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REFINEMENTS OF BOUNDS FOR NEUMAN MEANS IN TERMS OF ARITHMETIC AND CONTRAHARMONIC MEANS

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Abstract. In this paper, we present the sharp upper and lower bounds for the Neuman means S_{AC} and S_{CA} in terms of the the arithmetic mean A and contraharmonic mean C. The given results are the improvements of some known results.

1. Introduction

Let $a,b>0$ with $a\neq b$. Then the Schwab-Borchardt mean $SB(a,b)$ [1, 2] of a and b is given by

$$SB(a,b)=\begin{cases}\dfrac{\sqrt{b^2-a^2}}{\cos^{-1}(a/b)}, & a<b,\\[2ex] \dfrac{\sqrt{a^2-b^2}}{\cosh^{-1}(a/b)}, & a>b,\end{cases}$$

where $\cos^{-1}(x)$ and $\cosh^{-1}(x)=\log(x+\sqrt{x^2-1})$ are the inverse cosine and inverse hyperbolic cosine functions, respectively.

It is well known that the Schwab-Borchardt mean $SB(a,b)$ is strictly increasing in both a and b, nonsymmetric and homogeneous of degree 1 with respect to a and b. Many symmetric bivariate means are special cases of the Schwab-Borchardt mean. For example, $P(a,b)=(a-b)/[2\arcsin((a-b)/(a+b))]=SB[G(a,b),A(a,b)]$ is the first Seiffert mean, $T(a,b)=(a-b)/[2\arctan((a-b)/(a+b))]=SB[A(a,b),Q(a,b)]$ is the second Seiffert mean, $M(a,b)=(a-b)/[2\sinh^{-1}((a-b)/(a+b))]=SB[Q(a,b),A(a,b)]$ is the Neuman-Sándor mean, $L(a,b)=(a-b)/[2\tanh^{-1}((a-b)/(a+b))]=SB[A(a,b),G(a,b)]$ is the logarithmic mean, where $G(a,b)=\sqrt{ab}$, $A(a,b)=(a+b)/2$ and $Q(a,b)=\sqrt{(a^2+b^2)/2}$ are the geometric, arithmetic and quadratic means of a and b, respectively. Recently, the Schwab-Borchardt mean and the means derived from the Schwab-Borchardt mean have attracted the attention of numerous mathematicians. In particular, many remarkable inequalities for these means can be found in the literature [3-20].

Let

$$C(a,b)=(a^2+b^2)/(a+b) \tag{1.1}$$

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Paper JMI-09-71

be the contraharmonic means of a and b. Then it is well known that the inequalities

$$G(a,b) < L(a,b) < P(a,b) < A(a,b) < M(a,b) < T(a,b) < Q(a,b) < C(a,b)$$

hold for all $a,b>0$ with $a\neq b$.

Let $X(a,b)$ and $Y(a,b)$ be the symmetric bivariate means of a and b. Then the Neuman mean $S_{XY}(a,b)$ [21, 22] derived from the Schwab-Borchardt mean are given by

$$S_{XY}(a,b) = SB[X(a,b),Y(a,b)].$$

Let $a>b>0$, $v=(a-b)/(a+b)\in(0,1)$, $r\in(0,\log(2+\sqrt{3}))$ and $s\in(0,\pi/3)$ be the parameters such that $\cosh(r)=1/\cos(s)=1+v^2$. Then the following explicit formulas for the Neuman means S_{AC} and S_{CA} can be found in the literature [21].

$$S_{CA}(a,b) = A(a,b)\frac{\sinh(r)}{r}, \quad S_{AC}(a,b) = A(a,b)\frac{\tan(s)}{s}. \tag{1.2}$$

Neuman [21, 22] proved that the double inequalities

$$C^{1/3}(a,b)A^{2/3}(a,b) < S_{CA}(a,b) < \frac{1}{3}C(a,b)+\frac{2}{3}A(a,b), \tag{1.3}$$

$$A^{1/3}(a,b)C^{2/3}(a,b) < S_{AC}(a,b) < \frac{1}{3}A(a,b)+\frac{2}{3}C(a,b) \tag{1.4}$$

hold for all $a,b>0$ with $a\neq b$.

He et al. [23] found the best possible parameters α, β, λ and μ in the interval $[1/2,1]$ such that the double inequalities

$$C[\alpha a+(1-\alpha)b, \alpha b+(1-\alpha)a] < S_{AC}(a,b) < C[\beta a+(1-\beta)b, \beta b+(1-\beta)a],$$
$$C[\lambda a+(1-\lambda)b, \lambda b+(1-\lambda)a] < S_{CA}(a,b) < C[\mu a+(1-\mu)b, \mu b+(1-\mu)a]$$

hold for all $a,b>0$ with $a\neq b$.

In [24], the authors proved that the double inequalities

$$\alpha Q(a,b)+(1-\alpha)T(a,b) < S_{CA}(a,b) < \beta Q(a,b)+(1-\beta)T(a,b)$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\alpha\leqslant 0$ and $\beta\geqslant[\sqrt{3}\pi-4\log(2+\sqrt{3})]/[(\sqrt{2}\pi-4)\log(2+\sqrt{3})]=0.2975\cdots$.

Motivated by inequalities (1.3) and (1.4), it is natural to ask what are best possible parameters α_1, β_1, α_2 and β_2 such that the double inequalities

$$\left[\frac{1}{3}C(a,b)+\frac{2}{3}A(a,b)\right]^{\alpha_1}[C^{1/3}(a,b)A^{2/3}(a,b)]^{1-\alpha_1} < S_{CA}(a,b)$$
$$< \left[\frac{1}{3}C(a,b)+\frac{2}{3}A(a,b)\right]^{\beta_1}[C^{1/3}(a,b)A^{2/3}(a,b)]^{1-\beta_1},$$

$$\left[\frac{1}{3}A(a,b)+\frac{2}{3}C(a,b)\right]^{\alpha_2}[A^{1/3}(a,b)C^{2/3}(a,b)]^{1-\alpha_2} < S_{AC}(a,b)$$
$$< \left[\frac{1}{3}A(a,b)+\frac{2}{3}C(a,b)\right]^{\beta_2}[A^{1/3}(a,b)C^{2/3}(a,b)]^{1-\beta_2}$$

hold for all $a,b>0$ with $a\neq b$? The main purpose of this paper is to answer this question.

2. Lemmas

In order to prove our main results we need two lemmas, which we present in this section.

LEMMA 2.1. *Let* $p\in\mathbb{R}$, $\sigma_0=[3\log 3-6\log(\log(2+\sqrt{3}))-2\log 2]/[10\log 2-6\log 3]=0.7580\cdots$ *and*

$$\phi(x)=2(1-p)^2x^3+(-2p^2-17p+10)x^2+2(-p^2-7p+8)x+2(p^2-5p+4). \quad (2.1)$$

Then the following statements are true:

(1) If $p=4/5$*, then* $\phi(x)<0$ *for all* $x\in(1,2)$*;*

(2) If $p=\sigma_0$*, then there exists* $\lambda_1(=1.3857\cdots)\in(1,2)$ *such that* $\phi(x)>0$ *for* $x\in(1,\lambda_1)$ *and* $\phi(x)<0$ *for* $x\in(\lambda_1,2)$.

Proof. (1) If $p=4/5$, then (2.1) becomes

$$\phi(x)=\frac{2}{25}(x-1)(x^2-60x-16). \quad (2.2)$$

Therefore, Lemma 2.1(1) follows easily from (2.2).

(2) If $p=\sigma_0$, then numerical computations lead to

$$-2p^2-17p+10=-4.0368\cdots<0, \quad (2.3)$$

$$6(3p^2-16p+10)=-2.4316\cdots<0, \quad (2.4)$$

$$\phi(1)=-45p+36=1.8861\cdots>0, \quad (2.5)$$

$$\phi(2)=6p^2-138p+96=-5.1675\cdots<0, \quad (2.6)$$

It follows from (2.1), (2.3) and (2.4) that

$$\begin{aligned}\phi'(x)&=6(1-p)^2x^2+2(-2p^2-17p+10)x+2(-p^2-7p+8) \quad (2.7)\\&<24(1-p)^2+2(-2p^2-17p+10)+2(-p^2-7p+8)\\&=6(3p^2-16p+10)=-2.4316\cdots<0\end{aligned}$$

for $x\in(1,2)$.

Therefore, Lemma 2.1(2) follows from (2.5)–(2.7) and the numerical computations result $\phi(1.3856\cdots)>0$ and $\phi(1.3858\cdots)<0$. $\Box$

LEMMA 2.2. *Let* $q\in\mathbb{R}$, $\tau_0=(9\log 3-6\log\pi-4\log 2)/(6\log 5-6\log 3-4\log 2)=0.8432\cdots$ *and*

$$\varphi(x)=2(q^2-5q+4)x^3+2(-q^2-7q+8)x^2+(-2q^2-17q+10)x+2(1-q)^2. \quad (2.8)$$

Then the following statements are true:

(1) If $q = 4/5$, then $\varphi(x) > 0$ for all $x \in (1,2)$;

(2) If $q = \tau_0$, then there exists $\lambda_2 (= 1.3822\cdots) \in (1,2)$ such that $\varphi(x) < 0$ for $x \in (1,\lambda_2)$ and $\varphi(x) > 0$ for $x \in (\lambda_2, 2)$.

Proof. (1) If $q = 4/5$, then (2.8) becomes

$$\varphi(x) = \frac{2}{25}(x-1)(16x^2+60x-1). \tag{2.9}$$

Therefore, Lemma 2.2(1) follows easily from (2.9).

(2) If $q = \tau_0$, then numerical computations lead to

$$q^2 - 5q + 4 = 0.4947\cdots > 0, \tag{2.10}$$

$$-q^2 - 7q + 8 = 1.3860\cdots > 0, \tag{2.11}$$

$$-25q + 22 = 0.9182\cdots > 0, \tag{2.12}$$

$$\varphi(1) = -45q + 36 = -1.9471\cdots < 0, \tag{2.13}$$

$$\varphi(2) = 6q^2 - 174q + 150 = 7.5379\cdots > 0. \tag{2.14}$$

It follows from (2.8) and (2.10)–(2.12) that

$$\begin{aligned} \varphi'(x) &= 6(q^2-5q+4)x^2 + 4(-q^2-7q+8)x + (-2q^2-17q+10) \\ &> 6(q^2-5q+4) + 4(-q^2-7q+8) + (-2q^2-17q+10) \\ &= 3(-25q+22) > 0 \end{aligned} \tag{2.15}$$

for $x \in (1,2)$.

Therefore, Lemma 2.2(2) follows from (2.13)–(2.15) and the numerical computations result $\varphi(1.3821\cdots) < 0$ and $\varphi(1.3823\cdots) > 0$. □

3. Main results

THEOREM 3.1. *The double inequalities*

$$\left[\frac{1}{3}C(a,b) + \frac{2}{3}A(a,b)\right]^{\alpha_1} [C^{1/3}(a,b)A^{2/3}(a,b)]^{1-\alpha_1} < S_{CA}(a,b)$$
$$< \left[\frac{1}{3}C(a,b) + \frac{2}{3}A(a,b)\right]^{\beta_1} [C^{1/3}(a,b)A^{2/3}(a,b)]^{1-\beta_1}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \leqslant \sigma_0 = [3\log 3 - 6\log(\log(2+\sqrt{3})) - 2\log 2]/[10\log 2 - 6\log 3] = 0.7580\cdots$ and $\beta_1 \geqslant 4/5$.

Proof. Without loss of generality, we assume that $a > b > 0$. Let $v = (a-b)/(a+b)$, $u = v\sqrt{2+v^2}$, $x = \sqrt{1+u^2}$ and $p \in (0,1)$. Then $v \in (0,1)$, $u \in (0,\sqrt{3})$, $x \in (1,2)$, and (1.1) and (1.2) lead to

$$C(a,b) = A(a,b)\sqrt{1+u^2}, \quad S_{CA}(a,b) = A(a,b)\frac{u}{\sinh^{-1}(u)}. \tag{3.1}$$

It follows from (3.1) that

$$\frac{\log[S_{CA}(a,b)]-\log\left[C^{1/3}(a,b)A^{2/3}(a,b)\right]}{\log\left[\frac{1}{3}C(a,b)+\frac{2}{3}A(a,b)\right]-\log\left[C^{1/3}(a,b)A^{2/3}(a,b)\right]} = \frac{\log\frac{u}{\sinh^{-1}(u)}-\frac{1}{3}\log\sqrt{1+u^2}}{\log\left[\frac{1}{3}\sqrt{1+u^2}+\frac{2}{3}\right]-\frac{1}{3}\log\sqrt{1+u^2}}, \tag{3.2}$$

$$\begin{aligned}\log[S_{CA}(a,b)]-p\log\left[\frac{1}{3}C(a,b)+\frac{2}{3}A(a,b)\right]-(1-p)\log\left[C^{1/3}(a,b)A^{2/3}(a,b)\right]\\ =\log\frac{u}{\sinh^{-1}(u)}-p\log\left(\frac{\sqrt{1+u^2}+2}{3}\right)-\frac{(1-p)}{6}\log(1+u^2)\\ =\log\frac{\sqrt{x^2-1}}{\sinh^{-1}(\sqrt{x^2-1})}-p\log\frac{x+2}{3}-\frac{1-p}{3}\log x.\end{aligned} \tag{3.3}$$

Let

$$F(x)=\log\frac{\sqrt{x^2-1}}{\sinh^{-1}(\sqrt{x^2-1})}-p\log\frac{x+2}{3}-\frac{1-p}{3}\log x. \tag{3.4}$$

Then simple computations lead to

$$F(1)=0,\quad F(2)=\log\frac{\sqrt{3}}{\log(2+\sqrt{3})}-p\log\frac{4}{3}-\frac{1-p}{3}\log 2, \tag{3.5}$$

$$F'(x)=\frac{2(1-p)x^3+2(2+p)x^2+(1+2p)x+2(1-p)}{3x(x+2)(x^2-1)\sinh^{-1}(\sqrt{x^2-1})}f(x), \tag{3.6}$$

where

$$f(x)=\sinh^{-1}(\sqrt{x^2-1})-\frac{3x(x+2)\sqrt{x^2-1}}{2(1-p)x^3+2(2+p)x^2+(1+2p)x+2(1-p)}, \tag{3.7}$$

$$f(1)=0,\quad f(2)=\log(2+\sqrt{3})-\frac{4\sqrt{3}}{6-p}, \tag{3.8}$$

$$f'(x)=\frac{2(x+1)(x-1)^2}{\sqrt{x^2-1}[2(1-p)x^3+2(2+p)x^2+(1+2p)x+2(1-p)]^2}\phi(x), \tag{3.9}$$

where $\phi(x)$ is defined as in Lemma 2.1.

We divide the proof into two cases.

Case 1: $p=4/5$. Then it follows easily from Lemma 2.1(1), (3.3)–(3.6), (3.8) and (3.9) that

$$S_{CA}(a,b)<\left[\frac{1}{3}C(a,b)+\frac{2}{3}A(a,b)\right]^{4/5}[C^{1/3}(a,b)A^{2/3}(a,b)]^{1/5}. \tag{3.10}$$

Case 2: $p=\sigma_0=[3\log 3-6\log(\log(2+\sqrt{3}))-2\log 2]/[10\log 2-6\log 3]$. Then from (3.5) and (3.8) together with numerical computations we get

$$F(2)=0,\ \ f(2)=-0.004735\cdots<0. \tag{3.11}$$

Let $\lambda_1=1.3857\cdots$ be the number given in Lemma 2.1(2). We divide the discussion into two subcases.

Subcase 1: $x\in(1,\lambda_1]$. Then Lemma 2.1(2), (3.8) and (3.9) imply that

$$f(x)>0.$$

Subcase 2: $x\in(\lambda_1,2)$. Then Lemma 2.1(2) and (3.9) lead to the conclusion that $f(x)$ is strictly decreasing on the interval $[\lambda_1,2)$. Then from (3.11) and Subcase 1 we know that there exists $\lambda_0\in(\lambda_1,2)$ such that $f(x)>0$ for $x\in[\lambda_1,\lambda_0)$ and $f(x)<0$ for $x\in(\lambda_0,2)$.

It follows from Subcases 1 and 2 together with (3.6) that $F(x)$ is strictly increasing on $(1,\lambda_0]$ and strictly decreasing on $[\lambda_0,2)$. Therefore,

$$S_{CA}(a,b)>\left[\frac{1}{3}C(a,b)+\frac{2}{3}A(a,b)\right]^{\sigma_0}[C^{1/3}(a,b)A^{2/3}(a,b)]^{1-\sigma_0} \tag{3.12}$$

follows from (3.3)–(3.5) and (3.11) together with the piecewise monotonicity of $F(x)$.

Note that

$$\lim_{u\to 0^+}\frac{\log\frac{u}{\sinh^{-1}(u)}-\frac{1}{3}\log\sqrt{1+u^2}}{\log\left[\frac{1}{3}\sqrt{1+u^2}+\frac{2}{3}\right]-\frac{1}{3}\log\sqrt{1+u^2}}=\frac{4}{5}, \tag{3.13}$$

$$\lim_{u\to\sqrt{3}}\frac{\log\frac{u}{\sinh^{-1}(u)}-\frac{1}{3}\log\sqrt{1+u^2}}{\log\left[\frac{1}{3}\sqrt{1+u^2}+\frac{2}{3}\right]-\frac{1}{3}\log\sqrt{1+u^2}}=\sigma_0. \tag{3.14}$$

Therefore, Theorem 3.1 follows from (3.2) and (3.10) together with (3.12)–(3.14). □

THEOREM 3.2. *The double inequalities*

$$\left[\frac{1}{3}A(a,b)+\frac{2}{3}C(a,b)\right]^{\alpha_2}[A^{1/3}(a,b)C^{2/3}(a,b)]^{1-\alpha_2}<S_{AC}(a,b)$$
$$<\left[\frac{1}{3}A(a,b)+\frac{2}{3}C(a,b)\right]^{\beta_2}[A^{1/3}(a,b)C^{2/3}(a,b)]^{1-\beta_2}$$

holds for all $a,b>0$ *with* $a\neq b$ *if and only if* $\alpha_2\leqslant 4/5$ *and* $\beta_2\geqslant(9\log 3-6\log\pi-4\log 2)/(6\log 5-6\log 3-4\log 2)=0.8432\cdots$.

Proof. Without loss of generality, we assume that $a>b>0$. Let $v=(a-b)/(a+b)$, $u=v\sqrt{2+v^2}$, $x=\sqrt{1+u^2}$ and $q\in(0,1)$. Then $v\in(0,1)$, $u\in(0,\sqrt{3})$, $x\in(1,2)$, and (1.2) leads to

$$S_{AC}(a,b)=A(a,b)\frac{u}{\tan^{-1}(u)}. \tag{3.15}$$

It follows from (3.1) and (3.15) that

$$\frac{\log[S_{AC}(a,b)]-\log\left[A^{1/3}(a,b)C^{2/3}(a,b)\right]}{\log\left[\frac{1}{3}A(a,b)+\frac{2}{3}C(a,b)\right]-\log\left[A^{1/3}(a,b)C^{2/3}(a,b)\right]} \tag{3.16}$$

$$=\frac{\log\frac{u}{\tan^{-1}(u)}-\frac{2}{3}\log\sqrt{1+u^2}}{\log\left[\frac{2}{3}\sqrt{1+u^2}+\frac{1}{3}\right]-\frac{2}{3}\log\sqrt{1+u^2}},$$

$$\log[S_{AC}(a,b)]-q\log\left[\frac{1}{3}A(a,b)+\frac{2}{3}C(a,b)\right]-(1-q)\log\left[A^{1/3}(a,b)C^{2/3}(a,b)\right] \tag{3.17}$$

$$=\log\frac{u}{\tan^{-1}(u)}-q\log\left(\frac{2\sqrt{1+u^2}+1}{3}\right)-\frac{(1-q)}{3}\log(1+u^2)$$

$$=\log\frac{\sqrt{x^2-1}}{\tan^{-1}(\sqrt{x^2-1})}-q\log\frac{2x+1}{3}-\frac{2(1-q)}{3}\log x.$$

Let

$$G(x)=\log\frac{\sqrt{x^2-1}}{\tan^{-1}(\sqrt{x^2-1})}-q\log\frac{2x+1}{3}-\frac{2(1-q)}{3}\log x. \tag{3.18}$$

Then simple computations lead to

$$G(1)=0,\quad G(2)=\log\frac{3\sqrt{3}}{\pi}-q\log\frac{5}{3}-\frac{2(1-q)}{3}\log 2, \tag{3.19}$$

$$G'(x)=\frac{2(1-q)x^3+(2q+1)x^2+2(q+2)x+2(1-q)}{3x(2x+1)(x^2-1)\tan^{-1}(\sqrt{x^2-1})}g(x), \tag{3.20}$$

where

$$g(x)=\tan^{-1}(\sqrt{x^2-1})-\frac{3(2x+1)\sqrt{x^2-1}}{2(1-q)x^3+(2q+1)x^2+2(q+2)x+2(1-q)}, \tag{3.21}$$

$$g(1)=0,\quad g(2)=\frac{10\pi-2\pi q-15\sqrt{3}}{6(5-q)}, \tag{3.22}$$

$$g'(x)=\frac{2(x+1)(x-1)^2}{x\sqrt{x^2-1}[2(1-q)x^3+(2q+1)x^2+2(q+2)x+2(1-q)]^2}\varphi(x), \tag{3.23}$$

where $\varphi(x)$ is defined as in Lemma 2.2.

We divide the proof into two cases.

Case 1: $q=4/5$. Then it follows easily from Lemma 2.2(1), (3.18)–(3.20), (3.22) and (3.23) that

$$S_{AC}(a,b)>\left[\frac{1}{3}A(a,b)+\frac{2}{3}C(a,b)\right]^{4/5}[A^{1/3}(a,b)C^{2/3}(a,b)]^{1/5}. \tag{3.24}$$

Case 2: $q=\tau_0=(9\log 3-6\log\pi-4\log 2)/(6\log 5-6\log 3-4\log 2)$. Then from (3.19) and (3.22) together with numerical computations we get

$$G(2)=0,\ \ g(2)=0.0054\cdots>0. \tag{3.25}$$

Let $\lambda_2=1.3822\cdots$ be the number given in Lemma 2.2(2). We divide the discussion into two subcases.

Subcase 1: $x\in(1,\lambda_2]$. Then Lemma 2.2(2), (3.22) and (3.23) imply that

$$g(x)<0.$$

Subcase 2: $x\in(\lambda_2,2)$. Then Lemma 2.2(2) and (3.23) lead to the conclusion that $g(x)$ is strictly increasing on the interval $(\lambda_2,2]$. Then from (3.25) and Subcase 1 we know that there exists $\mu_0\in(\lambda_2,2)$ such that $g(x)<0$ for $x\in(\lambda_2,\mu_0)$ and $g(x)>0$ for $x\in(\mu_0,2)$.

It follows from Subcases 1 and 2 together with (3.20) that $G(x)$ is strictly decreasing on $(1,\mu_0]$ and strictly increasing on $[\mu_0,2)$. Therefore,

$$S_{AC}(a,b)<\left[\frac{1}{3}A(a,b)+\frac{2}{3}C(a,b)\right]^{\tau_0}[A^{1/3}(a,b)C^{2/3}(a,b)]^{1-\tau_0} \tag{3.26}$$

follows from (3.17)–(3.19) and (3.25) together with the piecewise monotonicity of $G(x)$.

Note that

$$\lim_{u\to0^+}\frac{\log\frac{u}{\tan^{-1}(u)}-\frac{2}{3}\log\sqrt{1+u^2}}{\log\left[\frac{2}{3}\sqrt{1+u^2}+\frac{1}{3}\right]-\frac{2}{3}\log\sqrt{1+u^2}}=\frac{4}{5}, \tag{3.27}$$

$$\lim_{u\to\sqrt{3}}\frac{\log\frac{u}{\tan^{-1}(u)}-\frac{2}{3}\log\sqrt{1+u^2}}{\log\left[\frac{2}{3}\sqrt{1+u^2}+\frac{1}{3}\right]-\frac{2}{3}\log\sqrt{1+u^2}}=\tau_0. \tag{3.28}$$

Therefore, Theorem 3.2 follows from (3.16) and (3.24) together with (3.26)–(3.28). □

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