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AN OPERATOR α–GEOMETRIC MEAN INEQUALITY

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Abstract. We square operator α-geometric mean inequality as follows: If $0 < m_1 \leqslant A \leqslant M_1$ and $0 < m_2 \leqslant B \leqslant M_2$ for some positive real numbers $m_1 < M_1$ and $m_2 < M_2$, then for every unital positive linear map Φ and $\alpha \in [0,1]$, the following inequality holds:

$$(\Phi(A)\sharp_\alpha \Phi(B))^2 \leqslant \left(\frac{(M_1+m_1)^2((M_1+m_1)^{-1}(M_2+m_2))^{2\alpha}}{4(m_2M_2)^\alpha(m_1M_1)^{(1-\alpha)}}\right)^2 \Phi^2(A\sharp_\alpha B).$$

1. Introduction

We continue the recent study on squaring operator inequalities; see [4, 5]. Let $\mathbb{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ with the identity I. Throughout the paper, a capital letter means an operator in $\mathbb{B}(\mathscr{H})$. An operator A is called positive if $\langle Ax, x\rangle \geqslant 0$ for all $x \in \mathscr{H}$, and we then write $A \geqslant 0$. An operator A is said to be strictly positive (i.e. $A > 0$) if it is a positive invertible operator. In this paper, the inequality between operators is in the sense of Löewner partial order, that is, $B \geqslant A$ means $B - A \geqslant 0$. A linear map $\Phi : \mathbb{B}(\mathscr{H}) \to \mathbb{B}(\mathscr{K})$ is called (strictly) positive if $(A > 0)$ $A \geqslant 0$ implies $(\Phi(A) > 0)$ $\Phi(A) \geqslant 0$. Φ is said to be unital if $\Phi(I) = I$. The operator norm is denoted by $\|\cdot\|$. For $A, B > 0$ and $\alpha \in [0,1]$, the geometric mean $A\sharp_\alpha B$ is defined by $A\sharp_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}$.

Seo [8, Theorem 3] gave α-geometric mean inequality as follows:

THEOREM 1.1. *Let $\Phi : \mathbb{B}(\mathscr{H}) \to \mathbb{B}(\mathscr{K})$ be a unital positive linear map and let A and B be positive operators such that $0 < m_1^2 \leqslant A \leqslant M_1^2$ and $0 < m_2^2 \leqslant B \leqslant M_2^2$ for some positive real numbers $m_1 < M_1$ and $m_2 < M_2$. Then for $\alpha \in [0,1]$*

$$\Phi(A)\sharp_\alpha \Phi(B) \leqslant K(m, M, \alpha)^{-1} \Phi(A\sharp_\alpha B) \tag{1.1}$$

where we suppose $(\frac{m_2}{M_1})^2 = m$, $(\frac{M_2}{m_1})^2 = M$ and the generalized Kantorovich constant $K(m,M,\alpha)$ [2, Definition 2.2] *is defined by*

$$K(m,M,\alpha) = \frac{mM^\alpha - Mm^\alpha}{(\alpha-1)(M-m)}\left(\frac{\alpha-1}{\alpha}\frac{M^\alpha - m^\alpha}{mM^\alpha - Mm^\alpha}\right)^\alpha$$

for any real number $\alpha \in \mathbb{R}$.

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Let $0 < m \leqslant A \leqslant M$ and Φ be positive unital linear map. Marshall and Olkin [6] proved the following operator Kantorovich inequality:

$$\Phi(A^{-1}) \leqslant \frac{(M+m)^2}{4Mm}\Phi(A)^{-1}. \tag{1.2}$$

It is surprising that Lin [5, Theorem 2.8] showed that the operator inequality (1.2) can be squared:

$$\Phi^2(A^{-1}) \leqslant \left(\frac{(M+m)^2}{4Mm}\right)^2 \Phi(A)^{-2}. \tag{1.3}$$

Inspired by Lin's idea in obtaining the inequality (1.3), we prove a second powering of the operator inequality (1.1) in this paper: It is well known that t^s $(0 \leqslant s \leqslant 1)$ is an operator monotone function and not so is t^2; see [7]. However, by the operator inequality (1.1) we can say that t^2 is order preserving in the following sense: If $0 < m_1 \leqslant A \leqslant M_1$ and $0 < m_2 \leqslant B \leqslant M_2$ for some positive real numbers $m_1 < M_1$ and $m_2 < M_2$, then for every unital positive linear map Φ and $\alpha \in [0,1]$, the following inequality holds:

$$(\Phi(A)\sharp_\alpha\Phi(B))^2 \leqslant \left(\frac{(M_1+m_1)^2((M_1+m_1)^{-1}(M_2+m_2))^{2\alpha}}{4(m_2M_2)^\alpha(m_1M_1)^{(1-\alpha)}}\right)^2 \Phi^2(A\sharp_\alpha B).$$

2. Main result

We start our work with the Lemmas which describes Ando's inequality.

LEMMA 2.1. [7, Theorem 1.17] *Let Φ be a unital strictly positive linear map and $A > 0$. Then*

$$\Phi^{-1}(A) \leqslant \Phi(A^{-1}). \tag{2.1}$$

LEMMA 2.2. [3] *Let Φ be a unital positive linear map and A, B be positive operators. Then for $\alpha \in [0,1]$*

$$\Phi(A\sharp_\alpha B) \leqslant \Phi(A)\sharp_\alpha\Phi(B). \tag{2.2}$$

LEMMA 2.3. [1] *Let $A,B > 0$. Then the following norm inequality holds:*

$$\|AB\| \leqslant \frac{1}{4}\|A+B\|^2. \tag{2.3}$$

Now we give our main result.

THEOREM 2.4. *Let $\Phi : \mathbb{B}(\mathscr{H}) \to \mathbb{B}(\mathscr{K})$ be a unital positive linear map and let A and B be positive operators such that $0 < m_1 \leqslant A \leqslant M_1$ and $0 < m_2 \leqslant B \leqslant M_2$ for some positive real numbers $m_1 < M_1$ and $m_2 < M_2$. Then for $\alpha \in [0,1]$*

$$(\Phi(A)\sharp_\alpha\Phi(B))^2 \leqslant \left(\frac{(M_1+m_1)^2((M_1+m_1)^{-1}(M_2+m_2))^{2\alpha}}{4(m_2M_2)^\alpha(m_1M_1)^{1-\alpha}}\right)^2 \Phi^2(A\sharp_\alpha B). \tag{2.4}$$

Proof. As (2.4) is equivalent to

$$\|(\Phi(A)\sharp_\alpha\Phi(B))\Phi^{-1}(A\sharp_\alpha B))\| \leqslant \frac{(M_1+m_1)^2((M_1+m_1)^{-1}(M_2+m_2))^{2\alpha}}{4(m_2M_2)^\alpha(m_1M_1)^{1-\alpha}}. \tag{2.5}$$

It is easy to obtain that

$$(M_1-A)(m_1-A)A^{-1} = A+m_1M_1A^{-1}-m_1-M_1 \leqslant 0,$$

and hence

$$m_1M_1\Phi(A^{-1})+\Phi(A) \leqslant M_1+m_1. \tag{2.6}$$

In the same way, we also have

$$m_2M_2\Phi(B^{-1})+\Phi(B) \leqslant M_2+m_2. \tag{2.7}$$

Whence, by the α-geometric means of both sides (2.6) and (2.7), we have

$$\begin{aligned}&(m_1M_1\Phi(A^{-1})+\Phi(A))\sharp_\alpha(m_2M_2\Phi(B^{-1})+\Phi(B))\\ &\leqslant (M_1+m_1)\sharp_\alpha(M_2+m_2),\end{aligned} \tag{2.8}$$

Since the α-geometric mean operation is subadditivity, we can see

$$\begin{aligned}&(m_2M_2)^\alpha(m_1M_1)^{1-\alpha}(\Phi^{-1}(A\sharp_\alpha B))+\Phi(A)\sharp_\alpha\Phi(B)\\ &\leqslant (m_1M_1\Phi(A^{-1}))\sharp_\alpha(m_2M_2\Phi(B^{-1}))+(\Phi(A)\sharp_\alpha\Phi(B)) \qquad \text{(by (2.1) and (2.2))}\\ &= (m_2M_2)^\alpha(m_1M_1)^{1-\alpha}(\Phi(A^{-1})\sharp_\alpha\Phi(B^{-1}))+(\Phi(A)\sharp_\alpha\Phi(B))\\ &\leqslant (m_1M_1\Phi(A^{-1})+\Phi(A))\sharp_\alpha(m_2M_2\Phi(B^{-1})+\Phi(B)).\end{aligned} \tag{2.9}$$

So

$$\begin{aligned}&\|(\Phi(A)\sharp_\alpha\Phi(B))((m_2M_2)^\alpha(m_1M_1)^{1-\alpha}\Phi^{-1}(A\sharp_\alpha B))\|\\ &\leqslant \frac{1}{4}\|(\Phi(A)\sharp_\alpha\Phi(B)+(m_2M_2)^\alpha(m_1M_1)^{1-\alpha}\Phi^{-1}(A\sharp_\alpha B))\|^2 \qquad \text{(by (2.3))}\\ &\leqslant \frac{1}{4}\|(m_1M_1\Phi(A^{-1})+\Phi(A))\sharp_\alpha(m_2M_2\Phi(B^{-1})+\Phi(B))\|^2 \qquad \text{(by (2.9))}\\ &\leqslant \frac{1}{4}((M_1+m_1)\sharp_\alpha(M_2+m_2))^2 \qquad \text{(by (2.8))}\\ &= \frac{(M_1+m_1)^2((M_1+m_1)^{-1}(M_2+m_2))^{2\alpha}}{4},\end{aligned}$$

which implies that

$$\|(\Phi(A)\sharp_\alpha\Phi(B))\Phi^{-1}(A\sharp_\alpha B))\| \leqslant \frac{(M_1+m_1)^2((M_1+m_1)^{-1}(M_2+m_2))^{2\alpha}}{4(m_2M_2)^\alpha(m_1M_1)^{1-\alpha}}.$$

Thus, (2.5) holds. This completes the proof of Theorem 2.4. □

REMARK 2.5. It is easy to know that the coefficient

$$K\left(\frac{m_2}{M_1},\frac{M_2}{m_1},\alpha\right)^{-1}$$

in (1.1) is smaller than

$$\frac{(M_1+m_1)^2((M_1+m_1)^{-1}(M_2+m_2))^{2\alpha}}{4(m_2M_2)^{\alpha}(m_1M_1)^{(1-\alpha)}}$$

in (2.4), but we obtain the relation between $(\Phi(A)\sharp_\alpha\Phi(B))^2$ and $\Phi^2(A\sharp_\alpha B)$.

CONJECTURE 2.6. Under the same condition as in Theorem 1.1, the following inequality holds:

$$(\Phi(A)\sharp_\alpha\Phi(B))^2 \leqslant K(m,M,\alpha)^{-2}\Phi^2(A\sharp_\alpha B).$$

REFERENCES

[1] R. BHATIA, F. KITTANEH, *Notes on matrix arithmetic-geometric mean inequalities*, Linear Algebra Appl. **308** (2000), 203–211.

[2] T. FURUTA, J. MIĆIĆ, J. E. PEČARIĆ AND Y. SEO, *Mond-Pečarić Method in Operator Inequalities*, Monographs in inequalities 1, Element, Zagreb, 2005.

[3] K. KUBO, T. ANDO, *Means of positive linear operators*, Math. Ann., **246** (1980), 205–224.

[4] M. LIN, *Squaring a reverse AM-GM inequality*, Studia Math. **215** (2013), 187–194.

[5] M. LIN, *On an operator Kantorovich inequality for positive linear maps*, J. Math. Anal. Appl. **402** (2013), 127–132.

[6] A. W. MARSHALL, I. OLKIN, *Matrix versions of Cauchy and Kantorovich inequalities*, Aequationes Math. **40** (1990), 89–93.

[7] J. PEČARIĆ, T. FURUTA, J. MIĆIĆ HOT AND Y. SEO, *Mond Pečarić method in operator inequalities*, Element, Zagreb, 2005.

[8] Y. SEO, *Reverses of Ando's inequality for positive linear maps*, Math. Inequal. Appl. **14** (2011), 905–910.

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