB\|-\frac{1}{2}\left\|A^{\frac{3}{4}}XB^{\frac{5}{4}}+A^{\frac{5}{4}}XB^{\frac{3}{4}}\right\|\right)$$
$$\leqslant \left\|A^{\frac{1}{2}}XB^{\frac{3}{2}}+A^{\frac{3}{2}}XB^{\frac{1}{2}}\right\|.$$

By (2), we get

$$2\|AXB\|+4\left(\int_{\frac{1}{2}}^{\frac{3}{2}}\left\|A^vXB^{2-v}+A^{2-v}XB^v\right\|dv-\|AXB\|-\frac{1}{2}\left\|A^{\frac{3}{4}}XB^{\frac{5}{4}}+A^{\frac{5}{4}}XB^{\frac{3}{4}}\right\|\right)$$
$$\leqslant \frac{2}{t+2}\left\|A^2X+tAXB+XB^2\right\|.$$

This completes the proof. □

REMARK 1. Theorem 1 is better than inequality (4). In fact, by Lemma 2 , we have

$$\varphi\left(\frac{3}{4}\right) \leqslant 2\int_{\frac{1}{2}}^{1} \varphi(v)\,dv, \tag{9}$$

$$\varphi\left(\frac{5}{4}\right) \leqslant 2\int_{1}^{\frac{3}{2}} \varphi(v)\,dv, \tag{10}$$

It follows from (9), (10) and $\varphi\left(\frac{3}{4}\right) = \varphi\left(\frac{5}{4}\right)$ that

$$2\varphi\left(\frac{3}{4}\right) \leqslant 2\int_{\frac{1}{2}}^{\frac{3}{2}} \varphi(v)\,dv,$$

that is

$$\int_{\frac{1}{2}}^{\frac{3}{2}} \varphi(v)\,dv \geqslant \varphi\left(\frac{3}{4}\right),$$

where $\varphi(v) = \left\|A^{v}XB^{2-v} + A^{2-v}XB^{v}\right\|$. Thus

$$\int_{\frac{1}{2}}^{\frac{3}{2}} \left\|A^{v}XB^{2-v} + A^{2-v}XB^{v}\right\| dv \geqslant \left\|A^{\frac{3}{4}}XB^{\frac{5}{4}} + A^{\frac{5}{4}}XB^{\frac{3}{4}}\right\|.$$

So

$$4\left(\int_{\frac{1}{2}}^{\frac{3}{2}} \left\|A^{v}XB^{2-v} + A^{2-v}XB^{v}\right\| dv - \|AXB\| - \frac{1}{2}\left\|A^{\frac{3}{4}}XB^{\frac{5}{4}} + A^{\frac{5}{4}}XB^{\frac{3}{4}}\right\|\right)$$

$$-2\left(\int_{\frac{1}{2}}^{\frac{3}{2}} \left\|A^{v}XB^{2-v} + A^{2-v}XB^{v}\right\| dv - 2\|AXB\|\right)$$

$$= 2\left(\int_{\frac{1}{2}}^{\frac{3}{2}} \left\|A^{v}XB^{2-v} + A^{2-v}XB^{v}\right\| dv - \left\|A^{\frac{3}{4}}XB^{\frac{5}{4}} + A^{\frac{5}{4}}XB^{\frac{3}{4}}\right\|\right) \geqslant 0.$$

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