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ON SOME INEQUALITIES FOR THE IDENTRIC, LOGARITHMIC AND RELATED MEANS

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Abstract. We offer new proofs, refinements as well as new results related to classical means of two variables, including the identric and logarithmic means.

1. Introduction

Since last few decades, the inequalities involving the classical means such as arithmetic mean A, geometric mean G, identric mean I, logarithmic mean L and weighted geometric mean S have been studied extensively by numerous authors, see e.g. [1, 2, 4, 7, 8, 15, 16, 17].

For two positive real numbers a and b, we define

$$A = A(a,b) = \frac{a+b}{2}, \quad G = G(a,b) = \sqrt{ab},$$

$$L = L(a,b) = \frac{a-b}{\log(a)-\log(y)}, \quad a \neq b,$$

$$I = I(a,b) = \frac{1}{e}\left(\frac{a^a}{b^b}\right)^{1/(a-b)}, \quad a \neq b,$$

$$S = S(a,b) = (a^a b^b)^{1/(a+b)}.$$

For the historical background of these means we refer the reader to [2, 4, 5, 12, 15, 16, 17]. Generalizations, or related means are studied in [3, 8, 7, 10, 12, 14, 18]. Connections of these means with trigonometric or hyperbolic inequalities are pointed out in [3, 13, 6, 14, 17].

The main results of this paper read as follows:

THEOREM 1.1. *For all distinct positive real numbers a and b, we have*

$$1 < \frac{I}{\sqrt{I(A^2,G^2)}} < \frac{2}{\sqrt{e}}. \tag{1.2}$$

Both bounds are sharp.

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THEOREM 1.3. *For all distinct positive real numbers a and b, we have*

$$1 < \frac{2I^2}{A^2+G^2} < c, \tag{1.4}$$

where $c = 1.14\ldots$. The bounds are best possible.

REMARK 1.5. **A.** The left side of (1.4) may be rewritten also as

$$I > Q(A,G), \tag{1.6}$$

where $Q(x,y) = \sqrt{(x^2+y^2)/2}$ denotes the root square mean of x and y. In 1995, Seiffert [25] proved the first inequality in (1.2) by using series representations, which is rather strong. Now we prove that, (1.6) is a refinement of the first inequality in (1.2). Indeed, by the known relation $I(x,y) < A(x,y) = (x+y)/2$, we can write

$$I(A^2,G^2) < (A^2+G^2)/2 = Q(A,G)^2,$$

so one has:

$$I > Q(A,G) > \sqrt{I(A^2,G^2)}. \tag{1.7}$$

As we have $I(x^2,y^2) > I(x,y)^2$ (see Sándor [15]), hence (1.7) offers also a refinement of

$$I > I(A,G). \tag{1.8}$$

Other refinements of (1.8) have been provided in a paper by Neuman and Sándor [10]. Similar inequalities involving the logarithmic mean, as well as Sándor's means X and Y, we quote [3, 13, 14]. In the second part of paper, similar results will be proved.

B. In 1991, Sándor [16] proved the inequality

$$I > (2A+G)/3. \tag{1.9}$$

It is easy to see that, the left side of (1.4) and (1.9) cannot be compared.

In 2001 Sándor and Trif [21] have proved the following inequality:

$$I^2 < (2A^2+G^2)/3. \tag{1.10}$$

The left side of (1.4) offers a good companion to (1.10). We note that the inequality (1.10) and the right side of (1.4) cannot be compared.

In [25], Seiffert proved the following relation:

$$L(A^2,G^2) > L^2, \tag{1.11}$$

which was refined by Neuman and Sándor [10] (for another proof, see [8]) as follows:

$$L(A,G) > L. \tag{1.12}$$

We will prove with a new method the following refinement of (1.11) and a counterpart of (1.12):

THEOREM 1.13. *We have*

$$L(A^2,G^2) = \frac{(A+G)}{2}L(A,G) > \frac{(A+G)}{2}L > L^2, \tag{1.14}$$

$$L(I,G) < L, \tag{1.15}$$

$$L < L(I,L) < L\cdot(I-L)/(L-G). \tag{1.16}$$

COROLLARY 1.17. *One has*

$$G\cdot I/L < \sqrt{I\cdot G} < L(I,G) < L, \tag{1.18}$$

$$(L(I,G))^2 < L\cdot L(I,G) < L(I^2,G^2) < L\cdot(I+G)/2. \tag{1.19}$$

REMARK 1.20. **A.** Relation (1.18) improves the inequality

$$G\cdot I/L < L(I,G),$$

due to Neuman and Sándor [10]. Other refinements of the inequality

$$L < (I+G)/2 \tag{1.21}$$

are provided in [19].

B. Relation (1.16) is indeed a refinement of (1.21), as the weaker inequality can be written as $(I-L)/(L-G) > 1$, which is in fact (1.21).

The mean S is strongly related to other classical means. For example, in 1993 Sándor [17] discovered the identity

$$S(a,b) = I(a^2,b^2)/I(a,b), \tag{1.22}$$

where I is the identric mean. Inequalities for the mean S may be found in [15, 17, 20].

The following result shows that I and $S(A,G)$ cannot be compared, but this is not true in case of I and $S(Q,G)$. Even a stronger result holds true.

THEOREM 1.23. *None of the inequalities $I > S(A,G)$ or $I < S(A,G)$ holds true. On the other hand, one has*

$$S(Q,G) > A > I, \tag{1.24}$$

$$I(Q,G) < A. \tag{1.25}$$

REMARK 1.26. By (1.24) and (1.25), one could ask if I and $I(Q,G)$ may be compared to each other. It is not difficult to see that, this becomes equivalent to one of the inequalities

$$\frac{y\log y}{y-1} < (\text{or} >)\frac{x}{\tanh(x)}, \quad x > 0, \tag{1.27}$$

where $y = \sqrt{\cosh(2x)}$. By using the Mathematica Software [11], we can show that (1.27) with "$<$" is not true for $x = 3/2$, while (1.27) with "$>$" is not true for $x = 2$.

2. Lemmas and proofs of the main results

The following lemma will be utilized in our proofs.

LEMMA 2.1. *For $b > a > 0$ there exists an $x > 0$ such that*

$$\frac{A}{G} = \cosh(x),\ \frac{I}{G} = e^{x/\tanh(x)-1}. \tag{2.2}$$

Proof. For any $a > b > 0$, one can find an $x > 0$ such that $a = e^x \cdot G$ and $b = e^{-x} \cdot G$. Indeed, it is immediate that such an x is (by considering $a/b = e^{2x}$), $x = (1/2)\log(a/b) > 0$. Now, as $A = G\cdot(e^x + e^{-x})/2 = G\cosh(x)$, we get $A/G = \cosh(x)$. Similarly, we get

$$I = G\cdot(1/e)\exp(x(e^x + e^{-x})/(e^x - e^{-x})),$$

which gives $I/G = e^{x/\tanh(x)-1}$. $\square$

Proof of Theorem 1.1. For $x > 0$, we have $I/G = e^{x/\tanh(x)-1}$ and $A/G = \cosh(x)$ by Lemma 2.1. Since

$$\log(I(a,b)) = \frac{a\log a - b\log b}{a-b} - 1,$$

we get

$$\log(\sqrt{I((A/G)^2,1)}) = \frac{\cosh(x)^2\log(\cosh(x))}{\cosh(x)^2 - 1} - \frac{1}{2}.$$

By using this identity, and taking the logarithms in the second identity of (2.2), the inequality

$$0 < \log(I/G) - \log(\sqrt{I(A/G)^2,1}) < \log 2 - 1/2$$

becomes

$$\frac{1}{2} < f(x) < \log 2, \tag{2.3}$$

where

$$f(x) = \frac{x}{\tanh(x)} - \frac{\log(\cosh(x))}{\tanh(x)^2}.$$

A simple computation (which we omit here) for the derivative of $f(x)$ gives:

$$\sinh(x)^3 f'(x) = 2\cosh(x)\log(\cosh(x)) - x\sinh(x). \tag{2.4}$$

The following inequality appears in [6]:

$$\log(\cosh(x)) > \frac{x}{2}\tanh(x),\ x > 0, \tag{2.5}$$

which gives $f'(x) > 0$, so $f(x)$ is strictly increasing in $(0,\infty)$. As $\lim_{x\to 0} f(x) = 1/2$, and $\lim_{x\to\infty} f(x) = \log 2$, the double inequality (2.3) follows. So we have obtained a new proof of (1.2). $\square$

We note that Seiffert's proof is based on certain infinite series representations. Also, our proof shows that the constants 1 and $2/\sqrt{e}$ in (1.2) are optimal.

LEMMA 2.6. *Let*

$$f(x) = \frac{2x}{\tanh(x)} - \log\left(\frac{\cosh(x)^2+1}{2}\right),\ x > 0.$$

Then

$$2 < f(x) < f(1.606\ldots) = 2.1312\ldots. \tag{2.7}$$

Proof. One has $(\cosh(x)^2+1)/2f'(x) = g(x)$, where

$$g(x) = \sinh(x)\cosh(x)^3 - x\cosh(x)^2 + \sinh(x)\cosh(x) - x$$
$$-\cosh(x)\sinh(x)^3 2\sinh(x)\cosh(x) - x\cosh(x)^2 - x,$$

by remarking that

$$\sinh(x)\cosh(x)^3 - \cosh(x)\sinh(x)^3 = \sinh(x)\cosh(x).$$

Now, a simple computation gives

$$g'(x) = \sinh(x)\cdot(3\sinh(x) - 2x\cosh(x)) = 3\sinh(x)\cosh(x)\cdot k(x),$$

where $k(x) = \tanh(x) - 2x/3$ As it is well known that the function $\tanh(x)/x$ is strictly decreasing, the equation $\tanh(x)/x = 2/3$ can have at most a single solution. As $\tanh(1) = 0.7615\ldots > 2/3$ and $\tanh(3/2) = 0.9051\ldots < 1 = (2/3)\cdot(3/2)$, we find that the equation $k(x) = 0$ has a single solution x_0 in $(1, 3/2)$, and also that $k(x) > 0$ for x in $(0, x_0)$ and $k(x) < 0$ in $(x_0, 3/2)$. This means that the function $g(x)$ is strictly increasing in the interval $(0, x_0)$ and strictly decreasing in (x_0, ∞). As $g(1) = 0.24... > 0$, clearly $g(x_0) > 0$, while $g(2) = -3.01... < 0$ implies that there exists a single zero x_1 of $g(x)$ in $(x_0, 2)$. In fact, as $g(3/2) = 0.21... > 0$, we get that x_1 is in $(3/2, 2)$.

From the above consideration we conclude that $g(x) > 0$ for $x \in (0, x_1)$ and $g(x) < 0$ for $x \in (x_1, \infty)$. Therefore, the point x_1 is a maximum point to the function $f(x)$. It is immediate that $\lim_{x\to 0} f(x) = 2$. On the other hand, we shall compute the limit of f(x) at ∞. Clearly $t = \cosh(x)$ tends to ∞ as x tends to ∞. Since $\log(t^2+1) - \log(t^2) = \log((t^2+1)/t^2)$ tends to $\log 1 = 0$, we have to compute the limit of $l(x) = 2x\cosh(x)/\sinh(x) - 2\log(\cosh(x)) + \log 2$. Here

$$2x\frac{\cosh(x)}{\sinh(x)} - 2\log(\cosh(x)) = 2\log\left(\frac{\exp(x\cosh(x)/\sinh(x))}{\cosh x}\right).$$

Now remark that $(x\cosh(x) - x\sinh(x))/\sinh(x)$ tends to zero, as $x\cosh(x) - x\sinh(x) = x\exp(-x)$. As $\exp(x)/\cosh x$ tends to 2, by the above remarks we get that the the limit of $l(x)$ is $2\log 2 + \log 2 = 3\log 2 > 2$. Therefore, the left side of inequality (2.7) is proved. The right side follows by the fact that $f(x) < f(x_1)$. By Mathematica Software® [11], we can find $x_1 = 1.606\ldots$ and $f(x_1) = 2.1312\ldots$. □

Proof of Theorem 1.3. By Lemma 2.1, one has $(I/G)^2 = \exp(2(x/\tanh(x) - 1))$, while $(A/G)^2 = \cosh(x)^2$, $x > 0$. It is immediate that, the left side of (2.7) implies the left side of (1.4). Now, by the right side of (2.7) one has $I^2 < \exp(c_1)(A^2+G^2)/2$, where $c_1 = f(x_1)-2 = 0.13\cdots$. Since $\exp(0.13\cdots) = 1.14$, we get also the right side of (1.4). $\square$

Proof of Theorem 1.13. The first relation of (1.14) follows from the identity

$$L(x^2,y^2) = ((x+y)/2)\cdot L(x,y),$$

which is a consequence of the definition of logarithmic mean, by letting $x = A$, $y = G$. The second inequality of (1.14) follows by (1.12), while the third one is a consequence of the known inequality

$$L < (A+G)/2. \tag{2.8}$$

A simple proof of (2.8) can be found in [12]. For (1.15), by the definition of logarithmic mean, one has

$$L(I,G) = (I-G)/\log(I/G),$$

and on base of the known identity

$$\log(I/G) = A/L-1$$

(see [15, 22]), we get

$$L(I,G) = ((I-G)/(A-L))L < L,$$

since the inequality $(I-G)/(A-L) < 1$ can be rewritten as

$$I+L < A+G$$

due to Alzer (see [15]).

The first inequality of (1.16) follows by the fact that L is a mean (i.e. if $x<y$ then $x < L(x,y) < y$), and the well known relation $L < I$ (see [15]) For the proof of last relation of (1.16) we will use a known inequality of Sándor ([15]), namely:

$$\log(I/L) > 1-G/L. \tag{2.9}$$

Write now that $L(I,L) = (I-L)/\log(I/L)$, and apply (2.9). Therefore, the proof of (1.16) is finished. $\square$

Proof of Corollary 1.17. The first inequality of (1.18) follows by the well known relation $L > \sqrt{GI}$ (see [2]), while the second relation is a consequence of the classical relation $L(x,y) > G(x,y)$ (see e.g. [15]) applied to $x = I$, $y = G$. The last relation is inequality (1.14).

The first inequality of (1.19) follows by (1.14), while the second one by $L(I^2,G^2) = L(I,G)\cdot(I+G)/2$ and inequality $L < (I+G)/2$. The last inequality follows in the same manner. $\square$

Proof of Theorem 1.23. Since the mean S is homogeneous, the relation $I > S(A,G)$ may be rewritten as $I/G > S(A/G,1)$, so by using logarithm and applying Lemma 2.1, this inequality may be rewritten as

$$\frac{x}{\tanh(x)} - 1 > \frac{\cosh(x)\log(\cosh(x))}{1+\cosh(x)}, \quad x > 0. \tag{2.10}$$

By using Mathematica Software® [11], one can see that inequality (2.10) is not true for $x > 2.284$. Similarly, the reverse inequality of (2.10) is not true, e.g. for $x < 2.2$. These show that, I and $S(A,G)$ cannot be compared to each other. In order to prove inequality (1.24), we will use the following result proved in [20]: The inequality

$$S > Q \tag{2.11}$$

holds true. By writing (2.11) as $S(a,b) > Q(a,b)$ for $a = Q$, $b = G$, and remarking that $Q(a,b) = \sqrt{(a^2+b^2)/2}$ and that $(Q^2+G^2)/2 = A^2$, we get the first inequality of (1.24). The second inequality is well known (see [15] for history and references).

By using $I(a,b) < A(a,b) = (a+b)/2$ for $a = Q$ and $b = G$ we get $I(Q,G) < (Q+G)/2$. On the other hand by inequality $(a+b)/2 < \sqrt{(a^2+b^2)/2}$ and $(Q^2+G^2)/2 = A^2$, inequality (1.25) follows as well. This completes the proof. □

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