THE LOGARITHMIC COEFFICIENT INEQUALITY FOR
CLOSE–TO–CONVEX FUNCTIONS OF COMPLEX ORDER

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Abstract. We prove that if \( n \geq 2 \) for each close-to-convex functions of complex order \( b \) in \( \mathcal{A} \) whose \( n - th \) logarithmic coefficients \( \gamma_n \) satisfies \( |\gamma_n| \leq An^{-1}\log n \), where \( A \) is an absolute constant.

1. Introduction

Let \( \mathcal{A} \) denote the class of functions \( f \) analytic in the unit disk \( \mathcal{U} = \{ z \in \mathbb{C}: |z| < 1 \} \) having the power series

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathcal{U}.
\] (1.1)

Let \( \mathcal{S} \) denote the class of functions \( f \in \mathcal{A} \) which are univalent in \( \mathcal{U} \) and \( \mathcal{S}^* \) point out the subset of \( \mathcal{S} \) consisting of those functions \( f \in \mathcal{S} \) for which \( f(\mathcal{U}) \) is starlike with respect to \( 0 \). It is well known that if \( f \in \mathcal{S}^* \), then

\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0,
\]

for all \( z \in \mathcal{U} \). Aouf and Nasr [1] introduced the class \( \mathcal{S}^*(b) \) of starlike functions of order \( b \), where \( b \) is a nonzero complex number, as follows:

\[
\text{Re} \left\{ 1 + \frac{1}{b} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right\} > 0, \quad z \in \mathcal{U}.
\]

Let \( \mathcal{S}_c \) denote the set of those functions \( f \in \mathcal{S} \) for which there exists a function \( g \in \mathcal{S}^* \) such that

\[
\text{Re} \left\{ \frac{zf'(z)}{g(z)} \right\} > 0,
\]


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for all $z \in \mathcal{U}$. The elements of $\mathcal{S}_c$ are called close-to-convex functions. Clearly, $\mathcal{S}^* \subset \mathcal{S}_c$. Al-Amiri and Fernando [2] introduced the class $\mathcal{S}_c(b)$ of close-to-convex functions of complex order $b$ as follows:

$$\text{Re} \left\{ 1 + \frac{1}{b} \left( \frac{zf'(z)}{g(z)} - 1 \right) \right\} > 0, \quad z \in \mathcal{U},$$

(1.2)

for some starlike function $g$.

Associated with each $f(z)$ in $\mathcal{S}$ is a well defined logarithmic function

$$\log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n z^n, \quad z \in \mathcal{U}.$$  

(1.3)

The numbers $\gamma_n$ are called the logarithmic coefficients of $f(z)$. Thus the Koebe function $k(z) = z(1-z)^{-2}$ has logarithmic coefficients $\gamma_n = \frac{1}{n}$. It is clear that $|\gamma_1| \leq 1$ for each $f(z) \in \mathcal{S}$. The problem of the best upper bounds for $|\gamma_n|$ is still open. In fact even the proper order of magnitude is still not known. It is known, however, for the starlike functions that the best bound is $|\gamma_n| \leq \frac{1}{n}$ and that this is not true in general [7, p. 151]; [6, p. 898]; [3, p. 140] and [8].

In the paper [4] it is pointed out that the inequality $|\gamma_n| \leq An^{-1} \log n$ (A is an absolute constant) which holds for circularly symmetric functions.

In a recent paper [9], it is presented that the inequality $|\gamma_n| \leq \frac{1}{n}$ holds also for close-to-convex functions. However, it is pointed out in [12] that there are some errors in the proof and, hence, the result is not substantiated. It is proved in [10] that there exists a function $f(z) \in \mathcal{S}_c$ such that $|\gamma_n| > \frac{1}{n}$. Furthermore, it is proved in [14] that the inequality $|\gamma_n| \leq An^{-1} \log n$ holds for close-to-convex functions, where $A$ is an absolute constant.

In the present paper, we study the logarithmic coefficients of the class $\mathcal{S}_c(b)$.

2. Main results

First, we give the following lemmas.

**Lemma 2.1.** [14] Let $f(z) \in \mathcal{S}$. Then, for $z = re^{i\theta}, \frac{1}{2} \leq r < 1$,

$$\frac{1}{2\pi} \int_0^{2\pi} \left| \frac{zf'(z)}{f(z)} \right|^2 \, d\theta \leq 1 + \frac{4}{1-r} \log \frac{1}{1-\sqrt{r}},$$

(2.1)

and

$$\frac{1}{2\pi} \int_0^r \left[ \int_0^{2\pi} \left| \frac{zf'(z)}{f(z)} \right|^2 \, d\theta \right] \, dr \leq 1 + 2 \log \frac{1}{1-r}.$$  

(2.2)

**Lemma 2.2.** [5] Let $f(z) \in \mathcal{S}, \tau \in \mathbb{C}$. Then, $z = re^{i\theta}, 0 < r < 1$,

$$\frac{\partial}{\partial \theta} \left( \text{arg} \left( \frac{f(z)}{z} \right)^\tau \right) = \tau \frac{\partial}{\partial \theta} \left( \text{arg} \left( \frac{f(z)}{z} \right) \right).$$

(2.3)
Proof. It is clear that
\[ \frac{zf'(z)}{f(z)} = \frac{1}{i} \frac{\partial}{\partial \theta} \left( \log \frac{f(z)}{z} \right) + 1. \]

(2.4)

It follows that
\[ \Re \frac{zf'(z)}{f(z)} = \Im \left\{ \frac{\partial}{\partial \theta} \left( \log \frac{f(z)}{z} \right) \right\} + 1 = \frac{\partial}{\partial \theta} \left( \arg \frac{f(z)}{z} \right) + 1. \]

(2.5)

Since
\[ \frac{zf'(z)}{f(z)} = \frac{1}{i\tau} \frac{\partial}{\partial \theta} \left( \log \left( \frac{f(z)}{z} \right)^\tau \right) + 1, \]

then
\[ \Re \frac{zf'(z)}{f(z)} = \frac{1}{\tau} \Im \left\{ \frac{\partial}{\partial \theta} \left( \log \left( \frac{f(z)}{z} \right)^\tau \right) \right\} + 1 = \frac{\partial}{\partial \theta} \left( \arg \frac{f(z)}{z} \right). \]

(2.6)

From (2.5) and (2.7) we obtain
\[ \frac{\partial}{\partial \theta} \left( \arg \frac{f(z)}{z} \right) = \tau \frac{\partial}{\partial \theta} \left( \arg \frac{f(z)}{z} \right). \]

\[ \square \]

Lemma 2.3. [2] Let \( f(z) \in \mathcal{S}_c(b) \). Then for \( |z| = r < 1 \) and \( |2b - 1| \leq 1 \)
\[ \frac{1 - |2b - 1|r}{1 + r} \leq \left| \frac{zf'(z)}{g(z)} \right| \leq \frac{1 + |2b - 1|r}{1 - r}. \]

(2.8)

Theorem 2.1. Let \( f(z) \in \mathcal{S}_c(b) \). Then for \( n \geq 2 \),
\[ |\gamma_n| \leq An^{-1} \log n \]

(2.9)

where \( A \) is an absolute constant, and the exponent \( -1 \) is the best possible.

Proof. If \( f(z) \in \mathcal{S}_c(b) \), then there exist \( g(z) \in \mathcal{S}_c \) such that \( \Re \left\{ 1 + \frac{1}{b} \left( \frac{zf'(z)}{g(z)} - 1 \right) \right\} > 0 \), \( b \neq 0 \), \( b \in \mathbb{C} \). Write \( h(z) = 1 + \frac{1}{b} \left( \frac{zf'(z)}{g(z)} - 1 \right) \), then \( \Re h(z) > 0 \). It is clear that
\[ h(z) = 2\Re h(z) - \overline{h(z)}. \]

From (1.3), we obtain
\[ \frac{zf'(z)}{f(z)} = 1 + z \left( \log \frac{f(z)}{z} \right)' = 1 + \sum_{k=1}^{\infty} 2k \gamma_k z^k. \]

(2.10)

Then, for \( z = re^{i\theta} \) \((0 < r < 1)\) and \( n = 2, 3, \ldots \)
\[ 2n \gamma_n = \frac{1}{2\pi i} \oint_{|z|=r} \frac{zf'(z)}{f(z)} z^{-n-1} dz. \]
Hence, we get
\[ |2n \gamma_n^n| = \frac{1}{2\pi} \left| \int_0^{2\pi} \frac{zf'(z)}{f(z)} e^{-i\theta} d\theta \right| = \frac{1}{2\pi} \left| \int_0^{2\pi} [b(h(z) - 1) + 1] \frac{g(z)}{f(z)} e^{-i\theta} d\theta \right| \]
\[ \leq \frac{|b|}{2\pi} \left| \int_0^{2\pi} \text{Re} h(z) \frac{g(z)}{f(z)} e^{-i\theta} d\theta \right| + \frac{|b|}{2\pi} \left| \int_0^{2\pi} \frac{g(z)}{f(z)} e^{-i\theta} d\theta \right| \]
\[ + \frac{|b - 1|}{2\pi} \left| \int_0^{2\pi} \frac{g(z)}{f(z)} e^{-i\theta} d\theta \right| \]
\[ = I_1 + I_2 + I_3. \]  
(2.11)

Now, we estimate two terms $I_1$ and $I_2$. Write
\[ \frac{zf'(z)}{f(z)} = u(re^{i\theta}) + iv(re^{i\theta}) \]  
(2.12)
a) For $I_1$:
\[ I_1 \leq \frac{|b|}{\pi} \left| \int_0^{2\pi} \text{Re} h(z) \frac{g(z)}{f(z)} d\theta \right| \leq \frac{|b|}{\pi} \left| \int_0^{2\pi} h(z) \frac{g(z)}{f(z)} d\theta \right| \]
\[ \leq \frac{1}{\pi} \left| \int_0^{2\pi} \frac{zf'(z)}{f(z)} e^{i \arg \frac{f(z)}{g(z)}} d\theta \right| + \frac{1 - |b|}{\pi} \left| \int_0^{2\pi} \frac{g(z)}{f(z)} d\theta \right| \]
\[ \leq \frac{1}{\pi} \left| \int_0^{2\pi} u(re^{i\theta}) e^{i \arg \frac{f(z)}{g(z)}} d\theta \right| + \frac{1}{\pi} \left| \int_0^{2\pi} v(re^{i\theta}) e^{i \arg \frac{f(z)}{g(z)}} d\theta \right| + \frac{|1 - |b||}{\pi} \left| \int_0^{2\pi} \frac{g(z)}{f(z)} d\theta \right| \]
\[ = 2(J_1 + J_2 + J_3). \]  
(2.13)

Applying the part of integration, (2.5) and (2.3), we have
\[ J_1 \leq \frac{1}{2\pi} \left| \int_0^{2\pi} e^{i \arg \frac{f(z)}{g(z)}} d\theta \right| + \frac{1}{2\pi} \left| \int_0^{2\pi} \frac{\partial}{\partial \theta} \left( \arg \frac{f(z)}{g(z)} \right) e^{i \arg \frac{f(z)}{g(z)}} d\theta \right| \]
\[ \leq 1 + \frac{1}{2\pi} \left| \int_0^{2\pi} \frac{\partial}{\partial \theta} \left( e^{i \arg \frac{f(z)}{g(z)}} \right) e^{-i \arg \frac{g(z)}{z}} d\theta \right| \]
\[ = 1 + \frac{1}{2\pi} \left| \int_0^{2\pi} e^{i \arg \frac{f(z)}{g(z)}} \frac{\partial}{\partial \theta} \left( \arg \frac{g(z)}{z} \right) d\theta \right| \]
\[ \leq 1 + \frac{1}{2\pi} \left| \int_0^{2\pi} \left( \frac{\partial}{\partial \theta} \arg g(z) \right) + \left| \frac{\partial}{\partial \theta} g(z) \right| \right| d\theta \]  
(2.14)
Since \( g(z) \in S^* \), we have (see [13])

\[
\frac{\partial}{\partial \theta} (\arg g(z)) > 0 \quad \text{and} \quad \int_0^{2\pi} \frac{\partial}{\partial \theta} (\arg g(z)) = 2\pi.
\]  
(2.15)

By applying (2.15), from (2.14) we get

\[
J_1 \leq 1 + \frac{1}{2\pi} \left( \int_0^{2\pi} \frac{\partial}{\partial \theta} (\arg g(z)) d\theta + \int_0^{2\pi} r d\theta \right)
= 1 + \frac{1}{2\pi} (2\pi + 2\pi r)
\leq 3.
\]  
(2.16)

By the Cauchy-Riemann condition, we obtain, for \( 0 < r_0 < r < 1 \)

\[
v(re^{i\theta}) - v(r_0 e^{i\theta}) = \int_{r_0}^{r} \frac{\partial v(re^{i\theta})}{\partial r} dr = -\int_{r_0}^{r} \frac{1}{r} \frac{\partial u(re^{i\theta})}{\partial \theta} dr.
\]  
(2.17)

By (2.17), we get

\[
J_2 \leq \frac{1}{2\pi} \left| \int_0^{2\pi} v(r_0 e^{i\theta}) e^{i \arg f(z)} d\theta \right| + \frac{1}{2\pi} \left| \int_0^{2\pi} \int_{r_0}^{r} \frac{1}{r} \frac{\partial u(re^{i\theta})}{\partial \theta} e^{i \arg f(z)} dr d\theta \right|
= J_{21} + J_{22}.
\]  
(2.18)

Taking \( r_0 = \frac{1}{2} \), it follows that

\[
J_{21} \leq \max_{\theta \in [0,2\pi]} \left| v(r_0 e^{i\theta}) \right| \leq \max_{\theta \in [0,2\pi]} \left| \frac{r_0 f'(r_0 e^{i\theta})}{f(r_0 e^{i\theta})} \right| \leq \frac{1 + r_0}{1 - r_0} = 3.
\]  
(2.19)

By the part of integration, we obtain

\[
J_{22} \leq \frac{1}{2\pi} \left| \int_{r_0}^{r} \int_0^{2\pi} \frac{1}{r} \frac{\partial u(re^{i\theta})}{\partial \theta} e^{i \arg f(z)} \left( \frac{\partial}{\partial \theta} \left( \frac{f(z)}{z} \right) - \frac{\partial}{\partial \theta} \left( \frac{g(z)}{z} \right) \right) d\theta dr \right|
= \left| \frac{zf'(z)}{f(z)} - 1 \right| + \left| \frac{zg'(z)}{g(z)} - 1 \right|.
\]  
(2.20)

By (2.5), we have

\[
\left| \frac{\partial}{\partial \theta} \left( \frac{f(z)}{z} \right) - \frac{\partial}{\partial \theta} \left( \frac{g(z)}{z} \right) \right| = \left| \left( \text{Re} \left( \frac{zf'(z)}{f(z)} \right) - 1 \right) - \left( \text{Re} \left( \frac{zg'(z)}{g(z)} \right) - 1 \right) \right|
\leq \left| \frac{zf'(z)}{f(z)} \right| + \left| \frac{zg'(z)}{g(z)} \right|.
\]  
(2.21)
By Schwarz inequality, Lemma 2.1 and (2.21), from (2.20) we get
\[
J_{22} \leq \frac{1}{2\pi} \int_{r_0}^{r} \int_{0}^{2\pi} \left| \frac{zf'(z)}{f(z)} \right|^2 d\theta dr + \left( \frac{1}{2\pi} \int_{r_0}^{r} \int_{0}^{2\pi} \left| \frac{zf'(z)}{f(z)} \right|^2 d\theta dr \right) \frac{1}{\pi} \int_{r_0}^{r} \int_{0}^{2\pi} \left| \frac{zg'(z)}{g(z)} \right|^2 d\theta dr \]
\[
\leq 4 \left( 1 + 2 \log \frac{1}{1-r} \right). \tag{2.22}
\]

By (2.19) and (2.22), from (2.18) we obtain
\[
J_2 \leq 7 + 8 \log \frac{1}{1-r}. \tag{2.23}
\]

By (2.16) and (2.23), from (2.13) we have
\[
I_1 \leq 20 + 16 \log \frac{1}{1-r} + 2J_3. \tag{2.24}
\]

b) For \(I_2\):
\[
I_2 \leq \frac{1}{2\pi} \int_{0}^{2\pi} \left| \frac{zf'(z)}{f(z)} \right|^2 e^{-2i\arg \frac{f(z)}{g(z)} e^{-in\theta}} \, d\theta + \frac{1 - \Re e}{2\pi} \int_{0}^{2\pi} \left| \frac{g(z)}{f(z)} \right| e^{-in\theta} \, d\theta \]
\[
= \frac{1}{2\pi} \int_{0}^{2\pi} \left| \frac{zf'(z)}{f(z)} \right|^2 e^{-2i\arg \frac{f(z)}{g(z)} e^{-in\theta}} \, d\theta + \frac{1 - \Re e}{2\pi} \int_{0}^{2\pi} \left| \frac{g(z)}{f(z)} \right| e^{-in\theta} \, d\theta. \tag{2.25}
\]

From (2.10) we get
\[
\frac{zf'(z)}{f(z)} e^{in\theta} = e^{in\theta} \left( 1 + \sum_{k=1}^{\infty} 2k \gamma_k z^k \right) = e^{in\theta} + \sum_{k=1}^{\infty} 2k \gamma_k z^k e^{(n+k)i\theta}
\]
\[
= \frac{1}{i} \frac{\partial}{\partial \theta} \left( \frac{e^{in\theta}}{n} + \sum_{k=1}^{\infty} \frac{2k \gamma_k z^k}{n+k} e^{(n+k)i\theta} \right) = \frac{1}{i} \frac{\partial}{\partial \theta} F(z). \tag{2.26}
\]

By the part of integration, we obtain
\[
I_2 \leq \frac{1}{\pi} \int_{0}^{2\pi} \left| F(z) e^{2i\arg \frac{f(z)}{g(z)}} \left( \frac{\partial}{\partial \theta} \left( \frac{\arg f(z)}{z} \right) - \frac{\partial}{\partial \theta} \left( \frac{\arg g(z)}{z} \right) \right) d\theta \right| \]
\[
+ \frac{1 - \Re e}{2\pi} \int_{0}^{2\pi} \left| \frac{g(z)}{f(z)} e^{-in\theta} \right| d\theta. \tag{2.27}
\]
By (2.5) and Schwartz inequality, it follows from (2.27)\[ L_2 \leq 2 \left( \frac{1}{2\pi} \int_0^{2\pi} |F(z)|^2 \, d\theta \right)^{1/2} \left( \frac{1}{2\pi} \int_0^{2\pi} \left( \left| z\frac{f'(z)}{f(z)} \right| + \left| \frac{zg'(z)}{g(z)} \right| \right)^2 \, d\theta \right)^{1/2} \\
+ \frac{1}{2\pi} \left| \int_0^{2\pi} \frac{g(z)}{f(z)} e^{-in\theta} \, d\theta \right| \\
= 2(L_1 L_2)^{1/2} + L_3. \tag{2.28}\]

Lebedev proves (see [11]) that if \( f(z) \in \mathcal{S} \) then
\[ \sum_{k=1}^{\infty} k |\gamma_k| r^{2k} \leq \log \frac{1}{1-r}. \tag{2.29}\]

By the definition of \( F(z) \) in (2.26), we obtain from (2.29)
\[ L_1 = \frac{1}{n^2} + 4 \sum_{k=1}^{\infty} \frac{k^2 |\gamma_k|^2 r^{2k}}{(n+k)^2} \leq \frac{1}{n^2} + 4 \sum_{k=1}^{\infty} k |\gamma_k|^2 r^{2k} \leq \frac{1}{n^2} + 4 \log \frac{1}{1-r}. \tag{2.30}\]

By Lemma 2.1, it follows that
\[ L_2 \leq \frac{1}{2\pi} \int_0^{2\pi} \left| z\frac{f'(z)}{f(z)} \right|^2 \, d\theta + 2 \left( \frac{1}{2\pi} \int_0^{2\pi} \left| z\frac{f'(z)}{f(z)} \right|^2 \, d\theta \right) \left( \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{zg'(z)}{g(z)} \right|^2 \, d\theta \right)^{1/2} \\
+ \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{zg'(z)}{g(z)} \right|^2 \, d\theta \\
\leq 4 \left( 1 + \frac{4}{1-r} \log \frac{1}{1-\sqrt{r}} \right). \tag{2.31}\]

Combining (2.30) and (2.31), from (2.28) we get
\[ L_2 \leq 4 \left( \frac{1}{n^2} + \frac{4}{n} \log \frac{1}{1-r} \right) \left( 1 + \frac{4}{1-r} \log \frac{1}{1-\sqrt{r}} \right)^{1/2} + L_3. \tag{2.32}\]

We obtain from (2.11), (2.24) and (2.32) that
\[ |2n\gamma_nr^n| \leq 20 + 16 \log \frac{1}{1-r} + 4 \left( \frac{1}{n^2} + \frac{4}{n} \log \frac{1}{1-r} \right) \left( 1 + \frac{4}{1-r} \log \frac{1}{1-\sqrt{r}} \right)^{1/2} \\
+ I_3 + 2J_3 + L_3. \tag{2.33}\]
From Lemma 2.1 and Lemma 2.3, we have

\[
I_3 + 2J_3 + L_3 = \frac{|b - 1|}{2\pi} \left| \int_0^{2\pi} \frac{g(z)}{f(z)} e^{-i\theta} d\theta \right| + \frac{2|1 - b|}{\pi} \left| \int_0^{2\pi} \frac{g(z)}{f(z)} d\theta \right|
\]

\[
+ \frac{1 - b}{2\pi} \left| \int_0^{2\pi} \frac{g(z)}{f(z)} e^{-i\theta} d\theta \right|
\]

\[
\leq \frac{3|b - 1|}{\pi} \left| \int_0^{2\pi} \frac{g(z)}{f(z)} d\theta \right| + \frac{3|b - 1|}{\pi} \left( \int_0^{2\pi} \left| \frac{zf'(z)}{f(z)} \right|^2 d\theta \right)^{\frac{1}{2}}
\]

\[
\leq 6|b - 1| \left( 1 + \frac{4}{1 - r} \log \frac{1}{1 - \sqrt{r}} \right)^{\frac{1}{2}} \left( \frac{1 + r}{1 - |2b - 1|} \right). \tag{2.34}
\]

Let \( r = 1 - \frac{1}{n}, \) \( n \geq 2, \) \( |2b - 1| \leq 1. \) We obtain from (2.33) and (2.34) that

\[
|\gamma_n| \leq \frac{1}{2n} \left( \frac{1 - \frac{1}{n}}{n} \right)^{-n} \left( 20 + 16 \log n + (1 + 8n \log n) \right)^{\frac{1}{2}}
\]

\[
\times \left[ 4 \left( \frac{1}{n^2} + \frac{4}{n} \log n \right)^{\frac{1}{2}} + \frac{6(2n - 1)|b - 1|}{n - (n - 1)|2b - 1|} \right], \tag{2.35}
\]

where \( A \) is an absolute constant. Thus, we have proved Theorem 2.1. \( \Box \)

**Remark 2.2.** If we take \( b = 1 \) in (2.35), we have results of [14].

**References**


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