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THE ISOPERIMETRIC INEQUALITY AND ITS STABILITY

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Abstract. In this paper, we deals with the isoperimetric-type inequalities for the closed convex curve in the Euclidean plane $\mathbb{R}^2$. In fact we establish a family of parametric inequalities involving the following geometric functionals associated to the given closed convex curve with a simple Fourier series proof: length of the curve, areas of the region included by the curve and the locus of curvature centers, and integral of the curvature radii of the curve and the locus of curvature centers. Using our isoperimetric-type inequalities, we also derive some new geometric Bonnesen-type inequalities. Furthermore, we investigate the stability property of such inequalities (near equality implies curve nearly circular).

1. Introduction and main results

Recall that the classical isoperimetric inequality in the Euclidean plane $\mathbb{R}^2$ states that:

THEOREM 1.1. (Isoperimetric inequality) *If γ is a simple closed curve of length L, enclosing a region of area A, then*

$$L^2 - 4\pi A \geqslant 0, \tag{1}$$

and the equality holds if and only if γ is a circle.

This famous fact was known to the ancient Greeks, but the first mathematical proof was only given in the 19th century by Steiner [1]. Since then, there have been many new proofs, sharpened forms, generalizations, and applications of this famous inequality.

Recently, three interesting reverse isoperimetric inequalities were respectively proved by S. L. Pan and H. Zhang in [2], by X. Gao in [3] and by S. L. Pan and J. N. Yang in [4] as follows:

THEOREM 1.2. (Pan-Zhang) *If γ is a simple closed curve of length L, enclosing a region of area A, then*

$$L^2 \leqslant 4\pi\left(A + |\tilde{A}|\right), \tag{2}$$

where $\tilde{A}$ is the area of the domain enclosed by the locus of curvature centers, and the equality in (2) *holds if and only if γ is a circle.*

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THEOREM 1.3. (Gao) *If γ is a simple closed curve of length L, enclosing a region of area A, then*

$$L^2 \leqslant 4\pi A + \pi|\tilde{A}|, \tag{3}$$

where $\tilde{A}$ is the area of the domain enclosed by the locus of curvature centers, and the equality in (3) *holds if and only if γ is a circle.*

THEOREM 1.4. (Pan-Yang) *Let γ be a $\mathscr{C}_+^2$ closed and strictly convex curve in the Euclidean plane $\mathbb{R}^2$ with length L and enclosing an area A, then*

$$\int_0^{2\pi} \rho(\theta)^2 d\theta \geqslant \frac{L^2 - 2\pi A}{\pi}, \tag{4}$$

where ρ is the radius of curvature and θ is the angle between x-axis and the outward normal vector at the corresponding point p, and the equality in (4) *holds if and only if γ is a circle.*

REMARK 1. It is obvious that if γ is a circle, then the locus of its curvature centers is only a point, and thus its area $\tilde{A} = 0$. Conversely, if $\tilde{A} = 0$, then from the classical isoperimetric inequality (1) and the reverse isoperimetric inequality (2), it follows that the area A and the length L of γ satisfy $L^2 = 4\pi A$, which implies that γ is a circle, and therefore the locus of curvature centers of γ is a point.

In this paper, we firstly prove the following interesting Gage-type isoperimetric inequality:

THEOREM 1.5. (Gage-Type) *Let γ be a $\mathscr{C}_+^2$ closed and strictly convex curve in the Euclidean plane $\mathbb{R}^2$ with length L and enclosing an area A, then*

$$\int_\gamma k^2 ds \geqslant \frac{4\pi L}{4A + |\tilde{A}|}, \tag{5}$$

where k is the curvature of γ, $\tilde{A}$ is the area of the domain enclosed by the locus of curvature centers and the equality in (5) *holds if and only if γ is a circle.*

Then we consider a family of parametric isoperimetric-type inequalities for closed convex plane curves, which is actually an improved version of the reverse isoperimetric inequalities (2), (3) and (4), and one of our main results is as follows:

THEOREM 1.6. (Main Theorem) *Let γ be a $\mathscr{C}_+^2$ closed and strictly convex curve in the Euclidean plane $\mathbb{R}^2$ with length L and enclosing an area A, let $\tilde{A}$ denote the area of the domain enclosed by the locus of curvature centers. Then for arbitrary constants α, β, λ, δ, σ, ω satisfying*

$$\begin{cases} \alpha, \lambda \geqslant 0 \\ 2\beta + 4\pi\delta + \sigma - \alpha \geqslant 0 \\ 8\beta - 8\lambda + 4\omega - \alpha \geqslant 0 \\ 6\beta + 24\lambda + 4\omega - \sigma \geqslant 0, \end{cases} \tag{6}$$

we have

$$\alpha \int_\gamma k^2 ds + \beta \int_0^{2\pi} \rho(\theta)^2 d\theta + \lambda \int_0^{2\pi} \rho_\beta^2(\theta) d\theta + \delta L^2 + \sigma A + \omega |\widetilde{A}| \geqslant 0, \tag{7}$$

where k is the curvature of γ, ρ and ρ_β respectively denote curvature radii of the curve γ and the locus of curvature centers. The equality in (7) *holds if γ is a circle and the parameters $\alpha, \beta, \lambda, \delta, \sigma, \omega$ satisfy*

$$\begin{cases} \alpha = 0 \\ 2\beta + 4\pi\delta + \sigma = 0. \end{cases}$$

Moreover if the equality in (7) *holds and the parameters $\alpha, \beta, \lambda, \delta, \sigma, \omega$ satisfy* (6), *then γ is a circle.*

REMARK 2. When $\alpha = \beta = \lambda = 0$, $\delta = -1$, $\sigma = \omega = 4\pi$ and $\alpha = \beta = \lambda = 0$, $\delta = -1$, $\sigma = 4\pi$, $\omega = \pi$, (6) satisfies clearly and the isoperimetric inequality (7) respectively turns into (2) and (3). If we select $\alpha = 0$, $\beta = 1$, $\lambda = 0$, $\delta = -\frac{1}{\pi}$, $\sigma = 2$, $\omega = 0$, then (6) also satisfies and we obtain (4). Hence (7) can also be regarded as a reverse isoperimetric-type inequality. Furthermore, if we select other values of the parameters $\alpha, \beta, \lambda, \delta, \sigma, \omega$ satisfying (6), then we can obtain some new geometric Bonnesen-type inequalities [5]:

THEOREM 1.7. *Let γ be a $\mathscr{C}_+^2$ closed and strictly convex curve in the Euclidean plane $\mathbb{R}^2$ with length L and enclosing an area A, let $\tilde{A}$ denote the area of the domain enclosed by the locus of curvature centers. Then we have*

$$\int_0^{2\pi} \rho(\theta)^2 d\theta \geqslant \frac{L^2}{\pi} - 2A + |\tilde{A}|, \tag{8}$$

$$\max_{\theta \in [0,2\pi]} \rho(\theta)^2 \geqslant \frac{1}{2\pi} \left(\frac{L^2}{\pi} - 2A + |\tilde{A}| \right), \tag{9}$$

$$10A \leqslant \int_\gamma k^2 ds + \int_0^{2\pi} \rho_\beta^2(\theta) d\theta + \frac{9}{4} |\widetilde{A}| \tag{10}$$

and

$$2L^2 \leqslant \int_0^{2\pi} \rho(\theta)^2 d\theta + \int_0^{2\pi} \rho_\beta^2(\theta) d\theta + 30A, \tag{11}$$

where k is the curvature of γ, ρ and ρ_β respectively denote curvature radii of the curve γ and the locus of curvature centers. Moreover, (8) *is actually an improved and sharp version of* (4). *The equalities in* (8) *and* (9) *hold if γ is a circle. Furthermore, if the equalities in* (8) *and* (9) *hold, then the Minkowski support function of γ is of the form*

$$p(\theta) = a_0 + a_1 \cos\theta + b_1 \sin\theta + a_2 \cos 2\theta + b_2 \sin 2\theta,$$

if the equalities in (10) *and* (11) *hold, then γ is a circle.*

REMARK 3. We can actually derive more new and interesting geometric Bonnesen-type inequalities by selecting the appropriate parameters α, β, λ, δ, σ, ω satisfying (6).

The stability problem associated with isoperimetric inequality is also interesting and significant. A well-known and the most frequently used example is the Steiner disc $S(K)$ (see section 4 for the definition of $S(K)$).

Recently in [6], S. L. Pan and H. P. Xu obtained the following stability estimates for the reverse isoperimetric inequality (2) by comparing a convex body K with its Steiner disk $S(K)$:

$$\begin{aligned} h_1(K,S(K))^2 &= \left(\max_u \left|p_K(u) - p_{S(K)}(u)\right|\right)^2 \\ &\leqslant \frac{4\pi^2-33}{96\pi^2}\left(4\pi\left(A(K)+|\tilde{A}(K)|\right)-L^2(K)\right) \end{aligned}$$

and

$$\begin{aligned} h_2(K,S(K))^2 &= \int_0^{2\pi} \left|p_K(\theta) - p_{S(K)}(\theta)\right|^2 d\theta \\ &\leqslant \frac{1}{18\pi}\left(4\pi\left(A(K)+|\tilde{A}(K)|\right)-L^2(K)\right), \end{aligned}$$

where $p_K(\theta)$ denotes the Minkowski support function of a given domain K, and $S(K)$ denotes the Steiner disc associated with K which satisfies

$$4\pi\left(A(S(K))+|\tilde{A}(S(K))|\right)-L^2(S(K))=0.$$

For arbitrary $\varepsilon > 0$ such that

$$\varphi(K) = 4\pi\left(A(K)+|\tilde{A}(K)|\right)-L^2(K) < \varepsilon,$$

by the stability estimates for inequality above it follows that

$$\max\left\{h_1(K,S(K))^2, h_2(K,S(K))^2\right\} \leqslant C|\varphi(K)-\varphi(S(K))| < C\varepsilon,$$

which implies that the reverse isoperimetric inequality (2) does have well stability property with respect to both Hausdorff distance and L^2-metric.

In this paper, we will also research the stability properties of our isoperimetric inequality (7) with respect to both Hausdorff distance and L^2-metric. The paper is organized as follows. In section 2, we recall some basic facts about the plane convex geometry. In section 3, we firstly prove the Gage-Type isoperimetric inequality (5), and then provide a simpler proof of Theorem 1.6 by using Fourier series, which is different from the approach in [2] and [4]. In section 4, we investigate stability properties of inequality (7) (near equality implies curve nearly circular). We believe that our trick could be used to derive more interesting isoperimetric inequalities.

2. Geometric quantities and their Fourier series

In this section, we recall some basic facts about the convex plane curve which will be used later. In this paper we always assume that γ is a closed and convex plane curve which is sufficiently regular, actually it should be a $\mathscr{C}_+^2$ closed and strictly convex curve in the plane $\mathbb{R}^2$, such that the curvature radii discussed can be defined and the Fourier series needed in the proof convergent uniformly. The details can be found in the classical literature [7].

Let $p(\theta)$ denote the Minkowski support function of curve $\gamma(\theta)$, where θ is the angle between x-axis and the outward normal vector at the corresponding point p. It gives us the parametrization of $\gamma(\theta)$ in terms of θ as follows:

$$\gamma(\theta) = (\gamma_1(\theta), \gamma_2(\theta)) = \left(p(\theta)\cos\theta - p'(\theta)\sin\theta, p(\theta)\sin\theta + p'(\theta)\cos\theta\right).$$

Therefore the curvature $k(\theta)$ and the radius of curvature $\rho(\theta)$ of $\gamma(\theta)$ can be calculated by

$$k(\theta) = \frac{d\theta}{ds} = \frac{1}{p(\theta)+p''(\theta)} > 0$$

and

$$\rho(\theta) = \frac{ds}{d\theta} = p(\theta)+p''(\theta) > 0,$$

where we use the fact that γ is a strictly convex plane curve. The length L of $\gamma(\theta)$ and the area A it bounds can be also calculated respectively by

$$L = \int_\gamma ds = \int_0^{2\pi} p(\theta)d\theta$$

and

$$A = \frac{1}{2}\int_\gamma p(\theta)\,ds = \frac{1}{2}\int_0^{2\pi}\left(p(\theta)^2 - p'(\theta)^2\right)d\theta.$$

At the same time, we could obtain the locus of centers of curvature of $\gamma(\theta)$ as follows

$$\begin{aligned}\beta(\theta) &= \gamma(\theta)+\rho(\theta)N(\theta)\\ &= \left(-p'(\theta)\sin\theta - p''(\theta)\cos\theta, p'(\theta)\cos\theta - p''(\theta)\sin\theta\right),\end{aligned}$$

then

$$\beta'(\theta) = -\left(p'(\theta)+p'''(\theta)\right)(\cos\theta, \sin\theta).$$

Therefore the curvature $k_\beta(\theta)$ and the radius of curvature $\rho_\beta(\theta)$ of the locus of curvature centers $\beta(\theta)$ can be calculated by

$$k_\beta(\theta) = \frac{d\theta}{ds} = \frac{1}{p'(\theta)+p'''(\theta)},$$

$$\rho_\beta(\theta) = \frac{ds}{d\theta} = p'(\theta)+p'''(\theta)$$

and

$$\int_0^{2\pi} \rho_\beta^2(\theta)\, d\theta = \int_0^{2\pi} \left(p'(\theta) + p'''(\theta)\right)^2 d\theta.$$

Moreover, the oriented area of the domain enclosed by $\beta(\theta)$ is given by

$$\tilde{A} = \frac{1}{2}\int_0^{2\pi} \left(p'(\theta)^2 - p''(\theta)^2\right) d\theta.$$

Since the Minkowski support function of a given domain K is always continuous, bounded and 2π-periodic, it has a Fourier series of the form

$$p(\theta) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta). \tag{12}$$

Differentiation of (12) with respect to θ gives us

$$p'(\theta) = \sum_{n=1}^{\infty} n(-a_n \sin n\theta + b_n \cos n\theta), \tag{13}$$

$$p''(\theta) = -\sum_{n=1}^{\infty} n^2 (a_n \cos n\theta + b_n \sin n\theta) \tag{14}$$

and

$$p'''(\theta) = -\sum_{n=1}^{\infty} n^3 (-a_n \sin n\theta + b_n \cos n\theta). \tag{15}$$

Thus by (12), (13), (14), (15) and the Parseval equality we could express these geometric quantities in terms of the Fourier coefficients of $p(\theta)$ as follows

$$\begin{aligned} \rho(\theta) &= p(\theta) + p''(\theta) \\ &= a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) - \sum_{n=1}^{\infty} n^2 (a_n \cos n\theta + b_n \sin n\theta), \end{aligned}$$

$$\begin{aligned} \int_0^{2\pi} \rho(\theta)^2 d\theta &= 2\left(\pi a_0^2 - \frac{\pi}{2}\sum_{n=2}^{\infty} \left(n^2-1\right)\left(a_n^2+b_n^2\right) + \frac{\pi}{2}\sum_{n=2}^{\infty} n^2\left(n^2-1\right)\left(a_n^2+b_n^2\right)\right) \\ &= 2\pi\left(a_0^2 + \frac{1}{2}\sum_{n=2}^{\infty}\left(n^2-1\right)^2\left(a_n^2+b_n^2\right)\right), \end{aligned}$$

$$L = 2\pi a_0, \tag{16}$$

$$A = \pi a_0^2 - \frac{\pi}{2}\sum_{n=2}^{\infty}\left(n^2-1\right)\left(a_n^2+b_n^2\right), \tag{17}$$

$$|\tilde{A}| = \frac{\pi}{2}\sum_{n=2}^{\infty} n^2\left(n^2-1\right)\left(a_n^2+b_n^2\right) \tag{18}$$

and

$$\begin{aligned}
&\int_0^{2\pi} \rho_\beta^2(\theta)\,d\theta \\
=&\int_0^{2\pi} \left(p'(\theta)+p'''(\theta)\right)^2 d\theta \\
=&\int_0^{2\pi} \left(\sum_{n=1}^{\infty} n\left(-a_n\sin n\theta+b_n\cos n\theta\right)-\sum_{n=1}^{\infty} n^3\left(-a_n\sin n\theta+b_n\cos n\theta\right)\right)^2 d\theta \\
=&\int_0^{2\pi} \left(\sum_{n=1}^{\infty} n\left(n^2-1\right)\left(-a_n\sin n\theta+b_n\cos n\theta\right)\right)^2 d\theta \\
=&\pi\sum_{n=1}^{\infty} n^2\left(n^2-1\right)^2\left(a_n^2+b_n^2\right).
\end{aligned}$$

3. Proof of the main theorems

In this section, we firstly prove the Gage-Type isoperimetric inequality (5).

Proof of Theorem 1.5. Applying the Hölder inequality we have

$$\begin{aligned}
2\pi &= \int_0^{2\pi} \frac{1}{\sqrt{p(\theta)+p''(\theta)}}\sqrt{p(\theta)+p''(\theta)}\,d\theta \\
&\leqslant \sqrt{\int_0^{2\pi} \frac{1}{p(\theta)+p''(\theta)}\,d\theta}\sqrt{\int_0^{2\pi} \left(p(\theta)+p''(\theta)\right)d\theta},
\end{aligned}$$

then

$$\begin{aligned}
4\pi^2 &\leqslant \int_0^{2\pi} \frac{1}{p(\theta)+p''(\theta)}\,d\theta \int_0^{2\pi} \left(p(\theta)+p''(\theta)\right)d\theta \\
&= \int_0^{2\pi} k(\theta)\,d\theta \int_0^{2\pi} \frac{1}{k(\theta)}\,d\theta \\
&= \int_\gamma k^2 ds \int_\gamma ds \\
&= L\int_\gamma k^2 ds.
\end{aligned}$$

Together with the reverse isoperimetric inequality (3) we have

$$\left(A+\frac{1}{4}|\widetilde{A}|\right)\int_\gamma k^2 ds \geqslant \frac{4\pi^2\left(A+\frac{1}{4}|\widetilde{A}|\right)}{L} = \frac{\pi\left(4A+|\widetilde{A}|\right)}{L} \geqslant \frac{\pi L^2}{L} = \pi L, \tag{19}$$

which implies that

$$\int_\gamma k^2 ds \geqslant \frac{4\pi L}{4A+|\tilde{A}|}.$$

Moreover, if γ is a circle, by the equality condition in (3), it follows that the equality in (5) holds clearly. Conversely, if the equality in (5) holds, then we actually have the equalities in (19) holds. That is to say that $L^2 = \pi\left(4A + |\widetilde{A}|\right)$, then by the equality conditions in (3), we have γ is a circle. $\square$

Now we turn to prove our main result Theorem 1.6.

Proof of Theorem 1.6. Firstly by the mean value inequality and the Gage-Type isoperimetric inequality (5) we have

$$A + \frac{1}{4}|\widetilde{A}| + \int_\gamma k^2 ds \geqslant 2\sqrt{\left(A + \frac{1}{4}|\widetilde{A}|\right)\int_\gamma k^2 ds} \geqslant 2\sqrt{\pi L},$$

thus to prove (7), we only need to prove that the following inequality satisfies under the condition (6):

$$\begin{aligned}
&\alpha\left(\int_\gamma k^2 ds + A + \frac{1}{4}|\widetilde{A}|\right) + \beta\int_0^{2\pi}\rho(\theta)^2 d\theta + \lambda\int_0^{2\pi}\rho_\beta^2(\theta)d\theta \\
&+\delta L^2 + (\sigma-\alpha)A + \left(\omega - \frac{\alpha}{4}\right)|\widetilde{A}| \\
\geqslant& 2\alpha\sqrt{\pi L} + \beta\int_0^{2\pi}\rho(\theta)^2 d\theta + \lambda\int_0^{2\pi}\rho_\beta^2(\theta)d\theta + \delta L^2 + (\sigma-\alpha)A + \left(\omega - \frac{\alpha}{4}\right)|\widetilde{A}| \\
\geqslant& 0
\end{aligned}$$

Then by using the expression of the geometric quantities in terms of the Fourier coefficients of $p(\theta)$ in section 2, we have

$$\begin{aligned}
&2\alpha\sqrt{\pi L} + \beta\int_0^{2\pi}\rho(\theta)^2 d\theta + \lambda\int_0^{2\pi}\rho_\beta^2(\theta)d\theta + \delta L^2 + (\sigma-\alpha)A + \left(\omega - \frac{\alpha}{4}\right)|\widetilde{A}| \\
=& 2\sqrt{2}\pi\alpha\sqrt{a_0} + \beta 2\pi\left(a_0^2 + \frac{1}{2}\sum_{n=2}^{\infty}\left(n^2-1\right)^2\left(a_n^2+b_n^2\right)\right) + \lambda\pi\sum_{n=2}^{\infty}n^2\left(n^2-1\right)^2\left(a_n^2+b_n^2\right) \\
&+\delta(2\pi a_0)^2 + (\sigma-\alpha)\left(\pi a_0^2 - \frac{\pi}{2}\sum_{n=2}^{\infty}\left(n^2-1\right)\left(a_n^2+b_n^2\right)\right) \\
&+\left(\omega - \frac{\alpha}{4}\right)\frac{\pi}{2}\sum_{n=2}^{\infty}n^2\left(n^2-1\right)\left(a_n^2+b_n^2\right) \\
=& 2\sqrt{2}\pi\alpha\sqrt{a_0} + \left(2\pi\beta + 4\pi^2\delta + \pi(\sigma-\alpha)\right)a_0^2 + \pi\beta\sum_{n=2}^{\infty}\left(n^2-1\right)^2\left(a_n^2+b_n^2\right) \\
&+\lambda\pi\sum_{n=2}^{\infty}n^2\left(n^2-1\right)^2\left(a_n^2+b_n^2\right) - \frac{\pi}{2}(\sigma-\alpha)\sum_{n=2}^{\infty}\left(n^2-1\right)\left(a_n^2+b_n^2\right) \\
&+\frac{\pi}{2}\left(\omega - \frac{\alpha}{4}\right)\sum_{n=2}^{\infty}n^2\left(n^2-1\right)\left(a_n^2+b_n^2\right)
\end{aligned}$$

$$=2\sqrt{2}\pi\alpha\sqrt{a_0}+\left(2\pi\beta+4\pi^2\delta+\pi(\sigma-\alpha)\right)a_0^2$$
$$+\pi\sum_{n=2}^{\infty}\left(\left(\beta+\lambda n^2\right)\left(n^2-1\right)-\frac{\sigma-\alpha}{2}+\frac{1}{2}\left(\omega-\frac{\alpha}{4}\right)n^2\right)\left(n^2-1\right)\left(a_n^2+b_n^2\right),$$

where we use the fact that $a_0=\frac{L}{2\pi}\geqslant 0$. Thus it follows from (6) that

$$2\sqrt{2}\pi\alpha\sqrt{a_0}+\left(2\pi\beta+4\pi^2\delta+\pi(\sigma-\alpha)\right)a_0^2\geqslant 0$$

and

$$\pi\sum_{n=2}^{\infty}\left(\left(\beta+\lambda n^2\right)\left(n^2-1\right)-\frac{\sigma-\alpha}{2}+\frac{1}{2}\left(\omega-\frac{\alpha}{4}\right)n^2\right)\left(n^2-1\right)\left(a_n^2+b_n^2\right)$$
$$\geqslant\pi\sum_{n=2}^{\infty}\left(3(\beta+4\lambda)-\frac{\sigma-\alpha}{2}+2\left(\omega-\frac{\alpha}{4}\right)\right)3\left(a_n^2+b_n^2\right)$$
$$=\frac{3}{2}\pi\sum_{n=2}^{\infty}(6\beta+24\lambda-\sigma+4\omega)\left(a_n^2+b_n^2\right)$$
$$\geqslant 0,$$

which implies that

$$2\alpha\sqrt{\pi L}+\beta\int_0^{2\pi}\rho(\theta)^2\,d\theta+\lambda\int_0^{2\pi}\rho_\beta^2(\theta)\,d\theta+\delta L^2+(\sigma-\alpha)A+\left(\omega-\frac{\alpha}{4}\right)|\widetilde{A}|\geqslant 0.$$

Then we completes the proof of inequality (7).

Furthermore, if γ is a circle, then the locus of its curvature centers is only a point, and thus its area $\tilde{A}=0$ and the curvature radius of the locus of curvature centers $\rho_\beta(\theta)=0$, together with the equality conditions in (2) and (4) we have

$$L^2=4\pi\left(A+|\tilde{A}|\right)=4\pi A$$

and

$$\int_0^{2\pi}\rho(\theta)^2 d\theta=\frac{L^2-2\pi A}{\pi}=2A.$$

Hence

$$\alpha\int_\gamma k^2 ds+\beta\int_0^{2\pi}\rho(\theta)^2\,d\theta+\lambda\int_0^{2\pi}\rho_\beta^2(\theta)\,d\theta+\delta L^2+\sigma A+\omega|\widetilde{A}|$$
$$=\alpha\int_\gamma k^2 ds+2\beta A+4\pi\delta A+\sigma A,$$

then for the parameters $\alpha,\beta,\lambda,\delta,\sigma,\omega$ satisfy

$$\begin{cases}\alpha=0\\ 2\beta+4\pi\delta+\sigma=0,\end{cases}$$

we have

$$\alpha \int_\gamma k^2 ds + \beta \int_0^{2\pi} \rho(\theta)^2 d\theta + \lambda \int_0^{2\pi} \rho_\beta^2(\theta) d\theta + \delta L^2 + \sigma A + \omega |\widetilde{A}| = 0.$$

On the other hand, if the equality in (7) holds, the inequalities in

$$\begin{aligned}
&\alpha \left(\int_\gamma k^2 ds + A + \frac{1}{4}|\widetilde{A}| \right) + \beta \int_0^{2\pi} \rho(\theta)^2 d\theta + \lambda \int_0^{2\pi} \rho_\beta^2(\theta) d\theta \\
&+ \delta L^2 + (\sigma - \alpha) A + \left(\omega - \frac{\alpha}{4} \right) |\widetilde{A}| \\
\geqslant & 2\alpha \sqrt{\pi L} + \beta \int_0^{2\pi} \rho(\theta)^2 d\theta + \lambda \int_0^{2\pi} \rho_\beta^2(\theta) d\theta + \delta L^2 + (\sigma - \alpha) A + \left(\omega - \frac{\alpha}{4} \right) |\widetilde{A}|,
\end{aligned}$$

are all equalities, in particular we have

$$\left(A + \frac{1}{4}|\widetilde{A}| \right) \int_\gamma k^2 ds = \pi L.$$

By the equality condition in (5) we have γ is a circle. Then we complete the proof of Theorem 1.6. □

Proof of Theorem 1.7. We prove these geometric Bonnesen-type inequalities by selecting the appropriate parameters $\alpha,\ \beta,\ \lambda,\ \delta,\ \sigma,\ \omega$ satisfying (6). Let $\alpha = 1$, $\beta = -\frac{1}{\pi}$, $\lambda = 2$, $\delta = -1$, we obtain (8). Moreover since

$$\begin{aligned}
&\int_0^{2\pi} \rho(\theta)^2 d\theta - \left(\frac{L^2}{\pi} - 2A + |\tilde{A}| \right) \\
=& \frac{\pi}{2} \sum_{n=2}^{\infty} \left(2\left(n^2 - 1\right) - 2 - n^2 \right) \left(n^2 - 1\right) \left(a_n^2 + b_n^2\right) \\
=& \frac{\pi}{2} \sum_{n=3}^{\infty} \left(n^2 - 4\right) \left(n^2 - 1\right) \left(a_n^2 + b_n^2\right)
\end{aligned}$$

and the coefficient a_n, and b_n deeply depend on the curve γ we choose, thus the parameters $\alpha = 1$, $\beta = -\frac{1}{\pi}$, $\lambda = 2$, $\delta = -1$ such that the inequality (8) is actually an improved and sharp version of (4). Moreover, by using (8), the inequality (9) satisfies clearly.

Furthermore if γ is a circle, by the equality conditions in (2) and (4), it follows that the equalities in (8) and (9) hold apparently. Conversely, if the equality in (8) holds, since

$$\int_0^{2\pi} \rho(\theta)^2 d\theta - \left(\frac{L^2}{\pi} - 2A + |\tilde{A}| \right) = \frac{\pi}{2} \sum_{n=3}^{\infty} \left(n^2 - 4\right) \left(n^2 - 1\right) \left(a_n^2 + b_n^2\right),$$

we have $a_n = b_n = 0$ for $n \geqslant 3$ and the Minkowski support function of γ is of the form $p(\theta) = a_0 + a_1 \cos\theta + b_1 \sin\theta + a_2 \cos 2\theta + b_2 \sin 2\theta$.

Moreover, since

$$\max_{\theta\in[0,2\pi]}\rho(\theta)^2 \geqslant \frac{1}{2\pi}\int_0^{2\pi}\rho(\theta)^2 d\theta \geqslant \frac{1}{2\pi}\left(\frac{L^2}{\pi}-2A+|\tilde{A}|\right),$$

if the equality in (9) holds, we have

$$\int_0^{2\pi}\rho(\theta)^2 d\theta = \frac{L^2}{\pi}-2A+|\tilde{A}|.$$

By the equality condition in (8), it follows that the Minkowski support function of γ is of the form $p(\theta)=a_0+a_1\cos\theta+b_1\sin\theta+a_2\cos 2\theta+b_2\sin 2\theta$.

On the other hand, let $\alpha=1,\ \beta=0,\ \lambda=1,\ \delta=1,\ \sigma=-10,\ \omega=\frac{9}{4}$, we obtain (10), let $\alpha=0,\ \beta=1,\ \lambda=1,\ \delta=-2,\ \sigma=30,\ \omega=0$, we can derive (11). Furthermore, if the equalities in (10) and (11) hold, then by using the equality conditions in (7), we have γ is a circle. □

4. The stability properties of the isoperimetric inequality

Let K and M be two convex domains with respective Minkowski support functions p_K and p_M. The most frequently used function to measure the deviation between K and M is the Hausdorff distance

$$h_1(K,M)=\max_u |p_K(u)-p_M(u)|.$$

Another such measure which appears to be of particular value with respect to stability problems is the measure that corresponds to the L^2-metric in function space, which is defined by

$$h_2(K,M)=\left(\int_0^{2\pi}|p_K(\theta)-p_M(\theta)|^2 d\theta\right)^{\frac{1}{2}},$$

where θ is the angle between x-axis and the outward normal vector at the corresponding point p. It is obvious that $h_1(K,M)=0$ or $h_2(K,M)=0$ if and only if $K=M$.

The definition of Steiner disc $S(K)$ which is well-known and the most frequently used example is as follows:

DEFINITION 4.1. The Steiner disc of a domain K, denoted by $S(K)$ is the circular disc with radius $\frac{L(K)}{2\pi}$ and center at the Steiner point $\vec{s}(K)$ which can be defined in terms of the Minkowski support function $p_K(\theta)$:

$$\vec{s}(K)=\frac{1}{\pi}\int_0^{2\pi}\vec{u}(\theta)\,p_K(\theta)d\theta,$$

where $\vec{u}(\theta)$ is a unit tangent vector at the corresponding point p, and $L(K)$ denotes the perimeter of the domain K.

We now consider the stability properties of (7) with respect to both Hausdorff distance h_1 and h_2 metric.

THEOREM 4.2. *Let K be a domain enclosed by a $\mathscr{C}_+^2$ closed and strictly convex plane curve γ with area $A(K)$ and perimeter $L(K)$, and let $\tilde{A}(K)$ denote the oriented area of the domain enclosed by the locus of curvature centers of γ, $S(K)$ denotes the Steiner disc associated with K. Then for arbitrary constants α, β, λ, δ, σ, ω satisfying*

$$\begin{cases} \alpha,\lambda \geqslant 0 \\ 2\beta+4\pi\delta+\sigma-\alpha \geqslant 0 \\ 8\beta-8\lambda+4\omega-\alpha \geqslant 0 \\ 6\beta+24\lambda+4\omega-\sigma \geqslant 0, \end{cases} \tag{20}$$

we have

$$\begin{aligned} & h_1(K,S(K))^2 \\ \leqslant & C(\alpha,\beta,\lambda,\sigma,\omega)\left(\alpha\int_\gamma k^2 ds+\beta\int_0^{2\pi}\rho(\theta)^2 d\theta+\lambda\int_0^{2\pi}\rho_\beta^2(\theta)\,d\theta+\delta L^2+\sigma A+\omega|\widetilde{A}|\right) \end{aligned} \tag{21}$$

where k is the curvature of γ, ρ and ρ_β respectively denote curvature radii of the curve γ and the locus of curvature centers,

$$\begin{aligned} & C(\alpha,\beta,\lambda,\sigma,\omega) \\ = & \max\left\{1,\sum_{n=2}^{\infty}\frac{1}{\pi\left(\left(\beta+\lambda n^2\right)\left(n^2-1\right)-\frac{\sigma-\alpha}{2}+\frac{1}{2}\left(\omega-\frac{\alpha}{4}\right)n^2\right)\left(n^2-1\right)}\right\}. \end{aligned}$$

The equality holds if γ is a circle and the parameters α, β, λ, δ, σ, ω satisfy

$$\begin{cases} \alpha=0 \\ 2\beta+4\pi\delta+\sigma=0. \end{cases}$$

Proof. We may assume $\overrightarrow{s}(K)=0$, because of (12) and (16), the support functions p_K and $p_{S(K)}$ have the following Fourier series:

$$p_K(\theta)=\frac{L(K)}{2\pi}+\sum_{n=2}^{\infty}\left(a_n\cos n\theta+b_n\sin n\theta\right) \tag{22}$$

and

$$p_{S(K)}(\theta)=\frac{L(K)}{2\pi}. \tag{23}$$

One can observe that (22) and (23) yield an explicit expression (in terms of the Fourier coefficients) for the quantity

$$
\begin{aligned}
&\alpha\int_\gamma k^2 ds+\beta\int_0^{2\pi}\rho(\theta)^2\,d\theta+\lambda\int_0^{2\pi}\rho_\beta^2(\theta)\,d\theta+\delta L^2+\sigma A+\omega|\widetilde{A}|\\
\geqslant&2\alpha\sqrt{\pi L}+\beta\int_0^{2\pi}\rho(\theta)^2\,d\theta+\lambda\int_0^{2\pi}\rho_\beta^2(\theta)\,d\theta+\delta L^2+(\sigma-\alpha)A+\left(\omega-\frac{\alpha}{4}\right)|\widetilde{A}|\\
=&2\sqrt{2}\pi\alpha\sqrt{a_0}+\left(2\pi\beta+4\pi^2\delta+\pi(\sigma-\alpha)\right)a_0^2\\
&+\pi\sum_{n=2}^{\infty}\left(\left(\beta+\lambda n^2\right)\left(n^2-1\right)-\frac{\sigma-\alpha}{2}+\frac{1}{2}\left(\omega-\frac{\alpha}{4}\right)n^2\right)\left(n^2-1\right)\left(a_n^2+b_n^2\right)
\end{aligned}
\tag{24}
$$

Since it is easily seen that

$$
|a_n\cos n\theta+b_n\sin n\theta|\leqslant\sqrt{a_n^2+b_n^2},
$$

it follows that

$$
\begin{aligned}
|p_K(\theta)-p_{S(K)}(\theta)|&=\left|\frac{L(K)}{2\pi}+\sum_{n=2}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)-\frac{L(K)}{2\pi}\right|\\
&\leqslant\sum_{n=2}^{\infty}|a_n\cos n\theta+b_n\sin n\theta|\\
&\leqslant\sum_{n=2}^{\infty}\sqrt{a_n^2+b_n^2}.
\end{aligned}
$$

Using Hölder's inequality, together with (24) we have

$$
\begin{aligned}
&h_1(K,S(K))^2\\
\leqslant&\left(\sum_{n=2}^{\infty}\sqrt{a_n^2+b_n^2}\right)^2\\
\leqslant&2\sqrt{2}\pi\alpha\sqrt{a_0}+\left(2\pi\beta+4\pi^2\delta+\pi(\sigma-\alpha)\right)a_0^2\\
&+\sum_{n=2}^{\infty}\frac{1}{\pi\left(\left(\beta+\lambda n^2\right)\left(n^2-1\right)-\frac{\sigma-\alpha}{2}+\frac{1}{2}\left(\omega-\frac{\alpha}{4}\right)n^2\right)\left(n^2-1\right)}\\
&\times\left(\pi\sum_{n=2}^{\infty}\left(\left(\beta+\lambda n^2\right)\left(n^2-1\right)-\frac{\sigma-\alpha}{2}+\frac{1}{2}\left(\omega-\frac{\alpha}{4}\right)n^2\right)\left(n^2-1\right)\left(a_n^2+b_n^2\right)\right)\\
\leqslant&\max\left\{1,\sum_{n=2}^{\infty}\frac{1}{\pi\left(\left(\beta+\lambda n^2\right)\left(n^2-1\right)-\frac{\sigma-\alpha}{2}+\frac{1}{2}\left(\omega-\frac{\alpha}{4}\right)n^2\right)\left(n^2-1\right)}\right\}\\
&\times\left(2\alpha\sqrt{\pi L}+\beta\int_0^{2\pi}\rho(\theta)^2\,d\theta+\lambda\int_0^{2\pi}\rho_\beta^2(\theta)\,d\theta+\delta L^2+(\sigma-\alpha)A+\left(\omega-\frac{\alpha}{4}\right)|\widetilde{A}|\right)\\
\leqslant&C(\alpha,\beta,\lambda,\sigma,\omega)\left(\alpha\int_\gamma k^2ds+\beta\int_0^{2\pi}\rho(\theta)^2\,d\theta+\lambda\int_0^{2\pi}\rho_\beta^2(\theta)\,d\theta+\delta L^2+\sigma A+\omega|\widetilde{A}|\right)
\end{aligned}
$$

for arbitrary constants α, β, λ, δ, σ, ω satisfying (20).

Furthermore, if γ is a circle, as in the proof of Theorem 1.6 we have

$$\begin{aligned}&\alpha\int_\gamma k^2 ds+\beta\int_0^{2\pi}\rho(\theta)^2 d\theta+\lambda\int_0^{2\pi}\rho_\beta^2(\theta)d\theta+\delta L^2+\sigma A+\omega|\widetilde{A}|\\=&\alpha\int_\gamma k^2 ds+2\beta A+4\pi\delta A+\sigma A,\end{aligned}$$

then for the parameters α, β, λ, δ, σ, ω satisfying

$$\begin{cases}\alpha=0\\ 2\beta+4\pi\delta+\sigma=0,\end{cases}$$

we have

$$\alpha\int_\gamma k^2 ds+\beta\int_0^{2\pi}\rho(\theta)^2 d\theta+\lambda\int_0^{2\pi}\rho_\beta^2(\theta)d\theta+\delta L^2+\sigma A+\omega|\widetilde{A}|=0.$$

It is obvious that $h_1(K,S(K))=0$, thus the equality in (21) holds. □

THEOREM 4.3. *Under the same assumptions of Theorem 4.2, then for arbitrary constants α, β, λ, δ, σ, ω satisfying*

$$\begin{cases}\alpha,\lambda\geqslant 0\\ 2\beta+4\pi\delta+\sigma-\alpha\geqslant 0\\ 8\beta-8\lambda+4\omega-\alpha\geqslant 0\\ 18\beta+72\lambda-3\sigma+12\omega-2\geqslant 0,\end{cases}\tag{25}$$

we have

$$h_2(K,S(K))^2\leqslant\alpha\int_\gamma k^2 ds+\beta\int_0^{2\pi}\rho(\theta)^2 d\theta+\lambda\int_0^{2\pi}\rho_\beta^2(\theta)d\theta+\delta L^2+\sigma A+\omega|\widetilde{A}|.\tag{26}$$

The equality holds if γ is a circle and the parameters α, β, λ, δ, σ, ω satisfy

$$\begin{cases}\alpha=0\\ 2\beta+4\pi\delta+\sigma=0.\end{cases}$$

Moreover if the equality in (26) *holds and the parameters α, β, λ, δ, σ, ω satisfy* (25)*, then γ is a circle.*

Proof. As in the proof of Theorem 4.2, we use Parseval's equality, (22) and (23) to deduce that

$$h_2(K,S(K))^2=\int_0^{2\pi}|p_K(\theta)-p_{S(K)}(\theta)|^2 d\theta=\pi\sum_{n=2}^{\infty}\left(a_n^2+b_n^2\right),$$

together with (24) one gets that

$$\begin{aligned}
&2\alpha\sqrt{\pi L}+\beta\int_0^{2\pi}\rho(\theta)^2\,d\theta+\lambda\int_0^{2\pi}\rho_\beta^2(\theta)\,d\theta+\delta L^2+(\sigma-\alpha)A+\left(\omega-\frac{\alpha}{4}\right)|\widetilde{A}|\\
&-h_2(K,S(K))^2\\
=&2\sqrt{2}\pi\alpha\sqrt{a_0}+\left(2\pi\beta+4\pi^2\delta+\pi(\sigma-\alpha)\right)a_0^2-\pi\sum_{n=2}^{\infty}\left(a_n^2+b_n^2\right)\\
&+\pi\sum_{n=2}^{\infty}\left(\left(\beta+\lambda n^2\right)\left(n^2-1\right)-\frac{\sigma-\alpha}{2}+\frac{1}{2}\left(\omega-\frac{\alpha}{4}\right)n^2\right)\left(n^2-1\right)\left(a_n^2+b_n^2\right)\\
=&2\sqrt{2}\pi\alpha\sqrt{a_0}+\left(2\pi\beta+4\pi^2\delta+\pi(\sigma-\alpha)\right)a_0^2\\
&+\pi\sum_{n=2}^{\infty}\left(\left(\left(\beta+\lambda n^2\right)\left(n^2-1\right)-\frac{\sigma-\alpha}{2}+\frac{1}{2}\left(\omega-\frac{\alpha}{4}\right)n^2\right)\left(n^2-1\right)-1\right)\left(a_n^2+b_n^2\right).
\end{aligned}$$

Hence for arbitrary constants α, β, λ, δ, σ, ω satisfying (25), we have

$$2\sqrt{2}\pi\alpha\sqrt{a_0}+\left(2\pi\beta+4\pi^2\delta+\pi(\sigma-\alpha)\right)a_0^2\geqslant 0$$

and

$$\begin{aligned}
&\pi\sum_{n=2}^{\infty}\left(\left(\left(\beta+\lambda n^2\right)\left(n^2-1\right)-\frac{\sigma-\alpha}{2}+\frac{1}{2}\left(\omega-\frac{\alpha}{4}\right)n^2\right)\left(n^2-1\right)-1\right)\left(a_n^2+b_n^2\right)\\
\geqslant&\pi\sum_{n=2}^{\infty}\left(3\left(3(\beta+4\lambda)-\frac{\sigma-\alpha}{2}+2\left(\omega-\frac{\alpha}{4}\right)\right)-1\right)\left(a_n^2+b_n^2\right)\\
=&\frac{\pi}{2}\sum_{n=2}^{\infty}(18\beta+72\lambda-3\sigma+12\omega-2)\left(a_n^2+b_n^2\right)\\
\geqslant&0
\end{aligned}$$

which implies that

$$\begin{aligned}
&h_2(K,S(K))^2\\
\leqslant&2\alpha\sqrt{\pi L}+\beta\int_0^{2\pi}\rho(\theta)^2\,d\theta+\lambda\int_0^{2\pi}\rho_\beta^2(\theta)\,d\theta+\delta L^2+(\sigma-\alpha)A+\left(\omega-\frac{\alpha}{4}\right)|\widetilde{A}|\\
\leqslant&\alpha\int_\gamma k^2ds+\beta\int_0^{2\pi}\rho(\theta)^2\,d\theta+\lambda\int_0^{2\pi}\rho_\beta^2(\theta)\,d\theta+\delta L^2+\sigma A+\omega|\widetilde{A}|.
\end{aligned}$$

Furthermore, if γ is a circle and the parameters α, β, λ, δ, σ, ω satisfy

$$\begin{cases}\alpha=0\\ 2\beta+4\pi\delta+\sigma=0,\end{cases}$$

as in the proof of Theorem 4.2, the equality in (26) holds. Conversely, if the equality in (26) holds and the parameters α, β, λ, δ, σ, ω satisfy (25), then as in the proof of Theorem 1.6, γ is a circle. This completes the proof of Theorem 4.3. □

REMARK 4. The combination of Theorem 4.2 and 4.3 leads to

$$\max\left\{h_1(K,S(K))^2, h_2(K,S(K))^2\right\}$$
$$\leqslant C(\alpha,\beta,\lambda,\sigma,\omega)\left(\alpha\int_\gamma k^2 ds+\beta\int_0^{2\pi}\rho(\theta)^2 d\theta+\lambda\int_0^{2\pi}\rho_\beta^2(\theta)d\theta+\delta L^2+\sigma A+\omega|\widetilde{A}|\right),$$

where

$$C(\alpha,\beta,\lambda,\sigma,\omega)$$
$$=\max\left\{1,\sum_{n=2}^{\infty}\frac{1}{\pi\left((\beta+\lambda n^2)(n^2-1)-\frac{\sigma-\alpha}{2}+\frac{1}{2}\left(\omega-\frac{\alpha}{4}\right)n^2\right)(n^2-1)}\right\},$$

which states that the isoperimetric inequality (7) does have well stability properties with respect to both Hausdorff distance and L^2-metric.

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REFERENCES

[1] J. STEINER, *Sur le maximum et le minimum des figures dans le plan, sur la sphère, et dans l'espace en général*, I and II, J. Reine Angew. Math. (Crelle), **24** (1842), 93–152 and 189–250.
[2] S. L. PAN, H. ZHANG, *A reverse isoperimetric inequality for convex plane curves*, Beitrage Algebra Geom. **48** (2007) 303–308.
[3] X. GAO, *A note on the reverse isoperimetric inequality*, Results. Math. **59** (2011), 83–90.
[4] S. L. PAN, J. N. YANG, *On a non-local perimeter-preserving curve evolution problem for convex plane curves*, Manuscripta Math. **127**, (2008), 469–484.
[5] R. SCHNEIDER, *Convex bodies: The Brunn-Minkowski theory*, Encyclopedia of Mathematics and its Applications, **44**, Cambridge University Press, Cambridge, 1993.
[6] S. L. PAN, H. P. XU, *Stability of a reverse isoperimetric inequality*, J. Math. Anal. Appl. **350** (2009), 348–353.
[7] H. GROEMER, *Geometric applications of Fourier series and spherical harmonics*, Encyclopedia of Mathematics and its Applications, **61**, Cambridge University Press, Cambridge, 1996.

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